On the Derived Functors of the Third Symmetric-Power Functor

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Abstract

We compute the derived functors of the third symmetric-power functor and their cross-effects for certain values. These calculations match predictions by the first named author and largely prove them in general.

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1 Introduction

Let R and S be rings. We recall that the construction of the left derived functors $L_kF : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ of any covariant right-exact functor $F : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ is achieved by applying three functors. The first functor constructs a projective resolution $P.$ of the R-module M that we wish to calculate the derived functor of. Then the functor F is applied to the resolution $P.$ giving the chain complex $F(P.)$. Lastly the kth derived functor L_kF is defined to be $H_k(F(P.))$, the kth homology of the chain complex $F(P.)$. However for a given module M the projective resolution of M is unique only up to chain-homotopy equivalence, so this construction crucially depends on the fact that F preserves chain-homotopies. In general, this fact does not hold when F is a nonlinear functor such as the ℓth symmetric-power functor, Sym^ℓ, or the ℓth exterior-power functor, Λ^ℓ. In the paper [2], Dold and Puppe overcome this problem and define the derived functors of non-linear functors by passing to the category of simplicial complexes using the Dold-Kan correspondence.

The Dold-Kan correspondence gives a pair of functors Γ and N that provide an equivalence between the category of non-negative chain complexes and the category of simplicial complexes; under this correspondence chain homotopies correspond to simplicial homotopies. Furthermore in the simplicial world all functors preserve simplicial homotopy (not just linear functors). Because of this the above definition of the derived functors of F becomes well defined for any functor when $F(P.)$ is replaced by the complex $NF\Gamma(P.)$.

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Now let R be a Noetherian commutative ring, let I be an ideal in R which is locally generated by a regular sequence of length d, let V be a finitely generated projective R/I-module and let $P.(V)$ be an R-projective resolution of V. In [5], the first named author explicitly calculates the modules $H_k N \operatorname{Sym}^\ell \Gamma P.(V)$ when $d = 1$ and ℓ is any positive integer, and also when $d = 2$ and $\ell = 2$ (cf. [5, Theorems 3.2 & 6.4]). As explained in [5, § 4], such calculations lie at the heart of a new approach to the seminal Adams-Riemann-Roch Theorem and hence to Grothendieck's Riemann-Roch theory.

In this paper, we look at the case when $d = 2$ and $\ell = 3$. Throughout this paper we use the symbol G_k to denote the derived functor

$$G_k: \begin{array}{ll} \mathcal{P}_{R/I} & \to R\text{-}\mathbf{mod} \\ V & \mapsto H_k N \operatorname{Sym}^3 \Gamma P.(V) \end{array}$$

where $\mathcal{P}_{R/I}$ denotes the category of finitely generated projective R/I-modules. Let L_1^3 denote the Schur functor indexed with Young diagram of shape $(2, 1)$ (cf. Definition 2.5). In [5, Example 6.6], the first named author made the following prediction about the functor G_k

$$G_k(V) \cong \begin{cases} \operatorname{Sym}^3(V) & k = 0 \\ L_1^3(V) \otimes I/I^2 & k = 1 \\ L_1^3(V) \otimes I/I^2 \otimes \Lambda^2(I/I^2) & k = 3 \\ D^3(V) \otimes \Lambda^2(I/I^2)^{\otimes 2} & k = 4 \\ 0 & k \geq 5 \end{cases}$$

and for the case when $k = 2$ he suggests that there exists an exact sequence:

$$0 \to D^2(V) \otimes V \otimes \Lambda^2(I/I^2) \to H_2 N \operatorname{Sym}^3 \Gamma(P.(V)) \\ \to \Lambda^3(V) \otimes \operatorname{Sym}^2(I/I^2) \to 0.$$

Note that the prediction for $G_3(V)$ is given in slightly different terms but, as will be explained after Proposition 2.6, this formulation is equivalent.

For each k let F_k stand for the prediction made for G_k to be (note that F_2 is defined only on objects, not on morphisms). We shall show that these predictions are true if V is the free module R/I of rank 1:

$$G_k(R/I) \cong F_k(R/I),$$

cf. Theorem 5.5. Moreover, we shall show that the following isomorphisms hold for the higher cross-effects (cf. Section 2 for the definition of cross-effect functors):

$$\mathrm{cr}_2(G_k)(R/I, R/I) \cong \mathrm{cr}_2(F_k)(R/I, R/I)$$
$$\mathrm{cr}_3(G_k)(R/I, R/I, R/I) \cong \mathrm{cr}_3(F_k)(R/I, R/I, R/I),$$

cf. Proposition 2.8 and Theorems 4.1 and 3.2.

[5, Theorem 1.5] implies that, if these isomorphisms commute with certain structural maps, then the predictions are true in general. To verify that the isomorphisms commute with the maps as required, we would need to not only complete some quite involved calculations, but we would also require a proper definition of the functor F_2, i.e., one which applies to objects and morphisms (rather than just to objects). Finding such a suitable candidate for F_2 remains an open problem.

2 Cross-effect functors

In [3], Eilenberg and Mac Lane introduced the theory of cross-effect functors. The theory of cross-effect functors are central to the calculations of this paper. In [5], the first named author proved a result (Theorem 1.5) which shows that two functors are isomorphic if they agree on certain data given by their cross-effect functors. In this section, we set the scene by introducing some of the theory of cross-effect functors from [3]. Then we introduce the aforementioned theorem from [5] (cf. Theorem 2.4) and exemplify it by showing that the Schur functor L_1^3 and the co-Schur functor $\tilde{L}_1^3$ are isomorphic. Furthermore we begin calculating the cross-effect functors of G_k and calculate $F_k(R/I), \mathrm{cr}_2(F_k)(R/I, R/I)$ and $\mathrm{cr}_3(F_k)(R/I, R/I, R/I)$.

Let $\mathcal{P}$ be an additive category, let $\mathcal{M}$ be an abelian category, and let $F : \mathcal{P} \to \mathcal{M}$ be a functor with $F(0) = 0$.

Definition 2.1. Let $k \geq 0$. For any $V_1, \ldots, V_k \in \mathcal{P}$ and $1 \leq i \leq k$, let

$$p_i : V_1 \oplus \ldots \oplus V_k \to V_i \to V_1 \oplus \ldots \oplus V_k$$

denote the ith projection. The k-functor $\mathrm{cr}_k(F) : \mathcal{P}^k \to \mathcal{M}$ defined by

$$(V_1, \ldots, V_k) \mapsto \mathrm{Im}\left(\sum_{j=1}^{k}(-1)^{k-j} \sum_{1\leq i_1 \leq \ldots \leq i_j \leq k} F(p_{i_1} + \ldots + p_{i_j})\right)$$

is called the *kth cross-effect of F*; here for any $V_1, \ldots, V_k, W_1, \ldots, W_k \in \mathcal{P}$ and $f_1 \in \mathrm{Hom}_{\mathcal{P}}(V_1, W_1), \ldots, f_k \in \mathrm{Hom}_{\mathcal{P}}(V_k, W_k)$, the map

$$\mathrm{cr}_k(F)(f_1, \ldots, f_k) : \mathrm{cr}_k(F)(V_1, \ldots, V_k) \to \mathrm{cr}_k(F)(W_1, \ldots, W_k)$$

is induced by $f_1 \oplus \ldots \oplus f_k \in \mathrm{Hom}_{\mathcal{P}}(V_1 \oplus \ldots \oplus V_k, W_1 \oplus \ldots \oplus W_k)$. The functor F is said to be *of degree less than k* if $\mathrm{cr}_k(F)$ is identically zero.

The cross-effect functors $\mathrm{cr}_k(F), k \geq 0$, have the following properties. We obviously have $\mathrm{cr}_0(F) \equiv 0$ and $\mathrm{cr}_1(F) = F$. Furthermore, we have $\mathrm{cr}_k(F)(V_1, \ldots, V_k) = 0$ if $V_i = 0$ for any $i \in \{1, \ldots, k\}$ (cf. [3, Theorem 9.2]). The most important property of cross-effects is given in the following proposition.

Proposition 2.2. For any $k, \ell \geq 1$ and $V_1, \ldots, V_\ell \in \mathcal{P}$, we have a canonical isomorphism

$$\mathrm{cr}_k(F)(\ldots, V_1 \oplus \ldots \oplus V_\ell, \ldots) \cong \bigoplus_{1\leq j \leq \ell} \bigoplus_{1 \leq i_1 < \ldots < i_j \leq \ell} \mathrm{cr}_{k+j-1}(\ldots, V_{i_1}, \ldots, V_{i_j}, \ldots)$$

which is functorial in $V_1, \ldots, V_\ell$. In particular, F is of degree $\leq k$, if and only if $\mathrm{cr}_k(F)$ is a k-additive functor.

Proof. Cf. [3, Theorem 9.1, Lemma 9.8, & Lemma 9.9]. Q.E.D.

From Proposition 2.2 we see that

$$F(V_1 \oplus V_2) \cong F(V_1) \oplus F(V_2) \oplus \mathrm{cr}_2(F)(V_1, V_2)$$

for all $V_1, V_2 \in \mathcal{P}$; i.e., $\mathrm{cr}_2(F)$ measures the deviation from linearity of the functor F. This isomorphism can also be used to define $\mathrm{cr}_2(F)$ (cf. [4, § 3]). Similarly, the higher cross-effects can be defined inductively by the isomorphism

$$\begin{aligned}\mathrm{cr}_k(F)(V_1, \ldots, V_{k-1}, V_k \oplus V_k') \cong\ & \mathrm{cr}_k(F)(V_1, \ldots, V_{k-1}, V_k) \\ & \oplus \mathrm{cr}_k(F)(V_1, \ldots, V_{k-1}, V_k') \\ & \oplus \mathrm{cr}_{k+1}(F)(V_1, \ldots, V_{k-1}, V_k, V_k')\end{aligned}$$

(cf. [4, § 7]). When actually calculating cross-effect functors it is this definition that we shall use.

Definition 2.3. Let $\ell \geq k \geq 1$, let $V_1, \ldots, V_k \in \mathcal{P}$, and let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{1, \ldots, \ell\}^k$ with $|\varepsilon| := \sum_{i=1}^k \varepsilon_i = \ell$. The composition

$$\begin{aligned}\Delta_\varepsilon : \mathrm{cr}_k(F)(V_1, \ldots, V_k) & \xrightarrow{\mathrm{cr}_k(F)(\Delta, \ldots, \Delta)} \mathrm{cr}_k(F)(V_1^{\varepsilon_1}, \ldots, V_k^{\varepsilon_k}) \\ & \overset{\pi}{\twoheadrightarrow} \mathrm{cr}_\ell(F)(V_1, \ldots, V_1, \ldots, V_k, \ldots, V_k)\end{aligned}$$

of the map $\mathrm{cr}_k(F)(\Delta, \ldots, \Delta)$ (induced by the diagonal maps $\Delta : V_i \to V_i^{\varepsilon_i}, i = 1, \ldots, k$) with the canonical projection π (according to Proposition 2.2) is called the *diagonal map associated with* ε. The analogous composition

$$\begin{aligned}+_\varepsilon : \mathrm{cr}_\ell(F)(V_1, \ldots, V_1, \ldots, V_k, \ldots, V_k) & \hookrightarrow \mathrm{cr}_k(F)(V_1^{\varepsilon_1}, \ldots, V_k^{\varepsilon_k}) \\ & \xrightarrow{\mathrm{cr}_k(F)(+, \ldots, +)} \mathrm{cr}_k(F)(V_1, \ldots, V_k)\end{aligned}$$

is called *plus map associated with* ε.

The maps Δ_ε and $+_\varepsilon$ form natural transformations between the functors $\mathrm{cr}_k(F)$ and $\mathrm{cr}_\ell(F)\circ(\Delta_{\varepsilon_1},\ldots,\Delta_{\varepsilon_k})$ from $\mathcal{P}^k$ to $\mathcal{M}$. One easily sees that the map Δ_ε can be decomposed into a composition of maps Δ_δ with $\delta\in\{1,2\}^i$ such that $|\delta| = i+1$ and $i\in\{k,\ldots,\ell-1\}$. The same holds for $+_\varepsilon$.

Theorem 2.4. Let A be a ring, $\mathcal{M}$ an abelian category, $d\in\mathbb{N}_+$, and let F and G be two be two functors of degree $\leq d$ with $F(0)=0=G(0)$ from the category of finitely generated A-modules to $\mathcal{M}$. Suppose that there exist isomorphisms

$$\alpha_i(A,\ldots,A) : \mathrm{cr}_i(F)(A,\ldots,A) \xrightarrow{\sim} \mathrm{cr}_i(G)(A,\ldots,A), \qquad i=1,\ldots,d,$$

which are compatible with the action of A in each component and which make the following diagrams commute for $i\in\{1,\ldots,d-1\}$ and $\varepsilon\in\{1,2\}^i$ with $|\varepsilon| = i+1$:

$$\begin{array}{ccc}
\mathrm{cr}_i(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_i(G)(A,\ldots,A) \\
\downarrow{\scriptstyle \Delta_\varepsilon} & & \downarrow{\scriptstyle \Delta_\varepsilon} \\
\mathrm{cr}_{i+1}(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_{i+1}(G)(A,\ldots,A)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{cr}_{i+1}(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_{i+1}(G)(A,\ldots,A) \\
\downarrow{\scriptstyle +_\varepsilon} & & \downarrow{\scriptstyle +_\varepsilon} \\
\mathrm{cr}_i(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_i(G)(A,\ldots,A).
\end{array}$$

Then the two functors F and G are isomorphic.

Proof. Cf. [5, Theorem 1.5]. Q.E.D.

Definition 2.5. Let A be a commutative ring. For any finitely generated A-module V we define $L_1^3(V)$ by the following exact sequences and we call L_1^3 the Schur functor indexed by the Young diagram of $(2,1)$.

$$\begin{array}{ccccccc}
\Lambda^3(V) & \longrightarrow & \Lambda^2(V)\otimes V & \longrightarrow & V\otimes \mathrm{Sym}^2(V) & \longrightarrow & \mathrm{Sym}^3(V) \\
 & & & \searrow \quad \nearrow & & & \\
 & & & L_1^3(V) & & & \\
 & & & \nearrow \quad \searrow & & & \\
 & & 0 & & 0 & &
\end{array}$$

And we define the co-Schur functor $\tilde{L}^3_1$ by the following exact sequence.

$$\begin{array}{ccccccc}
D^3(V) & \longrightarrow & D^2(V)\otimes V & \longrightarrow & V\otimes\Lambda^2(V) & \longrightarrow & \Lambda^3(V) \\
 & & & \searrow \quad \nearrow & & & \\
 & & & \tilde{L}^3_1(V) & & & \\
 & & & \nearrow \quad \searrow & & & \\
 & & 0 & & 0 & &
\end{array}$$

Here, the horizontal sequences are the well-known exact Koszul and co-Koszul complexes, cf. Definition 5.1.

We now give an example of how Theorem 2.4 can be used, by proving that the Schur functor L^3_1 and the co-Schur functor $\tilde{L}^3_1$ are isomorphic.

Proposition 2.6.

$$\mathrm{cr}_k(L^3_1)(A,\dots,A) \cong \mathrm{cr}_k(\tilde{L}^3_1)(A,\dots,A) \cong \begin{cases} 0 & k=1 \\ A\oplus A & k=2 \\ A\oplus A & k=3 \\ 0 & k\geq 4. \end{cases}$$

Moreover the two functors L^3_1 and $\tilde{L}^3_1$ are isomorphic.

Note this justifies the reformulation of the predictions for G_k as seen in the introduction, as the original prediction for G_3 in [5] was written as $G_3(V) \cong \tilde{L}^3_1(V)\otimes I/I^2\otimes\Lambda^2(I/I^2)$.

Proof. Let V and W be finitely generated projective A-modules. Definition 2.5 tells us the co-Schur functor $\tilde{L}^3_1(V)$ is given by the short exact sequence $0\to\tilde{L}^3_1(V)\to V\otimes\Lambda^2(V)\to\Lambda^3(V)\to 0$. In particular, this tells us that $\mathrm{cr}_1(\tilde{L}^3_1)(A)=0$, similarly we see that $L^3_1(A)=0$.

Next we want to compute $\mathrm{cr}_2(\tilde{L}^3_1)(A,A)$ and $\mathrm{cr}_2(L^3_1)(A,A)$. We have the short exact sequence $0\to L^3_1(V\oplus W)\to(V\oplus W)\otimes\Lambda^2(V\oplus W)\to\Lambda^3(V\oplus W)\to 0$. Using the canonical decomposition for Λ^n we get the following short exact sequence:

$$\begin{aligned}
&0\to\mathrm{cr}_2(\tilde{L}^3_1)(V,W)\to \\
&\quad V\otimes(V\otimes W)\oplus V\otimes\Lambda^2(W)\oplus W\otimes\Lambda^2(V)\oplus W\otimes(V\otimes W) \\
&\quad \to\Lambda^2(V)\otimes W\oplus V\otimes\Lambda^2(W)\to 0.
\end{aligned}$$

Therefore $\mathrm{cr}_2(\tilde{L}^3_1)(V,W)\cong V^{\otimes 2}\otimes W\oplus V\otimes W^{\otimes 2}$. Similarly we find that $\mathrm{cr}_2(L^3_1)(V,W)\cong V^{\otimes 2}\otimes W\oplus V\otimes W^{\otimes 2}$, so $\mathrm{cr}_2(\tilde{L}^3_1)$ and $\mathrm{cr}_2(L^3_1)$ are isomorphic as bi-functors. Because of the way that higher cross-effects are calculated from lower cross-effects we see that, for $k \geq 2$, the k-functors $\operatorname{cr}_k(\tilde{L}_1^3)$ and $\operatorname{cr}_k(L_1^3)$ will be isomorphic and the corresponding diagrams of $+$ and Δ maps will commute. We easily verify that $\operatorname{cr}_2(L_1^3)(A,A) \cong \operatorname{cr}_2(\tilde{L}_1^3)(A,A)$ and $\operatorname{cr}_3(L_1^3)(A,A,A) \cong \operatorname{cr}_3(\tilde{L}_1^3)(A,A,A)$ are free of rank 2 over A, as stated.

Since $\operatorname{cr}_1(\tilde{L}_1^3)(A) \cong \operatorname{cr}_1(L_1^3)(A) \cong 0$ the corresponding $+$ and Δ maps involving $\operatorname{cr}_1(\tilde{L}_1^3)(A)$ and $\operatorname{cr}_1(L_1^3)(A)$ will just be zero maps, and hence will commute with the isomorphism. So Theorem 2.4 implies that $\tilde{L}_1^3$ is isomorphic to L_1^3. Q.E.D.

The following lemma begins the work calculating the cross-effects of G_k that we shall complete in the following chapters. Let A and A' be simplicial R-modules; the reader should be aware that in the following lemma and its proof that $A \otimes A'$ means the simplicial module whose nth place is $A_n \otimes A'_n$.

Lemma 2.7.

$$\begin{aligned}\operatorname{cr}_2(G_k)(V,W) \cong{}& H_k N(\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)) \\ &\oplus H_k N(\Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(W))\end{aligned}$$

$$\operatorname{cr}_3(G_k)(V,W,X) \cong H_k N\big(\Gamma P.(V) \otimes \Gamma P.(W) \otimes \Gamma P.(X)\big).$$

Proof. First we calculate $G_k(V \oplus W)$ for R-modules V, W to give us an expression for $\operatorname{cr}_2(G_k)(V,W)$. To do this we use the fact that $P., \Gamma, N$ and H_k are linear functors and also the canonical decomposition $\operatorname{Sym}^3(V \oplus W) \cong \bigoplus_{\ell=0}^{3} \operatorname{Sym}^{3-\ell}(V) \otimes \operatorname{Sym}^{\ell}(W)$.

$$\begin{aligned}G_k(V \oplus W) &= H_k N \operatorname{Sym}^3 \Gamma P.(V \oplus W) \\ &\cong H_k N\Big(\operatorname{Sym}^3 \Gamma P.(V) \oplus \operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W) \\ &\qquad \oplus \Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(W) \oplus \operatorname{Sym}^3 \Gamma P.(W)\Big) \\ &\cong G_k(V) \oplus H_k N\Big(\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)\Big) \\ &\qquad \oplus H_k N\Big(\Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(W)\Big) \oplus G_k(W).\end{aligned}$$

And hence we get the desired expression for $\operatorname{cr}_2(G_k)(V,W)$.

We now complete the proof by using our expression for $\operatorname{cr}_2(G_k)(V,W)$, to get an expression for $\operatorname{cr}_3(G_k)(V,W,X)$. We start this by calculating $\operatorname{cr}_2(G_k)(V, W \oplus X)$ for R-modules V, W, X. To simplify our calculation we split up our expression for $\operatorname{cr}_2(G_k)(V, W \oplus X)$ into $H_k N\big(\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W \oplus X)\big)$ and $H_k N\big(\Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(W \oplus X)\big)$, calculate each part separately, then add them together afterwards.

$$\begin{aligned} H_kN\Big(\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W \oplus X)\Big) \\ &\cong H_kN\Big(\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)\Big) \\ &\quad \oplus H_kN\Big(\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(X)\Big). \end{aligned}$$

$$\begin{aligned} H_kN\Big(\Gamma P.(V)\otimes \operatorname{Sym}^2 \Gamma P.(W \oplus X)\Big) \\ &\cong H_kN\Big(\Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(W))\Big) \\ &\oplus H_kN\Big(\Gamma P.(V) \otimes \big(\Gamma P.(W) \otimes \Gamma P.(X)\big)\Big) \\ &\oplus H_kN\Big(\Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(X)\Big). \end{aligned}$$

This gives us the following expression for $\operatorname{cr}_2(G_k)(V, W \oplus X)$:

$$\begin{aligned} \operatorname{cr}_2(G_k)(V, W \oplus X) \cong &\ \operatorname{cr}_2(G_k)(V, W) \oplus \operatorname{cr}_2(G_k)(V, X) \\ &\oplus H_kN\Big(\Gamma P.(V) \otimes \big(\Gamma P.(W) \otimes \Gamma P.(X)\big)\Big). \end{aligned}$$

Therefore $\operatorname{cr}_3(G_k)(V, W, X) \cong H_kN\big(\Gamma P.(V)\otimes\Gamma P.(W)\otimes\Gamma P.(X)\big)$ as desired.

Q.E.D.

We finally calculate the cross-effects of the predictions evaluated at the free R/I-module of rank 1.

Proposition 2.8. We have the following R/I-module isomorphisms:

$$F_k(R/I) \cong \begin{cases} R/I & k = 0 \\ 0 & k = 1 \\ \Lambda^2(I/I^2) & k = 2 \\ 0 & k = 3 \\ \Lambda^2(I/I^2)^{\otimes 2} & k = 4 \end{cases}$$

$$\begin{aligned} &\operatorname{cr}_2(F_k)(R/I, R/I) \\ &\quad\cong \begin{cases} R/I \oplus R/I & k = 0 \\ I/I^2 \oplus I/I^2 & k = 1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k = 2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k = 3 \\ \Lambda^2(I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2)^{\otimes 2} & k = 4 \end{cases} \end{aligned}$$

$$\mathrm{cr}_3(F_k)(R/I, R/I, R/I)$$
$$\cong \begin{cases} R/I & k=0 \\ I/I^2 \oplus I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus (I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2) \oplus \mathrm{Sym}^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} & k=4. \end{cases}$$

Proof. These results follow from a simple application of Proposition 2.6, and the canonical decompositions for the symmetric-, exterior-, and divided-power functors. Q.E.D.

3 The iterated Eilenberg-Zilber Theorem and calculating $\mathrm{cr}_3(G_k)(R/I, R/I, R/I)$

Let A^1 and A^2 be simplicial R-modules. The Eilenberg-Zilber Theorem (cf. [6, § 28], in particular Corollary 29.6) tells us that $N\Delta(A^1 \otimes A^2)$ is chain homotopic to $\mathrm{Tot}(NA^1 \otimes NA^2)$ (where $\Delta(A^1 \otimes A^2)$ denotes the diagonal of the bisimplicial complex $A^1 \otimes A^2$). Let C^1, C'^1, C^2, C'^2 be chain complexes of R-modules. If C^1 is chain homotopic to C'^1 and C^2 is chain homotopic to C'^2 then $\mathrm{Tot}(C^1 \otimes C^2) \cong \mathrm{Tot}(C'^1 \otimes C'^2)$; because of this the Eilenberg-Zilber Theorem can be iterated to give us the following theorem.

Theorem 3.1 (The iterated Eilberg-Zilber Theorem)**.** Let $n \in \mathbb{N}$ with $n \geq 2$ and let $A^1, \ldots, A^n$ be simplicial complexes. Then the complexes $N\Delta(A^1 \otimes \ldots \otimes A^n)$ and $\mathrm{Tot}(NA^1 \otimes \ldots \otimes NA^n)$ are chain homotopic and (consequently) they are quasi-isomorphic:

$$H_k(N\Delta(A^1 \otimes \ldots \otimes A^n)) \cong H_k(\mathrm{Tot}(NA^1 \otimes \ldots \otimes NA^n)).$$

The following theorem, together with the canonical decomposition for exterior powers, shows that the third cross-effect functor for the derived functors of Sym^3 evaluated on $(R/I, R/I, R/I)$ matches the predictions (cf. Proposition 2.8).

Theorem 3.2.

$$\mathrm{cr}_3(G_k)(R/I, R/I, R/I) \cong \Lambda^k(I/I^2 \oplus I/I^2)$$

Proof. From Lemma 2.7 we know that

$$\mathrm{cr}_3(G_k)(V, W, X) \cong H_k N\big(\Gamma P.(V) \otimes \Gamma P.(W) \otimes \Gamma P.(X)\big).$$

Here $\Gamma P.(V) \otimes \Gamma P.(W) \otimes \Gamma P.(X)$ stands for the simplicial complex whose kth place is $\Gamma P_k(V) \otimes \Gamma P_k(W) \otimes \Gamma P_k(X)$. But we may consider this to be the diagonal of the tri-simplicial complex whose (k,ℓ,m)th place is $\Gamma P_k(V) \otimes \Gamma P_\ell(W) \otimes \Gamma P_m(X)$, and by doing so we write

$$\mathrm{cr}_3(G_k)(V,W,X) \cong H_k N\Delta\big(\Gamma P.(V) \otimes \Gamma P.(W) \otimes \Gamma P.(X)\big).$$

The iterated Eilenberg-Zilber Theorem tells us that

$$\begin{aligned} &H_k N\Delta\big(\Gamma P.(R/I) \otimes \Gamma P.(R/I) \otimes \Gamma P.(R/I)\big) \\ &\qquad\qquad \cong H_k \operatorname{Tot}\big(P.(R/I) \otimes P.(R/I) \otimes P.(R/I)\big). \end{aligned}$$

[5, Theorem 5.1] tells us that

$$H_k \operatorname{Tot}\big(P.(R/I) \otimes P.(R/I) \otimes P.(R/I)\big) \cong \Lambda^k(I/I^2 \oplus I/I^2),$$

as desired. Q.E.D.

4 The derived functors of Sym^2 and calculating $\mathrm{cr}_2(G_k)(R/I, R/I)$

The following theorem shows that the second cross-effect functor of the derived functors of Sym^3 evaluated on $(R/I, R/I)$ matches the predictions (cf. Proposition 2.8). Essential to the following proof is the information that has already been calculated about $H_k N \operatorname{Sym}^2 \Gamma P.$ in [5], and the use of the Hypertor functor to exploit this.

Theorem 4.1.

$$\mathrm{cr}_2(G_k)(R/I,R/I) \cong \begin{cases} R/I \oplus R/I & k=0 \\ I/I^2 \oplus I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2)^{\otimes 2} & k=4 \\ 0 & k \geq 5. \end{cases}$$

Proof. Lemma 2.7 tells us that

$$\begin{aligned} \mathrm{cr}_2(G_k)(V,W) &\cong H_k N(\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)) \\ &\quad \oplus H_k N(\Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(W)). \end{aligned}$$

Here $\operatorname{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)$ and $\Gamma P.(V) \otimes \operatorname{Sym}^2 \Gamma P.(W)$ stand for simplicial modules whose kth place is $\operatorname{Sym}^2 \Gamma P_k(V) \otimes \Gamma P_k(W)$ and $\Gamma P_k(V) \otimes \operatorname{Sym}^2 \Gamma P_k(W)$ respectively; however we may consider them to be the

$$\begin{array}{ccccc}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & \Lambda^2(I/I^2)^{\otimes 2} & 0 & \Lambda^2(I/I^2) \\
\cdots & 0 & I/I^2 \otimes \Lambda^2(I/I^2) & 0 & I/I^2 \\
\cdots & 0 & \Lambda^2(I/I^2) & 0 & R/I.
\end{array}$$

FIGURE 1.

diagonal of the bi-simplicial modules whose (k, ℓ)th place is $\mathrm{Sym}^2 \Gamma P_k(V) \otimes \Gamma P_\ell(W)$ and $\Gamma P_k(V) \otimes \mathrm{Sym}^2 \Gamma P_\ell(W)$ respectively, therefore we can write

$$\begin{aligned} \mathrm{cr}_2(G_k)(V, W) \cong\ & H_k N\Delta(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)) \\ & \oplus H_k N\Delta(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W)). \end{aligned}$$

We now start to calculate $H_k N\Delta(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W))$, later we shall calculate $H_k N\Delta(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W))$ and then add the two together to get an expression for $\mathrm{cr}_2(G_k)(V, W)$.
Using the Eilenberg-Zilber Theorem (Theorem 3.1) we see that

$$\begin{aligned} H_k N\Delta(\mathrm{Sym}^2(\Gamma P.(V) \otimes \Gamma P.(W)) &\cong H_k \operatorname{Tot}(N \mathrm{Sym}^2 \Gamma P.(V) \otimes N\Gamma P.(W)) \\ &\cong H_k \operatorname{Tot}(N \mathrm{Sym}^2 \Gamma P.(V) \otimes P.(W)). \end{aligned}$$

So we want to calculate $H_k \operatorname{Tot}(N \mathrm{Sym}^2 \Gamma P.(V) \otimes P.(W))$, but this is just the definition of the hypertor $\mathbf{Tor}_i^R(N \mathrm{Sym}^2 \Gamma P.(V), W)$. [7, Application 5.7.8] gives us a spectral sequence to calculate hypertor

$${}^{II}E^2_{pq} = \operatorname{Tor}_p(H_q(A), B) \Rightarrow \mathbf{Tor}^R_{p+q}(A_*, B).$$

[5, Theorem 6.4] tells us that

$$H_k N \mathrm{Sym}^2 \Gamma(P.(V)) \cong \begin{cases} \mathrm{Sym}^2(V) & k = 0 \\ \Lambda^2(V) \otimes I/I^2 & k = 1 \\ D^2(V) \otimes \Lambda^2(I/I^2) & k = 2 \\ 0 & k \geq 3. \end{cases}$$

We have $\mathrm{Sym}^2(R/I) \cong R/I, \Lambda^2(R/I) \cong 0$ and $D^2(R/I) \cong R/I$. Hence the terms in the second sheet of our spectral sequence are $\operatorname{Tor}_p(R/I, R/I)$ in the zeroth column, $\operatorname{Tor}_p(0, R/I) = 0$ in the first column, $\operatorname{Tor}_p(\Lambda(I/I^2), R/I)$ in the second column and 0 everywhere else. From [5, Example 5.2] (also cf. [5, Theorem 5.1]) we know that $\mathrm{Tor}_k(R/I, R/I) \cong \Lambda^k(I/I^2)$ and hence have $\mathrm{Tor}_k(V,W) \cong V \otimes W \otimes \Lambda^k(I/I^2)$. So the second sheet of the spectral sequence looks like Figure 1.

The differentials on this level of the spectral sequence are -2 in the p-direction and $+1$ in the q-direction, so each differential either comes from or goes to a zero module. Hence the differentials are all zero maps, i.e., the spectral sequence has already converged on the second level. Therefore

$$H_k N\Delta\big(\operatorname{Sym}^2 \Gamma P.(R/I) \otimes \Gamma P.(R/I)\big) \cong \begin{cases} R/I & k=0 \\ I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} & k=4 \\ 0 & k \geq 5. \end{cases}$$

By symmetry

$$\begin{aligned} &H_k N\Delta\big(\Gamma P.(R/I) \otimes \operatorname{Sym}^2 \Gamma P.(R/I)\big) \\ &\qquad \cong H_k N\Delta\big(\operatorname{Sym}^2 \Gamma P.(R/I) \otimes \Gamma P.(R/I)\big). \end{aligned}$$

Hence

$$\begin{aligned} &\mathrm{cr}_2(G_k)(R/I, R/I) \\ &\quad \cong \begin{cases} R/I \oplus R/I & k=0 \\ I/I^2 \oplus I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2)^{\otimes 2} & k=4 \\ 0 & k \geq 5. \end{cases} \end{aligned}$$

Q.E.D.

5 The Cauchy decomposition of $\operatorname{Sym}^3(P \otimes Q)$ and calculating $G_k(R/I)$

In this final section, we complete our calculations of G_k, but before we proceed we introduce the Cauchy decomposition given in [1, Chapter III] as it applies to the third symmetric power, and remind the reader of Koszul and co-Koszul complexes.

Let R be a ring. Let P and Q be finitely generated projective R-modules. We now summarize the Cauchy decomposition of $\operatorname{Sym}^3(P \otimes Q)$. This decomposition plays a central role in Theorem 5.5, our calculation of $G_k(R/I)$. A three step filtration

$$0 \subset M_{(3)}(\mathrm{Sym}^3(P \otimes Q)) \subset M_{(2,1)}(\mathrm{Sym}^3(P \otimes Q)) \subset \mathrm{Sym}^3(P \otimes Q)$$

is put on $\mathrm{Sym}^3(P \otimes Q)$. The $M_{(3)}(\mathrm{Sym}^3(P \otimes Q))$ part is defined to be the image of the determinant map

$$\Lambda^3 P \otimes \Lambda^3 Q \to \mathrm{Sym}^3(P \otimes Q)$$
$$p_1 \wedge p_2 \wedge p_3 \otimes q_1 \wedge q_2 \wedge q_3 \mapsto \begin{vmatrix} p_1 \otimes q_1 & p_1 \otimes q_2 & p_1 \otimes q_3 \\ p_2 \otimes q_1 & p_2 \otimes q_2 & p_2 \otimes q_3 \\ p_3 \otimes q_1 & p_3 \otimes q_2 & p_3 \otimes q_3 \end{vmatrix}$$

(note this is simply isomorphic to $\Lambda^3 P \otimes \Lambda^3 Q$). The $M_{(2,1)}(\mathrm{Sym}^3(P \otimes Q))$ part is defined to be equal to the previous part $M_{(3)}(\mathrm{Sym}^3(P \otimes Q))$ part plus the image of the following homomorphism:

$$\Lambda^2 P \otimes P \otimes \Lambda^2 Q \otimes Q \to \mathrm{Sym}^3(P \otimes Q)$$
$$p_1 \wedge p_2 \otimes p_3 \otimes q_1 \wedge q_2 \otimes q_3 \mapsto \begin{vmatrix} p_1 \otimes q_1 & p_1 \otimes q_2 \\ p_2 \otimes q_1 & p_2 \otimes q_2 \end{vmatrix} (p_3 \otimes q_3).$$

The quotients of this filtration are given by the short exact sequences

$$0 \to \Lambda^3 P \otimes \Lambda^3 Q \to M_{(2,1)}(\mathrm{Sym}^3(P \otimes Q)) \to L_1^3 P \otimes L_1^3 Q \to 0$$

and

$$0 \to M_{(2,1)}(\mathrm{Sym}^3(P \otimes Q)) \to \mathrm{Sym}^3(P \otimes Q) \to \mathrm{Sym}^3 P \otimes \mathrm{Sym}^3 Q \to 0.$$

We now properly introduce the definition of Koszul complexes, which we use for projective resolutions.

Definition 5.1. Let $f : P \to Q$ be a homomorphism between two finitely generated projective R-modules, and $n \in \mathbb{N}$. Let $\mathrm{Kos}^n(f)$ be the *Koszul complex*

$$0 \to \Lambda^n P \overset{d_{n-1}}{\to} \Lambda^{n-1} P \otimes Q \overset{d_{n-2}}{\to} \ldots \overset{d_1}{\to} P \otimes \mathrm{Sym}^{n-1} Q \overset{d_0}{\to} \mathrm{Sym}^n Q \to 0$$

where, for $k \in \{0, 1, \ldots, n-1\}$, the differential

$$d_k : \Lambda^{k+1} P \otimes \mathrm{Sym}^{n-k-1} Q \to \Lambda^k P \otimes \mathrm{Sym}^{n-k} Q$$

acts by

$$p_1 \wedge \ldots \wedge p_{k+1} \otimes q_{k+2} ... q_n$$
$$\mapsto \Sigma_{i=1}^{k+1} (-1)^{k+1-i} p_1 \wedge \ldots \wedge \hat{p}_i \wedge \ldots \wedge p_{k+1} \otimes f(p_i) q_{k+2} ... q_n.$$

Now let $f^* : Q^* \to P^*$ denote the dual map, then the part of the Koszul complex $\mathrm{Kos}^n(f^*)$ in the kth degree is $\Lambda^k Q^* \otimes \mathrm{Sym}^{n-k} P^*$. The dual of this chain complex is a co-chain complex with the part in the kth degree being $\left(\Lambda^k Q^* \otimes \mathrm{Sym}^{n-k} P^*\right)^* \cong \Lambda^k Q \otimes D^{n-k} P$, i.e.,

$$0 \leftarrow \Lambda^n Q \leftarrow \Lambda^{n-1} Q \otimes P \leftarrow \ldots \leftarrow Q \otimes D^{n-1} P \leftarrow D^n P \leftarrow 0.$$

We call this the *co-Koszul complex* and denote it by $\widetilde{\mathrm{Kos}^n}(f)$.

Remark 5.2. It is well known that the complexes $\mathrm{Kos}(f)$ and $\widetilde{\mathrm{Kos}}(f)$ are exact if f is an isomorphism.

The following two propositions will be useful in the proof of Theorem 5.5.

Proposition 5.3. Let $f : P \to Q$ be a homomorphism between two finitely generated projective R-modules. If we consider $P \to Q$ to be a chain complex concentrated in degrees 1 and 0 then we have quasi-isomorphisms

$$\mathrm{Kos}^n(f) \cong N \,\mathrm{Sym}^n \Gamma(P \to Q)$$

and

$$\widetilde{\mathrm{Kos}^n}(f) \cong N \Lambda^n \Gamma(P \to Q),$$

where Γ and N are the functors of the Dold-Kan correspondence.

Proof. Cf. [5, Proposition 2.4 & Remark 3.6]. Q.E.D.

Proposition 5.4. Let $A.$ and $B.$ be bounded complexes of finitely generated projective R-modules. Then we have a spectral sequence

$${}^{II}E^2_{p,q} = \bigoplus_{q=q'+q''} \mathrm{Tor}_p(H_{q'}(A.), H_{q''}(B.)) \Rightarrow \mathbf{Tor}_{p+q}(A., B.)$$

where $\mathbf{Tor}_n(A., B.)$ is defined as $H_n \,\mathrm{Tot}(A. \otimes B.)$. In particular, if $A.'$ and $B.'$ are complexes as above and $A \to A'$ and $B \to B'$ are quasi-isomorphisms then the induced morphism

$$\mathrm{Tot}(A. \otimes B.) \to \mathrm{Tot}(A.' \otimes B.')$$

is a quasi-isomorphism as well.

Proof. Cf. [7, Application 5.7.8]. Q.E.D.

The following theorem shows that if I is globally generated by a regular sequence then the derived functors of Sym^3 evaluated on R/I match the predictions.

Theorem 5.5. If I is generated by a regular sequence of length 2 then the module $G_k(R/I)$ is a free R/I-module of rank 1, for $k = 0, 2$ or 4 and otherwise of rank 0.

Proof. Let f, g be a regular sequence in R, and let I be generated by it. Also we let $K.$ denote the complex $\ldots \to 0 \to R \xrightarrow{f} R$ and $L.$ denote the Koszul complex $\ldots \to 0 \to R \xrightarrow{g} R$. We use the complex

$$\mathrm{Kos}^2(R \oplus R \overset{(f,g)}{\to} R) \cong \mathrm{Tot}(K. \otimes L.)$$

as a resolution of R/I, and see that

$$\begin{aligned} G_k(R/I) :=& H_k N \operatorname{Sym}^3 \Gamma \operatorname{Tot}(K. \otimes L.) \\ &\cong H_k N \operatorname{Sym}^3 \Gamma \operatorname{Tot}(N\Gamma K. \otimes N\Gamma L.). \end{aligned}$$

Theorem 3.1 tells us that $\mathrm{Tot}(N\Gamma K. \otimes N\Gamma L.)$ is chain homotopic to $N\Delta(\Gamma K. \otimes \Gamma L.)$. Applying Γ turns the notion of chain homotopy into simplicial homotopy, all functors preserve homotopy in the simplicial world and N changes the notion of simplicial homotopy into the notion of chain homotopy. So $N \operatorname{Sym}^3 \Gamma$ turns the chain homotopy between $\mathrm{Tot}(N\Gamma K. \otimes N\Gamma L.)$ and $N\Delta(\Gamma K. \otimes \Gamma L.)$ into a chain homotopy between $N \operatorname{Sym}^3 \Gamma \operatorname{Tot}(N\Gamma K. \otimes N\Gamma L.)$ and $N \operatorname{Sym}^3 \Gamma N\Delta(\Gamma K. \otimes \Gamma L.)$. Chain homotopic complexes are quasi-isomorphic, so continuing our calculation of $G_k(R/I)$ where we left off we get:

$$\begin{aligned} G_k(R/I) &\cong H_k N \operatorname{Sym}^3 \Gamma N\Delta(\Gamma K. \otimes \Gamma L.) \cong H_k N \operatorname{Sym}^3 \Delta(\Gamma K. \otimes \Gamma L.) \\ &\cong H_k N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.). \end{aligned}$$

Now we calculate $G_k(R/I)$ by calculating $H_k N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)$. We cannot calculate this directly, so instead we employ the short exact sequences that come from the Cauchy decomposition; these short exact sequences will allow us to get information about the homology modules of $N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)$ from easier to calculate homologies.

From the Cauchy decomposition we get the following short exact sequences of bisimplicial modules:

$$0 \to \Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L. \to M_{(2,1)}(\operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to L_1^3\Gamma K. \otimes L_1^3\Gamma L. \to 0$$

$$\begin{aligned} 0 \to M_{(2,1)}(\operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)) &\to \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \\ &\to \operatorname{Sym}^3 \Gamma K. \otimes \operatorname{Sym}^3 \Gamma L. \to 0. \end{aligned}$$

Applying $N\Delta$ to these gives us two short exact sequences of chain complexes. We can turn the homologies of these into two long exact sequences, this will allow us to get information about the homology modules of

$$M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.))$$

from the easier to calculate homology modules of $\Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L.$ and $L_1^3\Gamma K. \otimes L_1^3\Gamma L.$. This information about the homologies of

$$M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.))$$

together with the homologies of the easier to calculate homology modules of $\mathrm{Sym}^3 \Gamma K. \otimes \mathrm{Sym}^3 \Gamma L.$ will tell us the ranks of the homologies of $\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)$.

First we calculate the homologies of $L_1^3\Gamma K$. The definition of L_1^3 gives us the following short exact sequence for any finitely generated projective module P

$$0 \to L_1^3 P \to P \otimes \mathrm{Sym}^2 P \to \mathrm{Sym}^3 P \to 0,$$

which gives us the short exact sequence of simplicial complexes

$$0 \to L_1^3\Gamma K. \to \Gamma K. \otimes \mathrm{Sym}^2 \Gamma K. \to \mathrm{Sym}^3 \Gamma K. \to 0,$$

the middle term of this short exact sequence is the simplicial complex whose kth term is $\Gamma K_k \otimes \mathrm{Sym}^2 \Gamma K_k$. We think of this middle term instead as the diagonal of a bisimplicial complex whose (k,ℓ)th term is $\Gamma K_k \otimes \mathrm{Sym}^2 \Gamma K_\ell$. Now applying the functor N turns this into a short exact sequence of chain complexes

$$0 \to N L_1^3\Gamma K. \to N\Delta(\Gamma K. \otimes \mathrm{Sym}^2 \Gamma K.) \to N \,\mathrm{Sym}^3 \Gamma K. \to 0,$$

and from this we can create a long exact sequence that gives us information about $H_k N L_1^3 \Gamma K.$.

Applying the Eilenberg-Zilber Theorem, then with Propositions 5.3 and 5.4 we see that

$$\begin{aligned} H_k N\Delta(\Gamma K. \otimes \mathrm{Sym}^2 \Gamma K.) &\cong H_k \operatorname{Tot}(N\Gamma K. \otimes N \,\mathrm{Sym}^2 \Gamma K.) \\ &\cong H_k \operatorname{Tot}(K. \otimes \mathrm{Kos}^2(f)). \end{aligned}$$

Now

$$\begin{aligned} \mathrm{Kos}^2(f) &= (\Lambda^2(R) \to \Lambda^1(R) \otimes \mathrm{Sym}^1(R) \to \mathrm{Sym}^2(R)) \\ &= (0 \to R \xrightarrow{f} R) = K., \end{aligned}$$

and therefore

$$\begin{aligned} H_k N\Delta(\Gamma K. \otimes \operatorname{Sym}^2 \Gamma K.) &\cong H_k \operatorname{Tot}(K. \otimes K.) = H_k(P.(R/(f))^{\otimes 2}) \\ &= \Lambda^k((f)/(f)^2) \cong \begin{cases} R/(f) & k = 0, 1 \\ 0 & k > 1 \end{cases} \end{aligned}$$

with the last step given by the isomorphism

$$H_k(P.(R/J)^{\otimes m}) \cong \Lambda^k((J/J^2)^{m-1}),$$

cf. [5, Theorem 5.1]. Using Proposition 5.3 we get

$$H_k N \operatorname{Sym}^3 \Gamma K. \cong H_k(\operatorname{Kos}^3(f)),$$

and now

$$\begin{aligned} \operatorname{Kos}^3(f) &= (\Lambda^3(R) \to \Lambda^2(R) \otimes R \to \Lambda^1(R) \otimes \operatorname{Sym}^2(R) \to \operatorname{Sym}^3(R)) \\ &= (0 \to 0 \to R \xrightarrow{f} R) = K., \end{aligned}$$

hence

$$H_k N \operatorname{Sym}^3 \Gamma K. \cong \begin{cases} R/(f) & k = 0 \\ 0 & k \neq 0. \end{cases}$$

Therefore, the long exact sequence of homologies that we get from the short exact sequence

$$0 \to N L_1^3 \Gamma K. \to N\Delta(\Gamma K. \otimes \operatorname{Sym}^2 \Gamma K.) \to N \operatorname{Sym}^3 \Gamma K. \to 0$$

is given as in Figure 2, and so we get

$$H_k N L_1^3 \Gamma K. \cong \begin{cases} 0 & k \neq 1 \\ R/(f) & k = 1. \end{cases}$$

Similarly we get

$$H_k N L_1^3 \Gamma L. \cong \begin{cases} 0 & k \neq 1 \\ R/(g) & k = 1. \end{cases}$$

We work with the short exact sequence of simplicial modules

$$0 \to \Lambda^3 \Gamma K. \otimes \Lambda^3 \Gamma L. \to M_{(2,1)}(\operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to L_1^3 \Gamma K. \otimes L_1^3 \Gamma L. \to 0.$$

As above, we rewrite the left and right hand side of this exact sequence using Δ and we apply the functor N to get the following short exact sequence of chain complexes

$$\begin{aligned} 0 \to N\Delta(\Lambda^3 \Gamma K. \otimes \Lambda^3 \Gamma L.) &\to N M_{(2,1)}(\operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)) \\ &\to N\Delta(L_1^3 \Gamma K. \otimes L_1^3 \Gamma L.) \to 0. \end{aligned}$$

$$\begin{array}{ccccccc}
\cdots \to H_3 N L_1^3 \Gamma K. & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \\
\to H_2 N L_1^3 \Gamma K. & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \\
\to H_1 N L_1^3 \Gamma K. & \longrightarrow & R/(f) & \longrightarrow & 0 & \longrightarrow & \\
\to H_0 N L_1^3 \Gamma K. & \longrightarrow & R/(f) & \longrightarrow & R/(f) & \longrightarrow & 0
\end{array}$$

Figure 2.

The Eilenberg-Zilber Theorem tells us that

$$H_k N\Delta(\Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L.) \cong H_k \operatorname{Tot}(N\Lambda^3\Gamma K. \otimes N\Lambda^3\Gamma L.)$$

and

$$H_k N\Delta(L_1^3\Gamma K. \otimes L_1^3\Gamma L.) \cong H_k \operatorname{Tot}(NL_1^3\Gamma K. \otimes NL_1^3\Gamma L.).$$

By Propositions 5.3 and 5.4, we get

$$H_k \operatorname{Tot}(N\Lambda^3\Gamma K. \otimes N\Lambda^3\Gamma L.) \cong H_k \operatorname{Tot}(\widetilde{\operatorname{Kos}^3}(f) \otimes \widetilde{\operatorname{Kos}^3}(g)).$$

Since

$$\begin{aligned}\widetilde{\operatorname{Kos}^3}(f) &= (D^3(R) \to D^2(R) \otimes R \to R \otimes \Lambda^2(R) \to \Lambda^3(R)) \\ &= (R \xrightarrow{f} R \to 0 \to 0) = K.[-2],\end{aligned}$$

and similarly $\widetilde{\operatorname{Kos}^3}(g) = L.[-2]$, we obtain

$$\begin{aligned}H_k \operatorname{Tot}(N\Lambda^3\Gamma K. \otimes N\Lambda^3\Gamma L.) &\cong H_k \operatorname{Tot}(K.[-2] \otimes L.[-2]) \\ &\cong H_k(\operatorname{Tot}(K. \otimes L.)[-4]) \\ &\cong H_{k-4}(P.(R/I)) \cong \begin{cases} 0 & k \neq 4 \\ R/I & k = 4. \end{cases}\end{aligned}$$

In order to calculate $H_k \operatorname{Tot}(NL_1^3\Gamma K. \otimes NL_1^3\Gamma L.)$, we note that it is the hypertor functor $\mathbf{Tor}_k(NL_1^3\Gamma K., NL_1^3\Gamma L.)$ and use the hypertor spectral sequence of Proposition 5.4 taking $A. = NL_1^3\Gamma K.$ and $B. = NL_1^3\Gamma L.$. But since $H_k NL_1^3\Gamma K.$ and $H_k NL_1^3\Gamma L.$ are 0 unless $k = 1$ (see above) this spectral

$$\cdots \to 0 \to H_5 NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to 0 \to$$
$$R/I \to H_4 NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to 0 \to$$
$$0 \to H_3 NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to 0 \to$$
$$0 \to H_2 NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to R/I \to$$
$$0 \to H_1 NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to 0 \to$$
$$0 \to H_0 NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to 0 \to 0$$

FIGURE 3.

sequence collapses, with the only (potentially) non-zero terms being when $q' = q'' = 1$,, i.e., when $q = 2$. These (potentially) non-zero terms are $\mathrm{Tor}_p(R/(f), R/(g))$. Taking $K.$ as a projective resolution of $R/(f)$ then tensoring throughout by $R/(g)$ we get the chain complex

$$\big(0 \to R \otimes R/(g) \xrightarrow{f} R \otimes R/(g)\big) = \big(0 \to R/(g) \xrightarrow{f} R/(g)\big)$$

which has homology R/I at the zeroth place and 0 everywhere else. And so

$$H_k\big(\,\mathrm{Tot}(NL_1^3\Gamma K. \otimes NL_1^3\Gamma L.)\big) \cong \mathrm{Tor}_{k-2}(R/(f), R/(g)) \cong \begin{cases} 0 & k \neq 2 \\ R/I & k = 2. \end{cases}$$

The short exact sequence of chain complexes

$$0 \to N\Delta(\Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L.) \to NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \\ \to N\Delta(L_1^3\Gamma K. \otimes L_1^3\Gamma L.) \to 0$$

gives rise to the long exact sequence of homologies as given in Figure 3, and therefore we get

$$H_k NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \cong \begin{cases} R/I & k = 2, 4 \\ 0 & \text{otherwise.} \end{cases}$$

$$\cdots \to 0 \to H_5 N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \to 0$$
$$\to R/I \to H_4 N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \to 0$$
$$\to 0 \to H_3 N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \to 0$$
$$\to R/I \to H_2 N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \to 0$$
$$\to 0 \to H_1 N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \to 0$$
$$\to 0 \to H_0 N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \to R/I \to 0.$$

FIGURE 4.

We work with the short exact sequence of simplicial modules

$$0 \to M_{(2,1)}(\operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \\ \to \operatorname{Sym}^3 \Gamma K. \otimes \operatorname{Sym}^3 \Gamma L. \to 0$$

the term $\operatorname{Sym}^3 \Gamma K. \otimes \operatorname{Sym}^3 \Gamma L.$ is a simplicial module whose kth place is $\operatorname{Sym}^3 \Gamma K_k \otimes \operatorname{Sym}^3 \Gamma L_k$, but as above it is more useful to think of it as the diagonal of the bisimplicial complex whose (k, ℓ)th place is $\operatorname{Sym}^3 \Gamma K_k \otimes \operatorname{Sym}^3 \Gamma L_\ell$. Applying the functor N we get the following short exact sequence of chain complexes

$$0 \to NM_{(2,1)}(\operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to N \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \\ \to N\Delta(\operatorname{Sym}^3 \Gamma K. \otimes \operatorname{Sym}^3 \Gamma L.) \to 0.$$

Applying the Eilenberg-Zilber Theorem and Propositions 5.3 and 5.4 we see

$$H_k N\Delta(\operatorname{Sym}^3 \Gamma K. \otimes \operatorname{Sym}^3 \Gamma L.) \cong H_k \operatorname{Tot}(N \operatorname{Sym}^3 \Gamma K. \otimes N \operatorname{Sym}^3 \Gamma L.) \\ \cong H_k \operatorname{Tot}(\operatorname{Kos}^3(f) \otimes \operatorname{Kos}^3(g)).$$

Earlier in this proof we showed that $\mathrm{Kos}^3(f) = K.$ and similarly $\mathrm{Kos}^3(g) = L.$, so

$$H_k \operatorname{Tot}(\mathrm{Kos}^3(f) \otimes \mathrm{Kos}^3(g)) \cong H_k(\operatorname{Tot}(K. \otimes L.)) = H_k(P.(R/I)).$$

Hence

$$H_k N\Delta(\operatorname{Sym}^3 \Gamma K. \otimes \operatorname{Sym}^3 \Gamma L.) \cong \begin{cases} R/I & k = 0 \\ 0 & k \neq 0. \end{cases}$$

And so the short exact sequence of chain complexes

$$0 \to NM_{(2,1)}(\operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to N \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.) \\ \to N\Delta(\operatorname{Sym}^3(\Gamma K.) \otimes \operatorname{Sym}^3(\Gamma L.)) \to 0,$$

gives rise to the long exact sequence as given in Figure 4. Hence (as we know $G_k(R/I) \cong H_k N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.))$, we see that

$$G_k(R/I) \cong \begin{cases} R/I & k = 0, 2, 4 \\ 0 & \text{otherwise,} \end{cases}$$

as desired. Q.E.D.

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