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ON ALGEBRAIC MODELS FOR HOMOTOPY 3-TYPES

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(*communicated by Ronald Brown*)

Abstract

We explore the relations among quadratic modules, 2-crossed modules, crossed squares and simplicial groups with Moore complex of length 2.

Introduction

Crossed modules defined by Whitehead, [**23**], are algebraic models of connected (weak homotopy) 2-types. Crossed squares as introduced by Loday and Guin-Walery, [**22**], model connected 3-types. Crossed n-cubes model connected $(n+1)$-types, (cf. [**21**]). Conduché, [**10**], gave an alternative model for connected 3-types in terms of crossed modules of groups of length 2 which he calls '2-crossed module'. Conduché also constructed (in a letter to Brown in 1984) a 2-crossed module from a crossed square. Baues, [**3**], gave the notion of quadratic module which is a 2-crossed module with additional 'nilpotency' conditions. A quadratic module is thus a 'nilpotent' algebraic model of connected 3-types. Another algebraic model of connected 3-types is 'braided regular crossed module' introduced by Brown and Gilbert (cf. [**5**]). These notions are then related to simplicial groups. Conduché has shown that the category of simplicial groups with Moore complex of length 2 is equivalent to that of 2-crossed modules. Baues gives a construction of a quadratic module from a simplicial group in [**3**]. Berger, [**4**], gave a link between 2-crossed modules and double loop spaces.

Some light on the 2-crossed module structure was also shed by Mutlu and Porter, [**20**], who suggested ways of generalising Conduché's construction to higher n-types. Also Carrasco-Cegarra, [**9**], gives a generalisation of the Dold-Kan theorem to an equivalence between simplicial groups and a non-Abelian chain complex with a lot of extra structure, generalising 2-crossed modules.

The present article aims to show some relations among algebraic models of connected 3-types. Thus the main points of this paper are:

(i) to give a complete description of the passage from a crossed square to a 2-crossed module by using the 'Artin-Mazur' codiagonal functor and prove directly a 2-crossed module structure;

(ii) to give a functor from 2-crossed modules to quadratic modules based on Baues's work (cf. [**3**]);

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(iii) to give a full description of a construction of a quadratic module from a simplicial group by using the Peiffer pairing operators;

(iv) to give a construction of a quadratic module from a crossed square.

Therefore, the results of this paper can be summarized in the following commutative diagram

$$\begin{tikzcd}
& \mathbf{SimpGrp}_{\leqslant 2} \arrow[dl] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathbf{QM} & \mathbf{X_2Mod} \arrow[l] & \mathbf{Crs}^2 \arrow[l] \arrow[ll, bend left]
\end{tikzcd}$$

where the diagram is commutative, linking the constructions given below.

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1. Preliminaries

We refer the reader to May's book (cf. [**17**]) and Artin-Mazur's, [**1**], article for the basic properties of simplicial groups, bisimplicial groups, etc.

A simplicial group $\mathbf{G}$ consists of a family of groups G_n together with face and degeneracy maps $d_i^n : G_n \to G_{n-1}$, $0 \leqslant i \leqslant n$ $(n \neq 0)$ and $s_i^n : G_n \to G_{n+1}$, $0 \leqslant i \leqslant n$ satisfying the usual simplicial identities given by May. In fact it can be completely described as a functor $\mathbf{G} : \Delta^{op} \to \mathbf{Grp}$ where Δ is the category of finite ordinals.

Given a simplicial group $\mathbf{G}$, the Moore complex $(\mathbf{NG}, \partial)$ of $\mathbf{G}$, is the (non-Abelian) chain complex defined by;

$$NG_n = \ker d_0^n \cap \ker d_1^n \cap \cdots \cap \ker d_{n-1}^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_n^n by restriction.

The n^{th} *homotopy group* $\pi_n(\mathbf{G})$ of $\mathbf{G}$ is the n^{th} homology of the Moore complex of $\mathbf{G}$, i.e.

$$\pi_n(\mathbf{G}) \cong H_n(\mathbf{NG}, \partial) = \left(\bigcap_{i=0}^{n} \ker d_i^n \right) / d_{n+1}^{n+1} \left(\bigcap_{i=0}^{n} \ker d_i^{n+1} \right).$$

The Moore complex carries a lot of fine structure and this has been studied, e.g. by Carrasco and Cegarra (cf. [**9**]), Mutlu and Porter (cf. [**18, 19, 20**]).

Consider the product category $\Delta \times \Delta$ whose objects are pairs $([p], [q])$ and whose maps are pairs of weakly increasing maps. A (contravariant) functor $\mathbf{G}_{.,.} : (\Delta \times \Delta)^{op} \to \mathbf{Grp}$ is called a bisimplicial group. Hence $\mathbf{G}_{.,.}$ is equivalent to giving for each (p,q) a group $G_{p,q}$ and morphisms:

$$\begin{array}{ll} d_i^h : G_{p,q} \to G_{p-1,q} & \\ s_i^h : G_{p,q} \to G_{p+1,q} & 0 \leqslant i \leqslant p \\ d_j^v : G_{p,q} \to G_{p,q-1} & \\ s_j^v : G_{p,q} \to G_{p,q+1} & 0 \leqslant j \leqslant q \end{array}$$

such that the maps d_i^h, s_i^h commute with d_j^v, s_j^v and that d_i^h, s_i^h (resp. d_j^v, s_j^v) satisfy the usual simplicial identities.

We think of d_j^v, s_j^v as the vertical operators and d_i^h, s_i^h as the horizontal operators. If $\mathbf{G}.,.$ is a bisimplicial group, it is convenient to think of an element of $G_{p,q}$ as a product of a p-simplex and a q-simplex.

2. 2-Crossed Modules from Simplicial Groups

Crossed modules were initially defined by Whitehead as models for connected 2-types. As explained earlier, Conduché, [**10**], in 1984 described the notion of 2-crossed module as models for connected 3-types.

A crossed module is a group homomorphism $\partial : M \to P$ together with an action of P on M, written ${}^p m$ for $p \in P$ and $m \in M$, satisfying the conditions $\partial({}^p m) = p\partial(m)p^{-1}$ and ${}^{\partial m}m' = mm'm^{-1}$ for all $m, m' \in M, p \in P$. The last condition is called the 'Peiffer identity'.

The following definition of 2-crossed module is equivalent to that given by Conduché.

A 2-crossed module of groups consists of a complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with (a) actions of N on M and L so that ∂_2, ∂_1 are morphisms of N-groups, and (b) an N-equivariant function

$$\{\quad,\quad\} : M \times M \longrightarrow L$$

called a Peiffer lifting. This data must satisfy the following axioms:

$$\begin{array}{llcl}
\mathbf{2CM1}) & \partial_2\{m,m'\} & = & \left({}^{\partial_1 m}m'\right) mm'^{-1}m^{-1} \\
\mathbf{2CM2}) & \{\partial_2 l, \partial_2 l'\} & = & [l', l] \\
\mathbf{2CM3}) & (i)\ \ \{mm', m''\} & = & {}^{\partial_1 m}\{m', m''\}\{m, m'm''m'^{-1}\} \\
 & (ii)\ \ \{m, m'm''\} & = & \{m, m'\}^{mm'm^{-1}}\{m, m''\} \\
\mathbf{2CM4}) & \{m, \partial_2 l\}\{\partial_2 l, m\} & = & {}^{\partial_1 m}l l^{-1} \\
\mathbf{2CM5}) & {}^{n}\{m, m'\} & = & \{{}^n m, {}^n m'\}
\end{array}$$

for all $l, l' \in L$, $m, m', m'' \in M$ and $n \in N$.

Here we have used ${}^m l$ as a shorthand for $\{\partial_2 l, m\} l$ in condition **2CM3**)(ii) where l is $\{m, m''\}$ and m is $mm'(m)^{-1}$. This gives a new action of M on L. Using this notation, we can split **2CM4**) into two pieces, the first of which is tautologous:

$$\begin{array}{llcl}
\mathbf{2CM4}) & (a)\ \ \{\partial_2 l, m\} & = & {}^m l(l)^{-1}, \\
 & (b)\ \ \{m, \partial_2 l\} & = & ({}^{\partial_1 m} l)({}^m l^{-1}).
\end{array}$$

The old action of M on L, via ∂_1 and the N-action on L, is in general distinct from this second action with $\{m, \partial_2 l\}$ measuring the difference (by **2CM4**)(b)). An easy argument using **2CM2**) and **2CM4**)(b) shows that with this action, ${}^m l$, of M on L, (L, M, ∂_2) becomes a crossed module.

A morphism of 2-crossed modules can be defined in an obvious way. We thus define the category of 2-crossed modules denoting it by $\mathbf{X_2Mod}$.

The following theorem, in some sense, is known. We do not give the proof since it exists in the literature, [**10**], [**15**], [**18**], [**21**].

Theorem 2.1. *The category* $\boldsymbol{X_2Mod}$ *of 2-crossed modules is equivalent to the category* $\boldsymbol{SimpGrp}_{\leqslant 2}$ *of simplicial groups with Moore complex of length 2.* □

3. Cat^2-Groups and Crossed Squares

Although when first introduced by Loday and Walery, [**22**], the notion of crossed square of groups was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n-groups (cf. [**15**]).

A crossed square of groups is a commutative square of groups;

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

together with left actions of P on L, M, N and a function $h : M \times N \to L$. Let M and N act on M, N and L via P. The structure must satisfy the following axioms for all $l \in L$, $m, m' \in M$, $n, n' \in N$, $p \in P$;
(i) The homomorphisms $\mu, \nu, \lambda, \lambda'$ and $\mu\lambda$ are crossed modules and both λ, λ' are P-equivariant,
(ii) $h(mm', n) = h({}^m m', {}^m n)h(m, n)$,
(iii) $h(m, nn') = h(m, n)h({}^n m, {}^n n')$,
(iv) $\lambda h(m, n) = m {}^n m^{-1}$,
(v) $\lambda' h(m, n) = {}^m n n^{-1}$,
(vi) $h(\lambda l, n) = l {}^n l^{-1}$,
(vii) $h(m, \lambda' l) = {}^m l l^{-1}$,
$(viii)$ $h({}^p m, {}^p n) = {}^p h(m, n)$.

Recall from [**15**] that a cat^1-group is a triple (G, s, t) consisting of a group G and endomorphisms s, the source map, and t, the target map of G, satisfying the following axioms:

$$i) \quad st = t, ts = s, \qquad ii) \quad [\ker s, \ker t] = 1.$$

It was shown that in [**15**, Lemma 2.2] that setting $C = \ker s$, $B = \mathrm{Im} s$ and $\partial = t|_C$, then the conjugation action makes $\partial : C \to B$ into a crossed module. Conversely if $\partial : C \to B$ is a crossed module, then setting $G = C \rtimes B$ and letting s, t be defined by $s(c, b) = (1, b)$ and $t(c, b) = (1, \partial(c)b)$ for $c \in C$, $b \in B$, then (G, s, t) is a cat^1-group.

For a cat^2-group, we again have a group G, but this time with two independent cat^1-group structures on it. Explicitly:

A cat^2-group is a 5-tuple, (G, s_1, t_1, s_2, t_2), where (G, s_i, t_i), $i = 1, 2$, are cat^1-groups and

$$s_i s_j = s_j s_i, t_i t_j = t_j t_i, s_i t_j = t_j s_i$$

for $i, j = 1, 2$, $i \neq j$.

The following proposition was given by Loday (cf. [**15**]). We only present the sketch proof (see also [**20**]) of this result as we need some indication of proofs for later use.

Proposition 3.1. ([**15**]) *There is an equivalence of categories between the category of cat^2-groups and that of crossed squares.*

Proof: The cat^1-group (G, s_1, t_1) will give us a crossed module $\partial : C \to B$ with $C = \ker s$, $B = \mathrm{Im} s$ and $\partial = t|_C$, but as the two cat^1-group structures are independent, (G, s_2, t_2) restricts to give cat^1-group structures on C and B makes ∂ a morphism of cat^1-groups. We thus get a morphism of crossed modules

$$\begin{tikzcd}
\ker s_1 \cap \ker s_2 \arrow[r] \arrow[d] & \mathrm{Im} s_1 \cap \ker s_2 \arrow[d] \\
\ker s_1 \cap \mathrm{Im} s_2 \arrow[r] & \mathrm{Im} s_1 \cap \mathrm{Im} s_2
\end{tikzcd}$$

where each morphism is a crossed module for natural action, i.e. conjugation in G. It remains to produce an h-map, but it is given by the commutator within G since if $x \in \mathrm{Im} s_1 \cap \ker s_2$ and $y \in \ker s_1 \cap \mathrm{Im} s_2$ then $[x, y] \in \ker s_1 \cap \ker s_2$. It is easy to check the crossed square axioms.

Conversely, if

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & M \arrow[d] \\
N \arrow[r] & P
\end{tikzcd}$$

is a crossed square, then we can think of it as a morphism of crossed modules; $(L, N) \to (M, P)$.

Using the equivalence between crossed modules and cat^1-groups this gives a morphism

$$\partial : (L \rtimes N, s, t) \longrightarrow (M \rtimes P, s', t')$$

of cat^1-groups. There is an action of $(m, p) \in M \rtimes P$ on $(l, n) \in L \rtimes N$ given by

$$^{(m,p)}(l, n) = (^{m}(^{p}l)h(m, ^{p}n), ^{p}n).$$

Using this action, we thus form its associated cat^1-group with big group $(L \rtimes N) \rtimes (M \rtimes P)$ and induced endomorphisms s_1, t_1, s_2, t_2. $\square$

A generalisation of a crossed square to higher dimensions called a "crossed n-cube", was given by Ellis and Steiner (cf. [**14**]), but we use only the case $n = 2$.

The following result for groups was given by Mutlu and Porter (cf. [**18**]).

Let $\mathbf{G}$ be a simplicial group. Then the following diagram

$$\begin{tikzcd} NG_2/\partial_3 NG_3 \arrow[r, "\partial_2"] \arrow[d, "\partial_2'"'] & NG_1 \arrow[d, "\mu"] \\ \overline{NG}_1 \arrow[r, "\mu'"'] & G_1 \end{tikzcd}$$

is the underlying square of a crossed square. The extra structure is given as follows; $NG_1 = \ker d_0^1$ and $\overline{NG}_1 = \ker d_1^1$. Since G_1 acts on $NG_2/\partial_3 NG_3, \overline{NG}_1$ and NG_1, there are actions of $\overline{NG}_1$ on $NG_2/\partial_3 NG_3$ and NG_1 via μ', and NG_1 acts on $NG_2/\partial_3 NG_3$ and $\overline{NG}_1$ via μ. Both μ and μ' are inclusions, and all actions are given by conjugation. The h-map is

$$\begin{array}{rccl} h: & NG_1 \times \overline{NG}_1 & \longrightarrow & NG_2/\partial_3 NG_3 \\ & (x, \overline{y}) & \longmapsto & h(x,y) = [s_1x, s_1ys_0y^{-1}]\partial_3 NG_3. \end{array}$$

Here x and y are in NG_1 as there is a bijection between NG_1 and $\overline{NG}_1$. This example is clearly functorial and we denote it by:

$$\mathbf{M}(-,2) : \mathbf{SimpGrp} \longrightarrow \mathbf{Crs}^2.$$

This is the 2-dimensional case of a general construction of a crossed n-cube from a simplicial group given by Porter, [**21**], based on some ideas of Loday, [**15**].

4. 2-Crossed Modules from Crossed Squares

In this section we will give a description of the passage from crossed squares to 2-crossed modules by using the 'Artin-Mazur' codiagonal functor and prove directly the 2-crossed module structure; a similar construction has been done by Mutlu and Porter, [**20**], in terms of a bisimplicial nerve of a crossed square.

Conduché constructed (private communication to Brown in 1984) a 2-crossed module from a crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

as

$$L \xrightarrow{(\lambda^{-1},\lambda')} M \rtimes N \xrightarrow{\mu\nu} P.$$

We noted above that the category of crossed modules is equivalent to that of cat^1-groups. The corresponding equivalence in dimension 2 is reproved in Proposition 3.1.

We form the associated cat^2-group. This is

$$\begin{tikzcd}
(L \rtimes N) \rtimes (M \rtimes P) \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & M \rtimes P \arrow[d, shift right, "s_M"'] \arrow[d, shift left, "t_M"] \\
N \rtimes P \arrow[r, shift left, "s_N"] \arrow[r, shift right, "t_N"'] & P.
\end{tikzcd}$$

The source and target maps are defined as follows;

$$\begin{array}{ll}
s((l,n),(m,p)) = (n,p), & s'((l,n),(m,p)) = (m,p), \\
t((l,n),(m,p)) = ((\lambda' l)n, \mu(m)p), & t'((l,n),(m,p)) = ((\lambda l)^{(\nu n)}m, \nu(n)p), \\
s_N(n,p) = p, \quad t_N(n,p) = \nu(n)p, & s_M(m,p) = p, \quad t_M(m,p) = \mu(m)p
\end{array}$$

for $l \in L$, $m \in M$ and $p \in P$.

We take the binerve, that is the nerves in the both directions of the cat^2-group constructed. This is a bisimplicial group. The first few entries in the bisimplicial array are given below

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & (L^N) \rtimes ((L^N) \rtimes (M^P)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & M \rtimes (M \rtimes P) \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
((L^L) \rtimes N) \rtimes ((M^M) \rtimes P) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left] \arrow[d, shift right] & (L^N) \rtimes (M^P) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M \rtimes P \arrow[d, shift left] \arrow[d, shift right] \\
N \rtimes (N \rtimes P) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (N \rtimes P) \arrow[r, shift left] \arrow[r, shift right] & P
\end{tikzcd}$$

where $L^N = L \rtimes N$, $M^P = M \rtimes P$.

Some reduction has already been done. For example, the double semi-direct product represents the group of pairs of elements $((m_1,p_1),(m_2,p_2)) \in M \rtimes P$ where $\mu(m_1)p_1 = p_2$. This is the group $M \rtimes (M \rtimes P)$, where the action of $M \rtimes P$ on M is given by ${}^{(m,p)}m' = {}^{\mu(m)p}m'$.

We will recall the Artin-Mazur codiagonal functor ∇ (cf. [**1**]) from bisimplicial groups to simplicial groups.

Let $\mathbf{G_{.,.}}$ be a bisimplicial group. Put

$$G_{(n)} = \prod_{p+q=n} G_{p,q}$$

and define $\nabla_n \subset G_{(n)}$ as follow; An element $(x_0, \ldots, x_n)$ of $G_{(n)}$ with $x_p \in G_{p,n-p}$, is in ∇_n if and only if

$$d_0^v x_p = d_{p+1}^h x_{p+1}$$

for each $p = 0, \ldots, n-1$. Next, define the faces and degeneracies: for $j = 0, \ldots, n$, $D_j : \nabla_n \longrightarrow \nabla_{n-1}$ and $S_j : \nabla_n \longrightarrow \nabla_{n+1}$ by

$$D_j(x) = (d_j^v x_0, d_{j-1}^v x_1, \ldots, d_1^v x_{j-1}, d_j^h x_{j+1}, d_j^h x_{j+2}, \ldots, d_j^h x_n)$$

$$S_j(x) = (s_j^v x_0, s_{j-1}^v x_1, \ldots, s_0^v x_j, s_j^h x_j, s_j^h x_{j+1}, \ldots, s_j^h x_n).$$

Thus $\nabla(\mathbf{G}.,.) = \{\nabla_n : D_j, S_j\}$ is a simplicial group.

We now examine this construction in low dimension:

EXAMPLE:

For $n = 0$, $G_{(0)} = G_{0,0}$. For $n = 1$, we have

$$\nabla_1 \subset G_{(1)} = G_{1,0} \times G_{0,1}$$

where

$$\nabla_1 = \{(g_{1,0}, g_{0,1}) : d_0^v(g_{1,0}) = d_1^h(g_{0,1})\}$$

together with the homomorphisms

$$\begin{aligned} D_0^1(g_{1,0}, g_{0,1}) &= (d_0^v g_{1,0}, d_0^h g_{0,1}), \\ D_1^1(g_{1,0}, g_{0,1}) &= (d_1^v g_{1,0}, d_1^h g_{0,1}), \\ S_0^0(g_{0,0}) &= (s_0^v g_{0,0}, s_0^h g_{0,0}). \end{aligned}$$

For $n = 2$, we have

$$\nabla_2 \subset G_{(2)} = \prod_{p+q=2} G_{p,q} = G_{2,0} \times G_{1,1} \times G_{0,2}$$

where

$$\nabla_2 = \{(g_{2,0}, g_{1,1}, g_{0,2}) : d_0^v(g_{2,0}) = d_1^h(g_{1,1}),\ d_0^v(g_{1,1}) = d_2^h(g_{0,2})\}.$$

Now, we use the Artin-Mazur codiagonal functor to obtain a simplicial group $\mathbf{G}$ (of some complexity).

The base group is still $G_0 \cong P$. However the group of 1-simplices is the subset of

$$G_{1,0} \times G_{0,1} = (M \rtimes P) \times (N \rtimes P),$$

consisting of $(g_{1,0}, g_{0,1}) = ((m,p),(n,p'))$ where $\mu(m)p = p'$, i.e.,

$$G_1 = \{((m,p),(n,p')) : d_0^v(m,p) = \mu(m)p = p' = d_1^h(n,p')\}.$$

We see that the composite of two elements

$$(m_1, p_1, n_1, \mu(m_1)p_1) \text{ and } (m_2, p_2, n_2, \mu(m_2)p_2)$$

becomes

$$(m_1\ {}^{p_1}m_2, p_1p_2, n_1\ {}^{\mu(m_1)p_1}n_2, \mu(m_1\ {}^{p_1}m_2)p_1p_2)$$

(by the inter-change law). The subgroup G_1 of these elements is isomorphic to $N \rtimes (M \rtimes P)$, where M acts on N via P, ${}^m n = {}^{\mu m} n$. Indeed, one can easily show that the map

$$\begin{array}{cccc} f: & G_1 & \longrightarrow & N \rtimes (M \rtimes P) \\ & (m, p, n, \mu(m)p) & \longmapsto & (n, m, p) \end{array}$$

is an isomorphism.

Identifying G_1 with $N \rtimes (M \rtimes P)$, d_0 and d_1 have the descriptions

$$\begin{aligned} d_0(n,m,p) &= \upsilon(n)\mu(m)p \\ d_1(n,m,p) &= p. \end{aligned}$$

We next turn to the group of 2-simplices: this is the subset G_2 of

$$G_{2,0} \times G_{1,1} \times G_{0,2} = M \rtimes (M \rtimes P) \times ((L \rtimes N) \rtimes (M \rtimes P)) \times (N \rtimes (N \rtimes P))$$

whose elements

$$((m_2, m_1, p), (l, n, m, p'), (n_2, n_1, p''))$$

are such that

$$\begin{aligned} d_0^v(m_2, m_1, p) &= d_1^h(l, n, m, p') \\ d_0^v(l, n, m, p') &= d_2^h(n_2, n_1, p''). \end{aligned}$$

This gives the relations between the individual coordinates implying that $m_1 = m$, $\mu(m_2)p = p'$, $\lambda'(l)n = n_2$ and $\mu(m)p' = p''$. Thus the elements of G_2 have the form

$$((m_2, m_1, p), ((l, n), (m_1, \mu(m_2)p)), (\lambda'(l)n, n_1, \mu(m_1m_2)p)).$$

We then deduce the isomorphism

$$f : G_2 \longrightarrow (L \rtimes (N \rtimes M)) \rtimes (N \rtimes (M \rtimes P))$$

given by

$$\begin{aligned} &((m_2, m_1, p), ((l, n), (m_1, \mu(m_2)p)), ((\lambda'l)n, n_1, \mu(m_1m_2)p)) \\ &\qquad\qquad\qquad\qquad \longmapsto ((l, (n, m_1)), (n_1, (m_2, p)))\,. \end{aligned}$$

Therefore we can get a 2-truncated simplicial group $\mathbf{G}^{(2)}$ that looks like

$$\mathbf{G}^{(2)} : (L \rtimes (N \rtimes M)) \rtimes (N \rtimes (M \rtimes P)) \underset{s_0^1, s_1^1}{\overset{d_0^2, d_1^2, d_2^2}{\rightrightarrows}} N \rtimes (M \rtimes P) \underset{s_0^0}{\overset{d_0^1, d_1^1}{\rightrightarrows}} P$$

with the faces and degeneracies;

$$d_0^1(n, m, p) = \nu(n)\mu(m)p, \quad d_1^1(n, m, p) = p, \quad s_0(p) = (1, 1, p)$$

and

$$\begin{aligned} d_0^2((l, (n, m_1)), (n_1, (m_2, p))) &= (n_1, (\lambda l)^{\nu(n)}m_1, \nu(n)\mu(m_2)p), \\ d_1^2((l, (n, m_1)), (n_1, (m_2, p))) &= (n_1(\lambda' l)n, m_1m_2, p), \\ d_2^2((l, (n, m_1)), (n_1, (m_2, p))) &= (n, m_2, p), \\ s_0^1(n, m, p) &= ((1, (1, m)), (n, (1, p))), \\ s_1^1(n, m, p) &= ((1, (n, 1)), (1, (m, p))). \end{aligned}$$

For the verification of the simplicial identities, see appendix.

Remark:

The construction given above may be shortened in terms of the $\overline{W}$ construction or 'bar' construction (cf. [**1**], [**8**]), but we have not attempted this method.

Loday, [**15**], defined the mapping cone of a complex as analogous to the construction of the Moore complex of a simplicial group. (for further work see also [**11**]). We next describe the mapping cone of a crossed square of groups as follows:

Proposition 4.1. *The Moore complex of the simplicial group* $\mathbf{G}^{(2)}$ *is the mapping cone, in the sense of Loday, of the crossed square. Furthermore, this mapping cone complex has a 2-crossed module structure of groups.*

Proof: Given the 2-truncated simplicial group $\mathbf{G}^{(2)}$ described above, look at its Moore complex; we have $NG_0 = G_0 = P$. The second term of the Moore complex is $NG_1 = \ker d_0^1$. By the definition of d_0^1, $(n, m, p) \in \ker d_0^1$ if and only if $p = \mu(m)^{-1}\nu(n)^{-1}$. Since $d_0^1(n^{-1}, m^{-1}, \mu(m)\nu(n)) = \nu(n)^{-1}\mu(m)^{-1}\mu(m)\nu(n) = 1$, we have $(n^{-1}, m^{-1}, \mu(m)\nu(n)) \in \ker d_0^1$. Furthermore there is an isomorphism $f_1 : NG_1 \longrightarrow M \rtimes N$ given by

$$(n^{-1}, m^{-1}, \mu(m)\nu(n)) \mapsto (m, n).$$

We note that via this isomorphism, the map $\partial_1 : M \rtimes N \to P$ is given by $\partial_1(m, n) = \mu(m)\nu(n)$.

Now we investigate the intersection of the kernels of d_0^2 and d_1^2. Let

$$\mathbf{x} = ((l, (n, m_1)), (n_1, (m_2, p))) \in (L \rtimes (N \rtimes M)) \rtimes (N \rtimes (M \rtimes P)).$$

If $\mathbf{x} \in \ker d_0^2$, by the definition of d_0^2, we have

$$n_1 = 1,\ (\lambda l)^{\nu(n)} m_1 = 1,\ \nu(n)\mu(m_2)p = 1.$$

If $\mathbf{x} \in \ker d_1^2$, by the definition of d_1^2, we have

$$n_1(\lambda' l)n = 1,\ m_1 m_2 = 1,\ p = 1.$$

¿From these equalities we have $n = (\lambda' l)^{-1}$, and from

$$\begin{aligned} 1 &= (\lambda l)^{\nu(n)} m_1 \\ &= (\lambda l)^{\nu((\lambda' l)^{-1})} m_1 \\ &= (\lambda l)^{\mu \lambda l^{-1}} m_1 \qquad (\mu\lambda = \nu\lambda') \\ &= (\lambda l)(\lambda l)^{-1} m_1 (\lambda l) \\ &= m_1(\lambda l), \end{aligned}$$

we have $m_1 = m_2^{-1} = (\lambda l)^{-1}$ and $p = 1$. Therefore, $\mathbf{x} \in \ker d_0^2 \cap \ker d_1^2$ if and only if

$$\mathbf{x} = ((l, (\lambda' l^{-1}, \lambda l^{-1})), (1, \lambda l, 1)).$$

Thus we get $\ker d_0^2 \cap \ker d_1^2 \cong L$.

¿From these calculations, we have

$$d_2|_{\ker d_0^2 \cap \ker d_1^2}((l, (\lambda' l^{-1}, \lambda l^{-1})), (1, \lambda l, 1)) = (\lambda' l^{-1}, \lambda l, 1).$$

Of course $(\lambda' l^{-1}, \lambda l, 1) \in NG_1$ since

$$d_0^1(\lambda' l^{-1}, \lambda l, 1) = \nu\lambda' l^{-1}\mu(\lambda l)1 = 1.$$

By using above isomorphism f_1 and $d_2|_{\ker d_0^2 \cap \ker d_1^2}$, we can identify the map ∂_2 on

L by

$$\begin{aligned}\partial_2(l) &= f_1 d_2|_{\ker d_0^2 \cap \ker d_1^2}((l,(\lambda' l^{-1}, \lambda l^{-1})),(1,\lambda l,1))\\ &= f_1(\lambda' l^{-1}, \lambda l, 1)\\ &= (\lambda l^{-1}, \lambda' l) \in M \rtimes N.\end{aligned}$$

It can be seen that ∂_2 and ∂_1 are homomorphisms and

$$\begin{aligned}\partial_1\partial_2(l) &= \partial_1(\lambda l^{-1}, \lambda' l)\\ &= \mu(\lambda l)\nu(\lambda' l^{-1})\\ &= 1 \quad (\text{by } \nu\lambda' = \mu\lambda).\end{aligned}$$

Thus, if given a crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

its mapping cone complex is

$$L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} P$$

where $\partial_2 l = (\lambda l^{-1}, \lambda' l)$ and $\partial_1(m,n) = \mu(m)\nu(n)$. The semi-direct product $M \rtimes N$ can be formed by making N acts on M via P, ${}^n m = {}^{\nu(n)}m$, where the P-action is the given one.

These elementary calculations are useful as they pave the way for the calculation of the Peiffer commutator of $x = (m,n)$ and $y = (c,a)$ in the above complex;

$$\begin{aligned}\langle x,y\rangle &= {}^{\partial_1 x}yxy^{-1}x^{-1}\\ &= {}^{\mu(m)\nu(n)}(c,a)(m,n)({}^{a^{-1}}c^{-1}, a^{-1})({}^{n^{-1}}m^{-1}, n^{-1})\\ &= ({}^{\mu(m)\nu(n)}c, {}^{\mu(m)\nu(n)}a)(m^{\nu(na^{-1})}(c^{-1})^{\nu(n^{-1}n^{-1})}m^{-1}, na^{-1}n^{-1})\end{aligned}$$

which on multiplying out and simplifying is

$$({}^{\nu(nan^{-1})}mm^{-1}, {}^{\mu(m)}(nan^{-1})(na^{-1}n^{-1}))$$

(Note that any dependence on c vanishes!)

Conduché (unpublished work) defined the Peiffer lifting for this structure by

$$\{x,y\} = \{(m,n),(c,a)\} = h(m, nan^{-1}).$$

For the axioms of 2-crossed module see appendix. □

We thus have two ways of going from simplicial groups to 2-crossed modules

(i) ([**18**]) directly to get

$$NG_2/\partial_3 NG_3 \longrightarrow NG_1 \longrightarrow NG_0,$$

(ii) indirectly via the square axiom $\mathbf{M}(\mathbf{G},2)$ and then by the above construction to get

$$NG_2/\partial_3 NG_3 \longrightarrow \ker d_0 \rtimes \ker d_1 \longrightarrow G_1,$$

and they clearly give the same homotopy type. More precisely G_1 decomposes as $\ker d_1 \rtimes s_0G_0$ and the $\ker d_0$ factor in the middle term of (ii) maps down to that in this decomposition by the identity map. Thus d_0 induces a quotient map from (ii) to (i) with kernel isomorphic to

$$1 \longrightarrow \ker d_0 \xrightarrow{=} \ker d_0$$

which is thus contractible.

Note: The construction given above from a crossed square to a 2-crossed module preserves the homotopy type. In fact, Ellis (cf. [**13**]) defined the homotopy groups of the crossed square is the homology groups of the complex

$$L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} P \longrightarrow 1$$

where ∂_1 and ∂_2 are defined above.

5. Quadratic Modules from 2-Crossed Modules

Quadratic modules of groups were initially defined by Baues, [**2, 3**], as models for connected 3-types. In this section we will define a functor from the category $\mathbf{X_2Mod}$ of 2- crossed modules to that of quadratic modules $\mathbf{QM}$. Before giving the definition of quadratic module we should recall some structures.

Recall that *a pre-crossed module* is a group homomorphism $\partial : M \to N$ together with an action of N on M, written ${}^n m$ for $n \in N$ and $m \in M$, satisfying the condition $\partial({}^n m) = n\partial(m)n^{-1}$ for all $m \in M$ and $n \in N$.

A nil(2)-module is a pre-crossed module $\partial : M \to N$ with an additional "nilpotency"condition. This condition is $P_3(\partial) = 1$, where $P_3(\partial)$ is the subgroup of M generated by Peiffer commutator $\langle x_1, x_2, x_3\rangle$ of length 3.

The Peiffer commutator in a pre-crossed module $\partial : M \to N$ is defined by

$$\langle x, y\rangle = ({}^{\partial x}y)xy^{-1}x^{-1}$$

for $x, y \in M$.

For a group G, the group

$$G^{ab} = G/[G, G]$$

is the abelianization of G and

$$\partial^{cr} : M^{cr} = M/P_2(\partial) \to N$$

is the crossed module associated to the pre-crossed module $\partial : M \to N$. Here $P_2(\partial) = \langle M, M\rangle$ is the Peiffer subgroup of M.
The following definition is due to Baues (cf. [**3**]).

Definition 5.1. A quadratic module $(\omega, \delta, \partial)$ *is a diagram*

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N \end{tikzcd}$$

of homomorphisms between groups such that the following axioms are satisfied.
QM1) *The homomorphism $\partial : M \to N$ is a nil(2)-module with Peiffer commutator map w defined above. The quotient map $M \twoheadrightarrow C = (M^{cr})^{ab}$ is given by $x \mapsto \overline{x}$, where $\overline{x} \in C$ denotes the class represented by $x \in M$ and $C = (M^{cr})^{ab}$ is the abelianization of the associated crossed module $M^{cr} \to N$.*
QM2) *The boundary homomorphisms ∂ and δ satisfy $\partial\delta = 1$ and the quadratic map ω is a lift of the Peiffer commutator map w, that is $\delta\omega = w$ or equivalently*

$$\delta\omega(\overline{x} \otimes \overline{y}) = ({}^{\partial x}y)xy^{-1}x^{-1} = \langle x, y\rangle$$

for $x, y \in M$.
QM3) *L is an N-group and all homomorphisms of the diagram are equivariant with respect to the action of N. Moreover, the action of N on L satisfies the formula ($a \in L, x \in M$)*

$${}^{\partial x}a = \omega(\left(\overline{x} \otimes \overline{\delta a}\right)\left(\overline{\delta a} \otimes \overline{x}\right))a.$$

QM4) *Commutators in L satisfy the formula ($a, b \in L$)*

$$\omega(\overline{\delta a} \otimes \overline{\delta b}) = [b, a].$$

A map $\varphi : (\omega, \delta, \partial) \to (\omega', \delta', \partial')$ between quadratic modules is given by a commutative diagram, $\varphi = (l, m, n)$

$$\begin{tikzcd}
C \otimes C \arrow[r, "\omega"] \arrow[d, "\varphi_* \otimes \varphi_*"'] & L \arrow[r, "\delta"] \arrow[d, "l"'] & M \arrow[r, "\partial"] \arrow[d, "m"'] & N \arrow[d, "n"'] \\
C' \otimes C' \arrow[r, "\omega'"'] & L' \arrow[r, "\delta'"'] & M' \arrow[r, "\partial'"'] & N'
\end{tikzcd}$$

where (m, n) is a morphism between pre-crossed modules which induces $\varphi_* : C \to C'$ and where l is an n-equivariant homomorphism. Let **QM** be the category of quadratic modules and of maps as in above diagram.

Now, we construct a functor from the category of 2-crossed modules to that of quadratic modules.

Let

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

be a 2-crossed module. Let P_3 be the subgroup of C_1 generated by elements of the form

$$\langle\langle x, y\rangle, z\rangle \text{ and } \langle x, \langle y, z\rangle\rangle$$

with $x, y, z \in C_1$. We obtain $\partial_1(\langle\langle x, y\rangle, z\rangle) = 1$ and $\partial_1(\langle x, \langle y, z\rangle\rangle) = 1$, since ∂_1 is a pre-crossed module.

Let P_3' be the subgroup of C_2 generated by elements of the form

$$\{\langle x, y\rangle, z\} \text{ and } \{x, \langle y, z\rangle\}$$

for $x, y, z \in C_1$, where $\{-, -\}$ is the Peiffer lifting map. Then there are quotient groups

$$M = C_1/P_3$$

and

$$L = C_2/P'_3.$$

Then, $\partial : M \to C_0$ given by $\partial(xP_3) = \partial_1(x)$ is a well defined group homomorphism since $\partial_1(P_3) = 1$. We thus get the following commutative diagram

$$\begin{tikzcd} C_1 \arrow[rr, "\partial_1"] \arrow[dr, "q_1"'] & & C_0 \\ & M \arrow[ur, "\partial"'] & \end{tikzcd}$$

where $q_1 : C_1 \to M$ is the quotient map.

Furthermore, from the first axiom of 2-crossed module **2CM1**), we can write $\partial_2\{\langle x, y\rangle , z\} = \langle\langle x, y\rangle , z\rangle$ and $\partial_2\{x, \langle y, z\rangle\} = \langle x, \langle y, z\rangle\rangle$. Therefore, the map $\delta : L \to M$ given by $\delta(lP'_3) = (\partial_2 l)P_3$ is a well defined group homomorphism since $\partial_2(P'_3) = P_3$.

Thus we get the following commutative diagram;

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"] & M \arrow[r, "\partial"] & N \arrow[d, equal] \\ C_2 \arrow[u, "q_2"] \arrow[r, "\partial_2"'] & C_1 \arrow[u, "q_1"] \arrow[r, "\partial_1"'] & C_0 \end{tikzcd}$$

where q_1 and q_2 are the quotient maps and $C = (M^{cr})^{ab}$ is a quotient of C_1. The quadratic map is given by the Peiffer lifting map

$$\{-, -\} : C_1 \times C_1 \longrightarrow C_2,$$

namely

$$\omega(\overline{x'} \otimes \overline{y'}) = q_2(\{x, y\})$$

for $x', y' \in M$ and $x, y \in C_1$.

Proposition 5.2. *The diagram*

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N \end{tikzcd}$$

is a quadratic module of groups.

Proof: For the axioms, see appendix. □

Proposition 5.3. *The homotopy groups of the 2-crossed module are isomorphic to that of its associated quadratic module.*

Proof: Consider the 2-crossed module

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \tag{1}$$

and its associated quadratic module

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N = C_0. \end{tikzcd} \tag{2}$$

The homotopy groups of (1) are

$$\pi_i = \begin{cases} C_0/\partial_1(C_1) & i = 1, \\ \ker \partial_1/\mathrm{Im}\partial_2 & i = 2, \\ \ker \partial_2 & i = 3, \\ 0 & i = 0 \text{ or } i > 3. \end{cases}$$

The homotopy groups of (2) are

$$\pi_i' = \begin{cases} C_0/\partial(M) & i = 1, \\ \ker \partial/\mathrm{Im}\delta & i = 2, \\ \ker \delta & i = 3, \\ 0 & i = 0 \text{ or } i > 3. \end{cases}$$

We claim that $\pi_i = \pi_i'$ for all $i \geqslant 0$. In fact, since $\partial(M) \cong \partial_1(C_1)$, clearly $\pi_1 \cong \pi_1'$. Also $\ker \partial = \dfrac{\ker \partial_1}{P_3}$, $\mathrm{Im}\delta \cong \dfrac{\mathrm{Im}\partial_2}{P_3}$ so that $\pi_2' = \dfrac{\ker \partial_1/P_3}{\mathrm{Im}\partial_2/P_3} \cong \dfrac{\ker \partial_1}{\mathrm{Im}\partial_2} \cong \pi_2$. Consider now $\pi_3' = \{xP_3' : \partial_2(x) \in P_3\}$. We show that given $xP_3' \in \pi_3'$, there is $x'P_3' \in \pi_3'$ with $xP_3' = x'P_3'$ and $x' \in \ker \partial_2$. In fact, observe that since $\partial_2\{\langle x, y\rangle, z\} = \langle\langle x, y\rangle, z\rangle$, $\partial_2\{x, \langle y, z\rangle\} = \langle x, \langle y, z\rangle\rangle$, we have $\partial_2(P_3') = P_3$. Hence $\partial_2(x) \in P_3$ implies $\partial_2(x) = \partial_2(w)$, $w \in P_3'$; thus $\partial_2(xw^{-1}) = 1$; then take $x' = xw^{-1}$, so that $xP_3' = x'P_3'$ and $\partial_2(x') = 1$. Define $\alpha : \pi_3' \to \pi_3$, $\alpha(xP_3') = \alpha(x'P_3') = x'$ and $\beta : \pi_3 \to \pi_3'$, $\beta(x) = xP_3$. Clearly α and β are inverse bijections, proving the claim. It follows that (1) and (2) represent the same homotopy type. □

6. Simplicial Groups and Quadratic Modules

Baues gives a construction of a quadratic module from a simplicial group in Appendix B to Chapter IV of [**3**]. The quadratic modules can be given by using higher dimensional Peiffer elements in verifying the axioms.

This section is a brief résumé defining a variant of the Carrasco-Cegarra pairing operators that are called *Peiffer Pairings* (cf. [**9**]). The construction depends on a variety of sources, mainly Conduché, [**10**], Mutlu and Porter, [**18, 19, 20**]. We define a normal subgroup N_n of G_n and a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ (cf. [**18**]) with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to the lexicographic ordering in $S(n)$ where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, \ldots, j_1) \in S(n)$. The pairings that we will need,

$$\{F_{\alpha,\beta} : NG_{n-\sharp\alpha} \times NG_{n-\sharp\beta} \to NG_n : (\alpha,\beta) \in P(n), n \geqslant 0\}$$

are given as composites by the diagram

$$\begin{tikzcd} NG_{n-\sharp\alpha} \times NG_{n-\sharp\beta} \arrow[r, "F_{\alpha,\beta}"] \arrow[d, "s_\alpha \times s_\beta"'] & NG_n \\ G_n \times G_n \arrow[r, "\mu"'] & G_n \arrow[u, "p"'] \end{tikzcd}$$

where $s_\alpha = s_{i_r}, \ldots, s_{i_1} : NG_{n-\sharp\alpha} \to G_n$, $s_\beta = s_{j_s}, \ldots, s_{j_1} : NG_{n-\sharp\beta} \to G_n$, $p : G_n \to NG_n$ is defined by composite projections $p(x) = p_{n-1} \ldots p_0(x)$, where $p_j(z) = zs_jd_j(z)^{-1}$ with $j = 0, 1, \ldots, n-1$ and $\mu : G_n \times G_n \to G_n$ is given by the commutator map and $\sharp\alpha$ is the number of the elements in the set of α; similarly for $\sharp\beta$. Thus

$$F_{\alpha,\beta}(x_\alpha, y_\beta) = p[s_\alpha(x_\alpha), s_\beta(x_\beta)].$$

Definition 6.1. *Let N_n or more exactly N_n^G be the normal subgroup of G_n generated by elements of the form $F_{\alpha,\beta}(x_\alpha, y_\beta)$ where $x_\alpha \in NG_{n-\sharp\alpha}$ and $y_\beta \in NG_{n-\sharp\beta}$.*

This normal subgroup N_n^G depends functorially on G, but we will usually abbreviate N_n^G to N_n, when no change of group is involved. Mutlu and Porter (cf. [**18**]) illustrate this normal subgroup for $n = 2, 3, 4$, but we only consider for $n = 3$.

Example 6.2. *For all $x_1 \in NG_1, y_2 \in NG_2$, the corresponding generators of N_3 are:*

$$\begin{aligned} F_{(1,0)(2)}(x_1, y_2) &= [s_1s_0x_1, s_2y_2][s_2y_2, s_2s_0x_1], \\ F_{(2,0)(1)}(x_1, y_2) &= [s_2s_0x_1, s_1y_2][s_1y_2, s_2s_1x_1][s_2s_1x_1, s_2y_2][s_2y_2, s_2s_0x_1] \end{aligned}$$

and for all $x_2 \in NG_2, y_1 \in NG_1$,

$$F_{(0)(2,1)}(x_2, y_1) = [s_0x_2, s_2s_1y_1][s_2s_1y_1, s_1x_2][s_2x_2, s_2s_1y_1]$$

whilst for all $x_2, y_2 \in NG_2$,

$$\begin{aligned} F_{(0)(1)}(x_2, y_2) &= [s_0x_2, s_1y_2][s_1y_2, s_1x_2][s_2x_2, s_2y_2], \\ F_{(0)(2)}(x_2, y_2) &= [s_0x_2, s_2y_2], \\ F_{(1)(2)}(x_2, y_2) &= [s_1x_2, s_2y_2][s_2y_2, s_2x_2]. \end{aligned}$$

The following theorem is proved by Mutlu and Porter (cf. [**19**]).

Theorem 6.3. *Let* **G** *be a simplicial group and for $n > 1$, let D_n the subgroup of G_n generated by degenerate elements. Let N_n be the normal subgroup generated by elements of the form $F_{\alpha,\beta}(x_\alpha, y_\beta)$ with $(\alpha,\beta) \in P(n)$ where $x_\alpha \in NG_{n-\sharp\alpha}$ and $y_\beta \in NG_{n-\sharp\beta}$. Then*

$$NG_n \cap D_n = N_n \cap D_n$$

□

Baues defined a functor from the category of simplicial groups to that of quadratic modules (cf. [**3**]). Now we will reconstruct this functor by using the $F_{\alpha,\beta}$ functions. We will use the $F_{\alpha,\beta}$ functions in verifying the axioms of quadratic module.

Let $\mathbf{G}$ be a simplicial group with Moore complex $\mathbf{NG}$. Suppose that $G_3 = D_3$. Notice that $P_3(\partial_1)$ is the subgroup of NG_1 generated by triple brackets

$$\langle x, \langle y, z\rangle\rangle \text{ and } \langle\langle x, y\rangle, z\rangle$$

for $x, y, z \in NG_1$. Let $P_3'(\partial_1)$ be the subgroup of $NG_2/\partial_3 NG_3$ generated by elements of the form

$$\omega(\langle x, y\rangle, z) = s_0(\langle x, y\rangle)s_1 z s_0(\langle x, y\rangle)^{-1}s_1(\langle x, y\rangle)s_1 z^{-1}s_1(\langle x, y\rangle)^{-1}$$

and

$$\omega(x, \langle y, z\rangle) = s_0 x s_1(\langle y, z\rangle)s_0 x^{-1}s_1 x s_1(\langle y, z\rangle)^{-1}s_1 x^{-1}.$$

Then we have quotient groups

$$M = NG_1/P_3(\partial_1)$$

and

$$L = (NG_2/\partial_3 NG_3)/P_3'(\partial_1).$$

We obtain $\overline{\partial_2}\omega(\langle x, y\rangle, z) = \langle\langle x, y\rangle, z\rangle$ and $\overline{\partial_2}\omega(x, \langle y, z\rangle) = \langle x, \langle y, z\rangle\rangle$. Thus, $\delta : L \to M$ given by $\delta(\overline{a}P_3'(\partial_1)) = \overline{\partial_2}(\overline{a})P_3(\partial_1)$ is a well defined group homomorphism, where $\overline{a}$ is a coset in $NG_2/\partial_3 NG_3$.

Therefore, we obtain the following diagram,

$$\begin{tikzcd}
 & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\
L \arrow[r, "\delta"] & M \arrow[r, "\partial"] & N \\
NG_2/\partial_3 NG_3 \arrow[u, "q_2"] \arrow[r, "\overline{\partial_2}"'] & NG_1 \arrow[u, "q_1"] \arrow[r, "\partial_1"'] & NG_0 \arrow[u, equal]
\end{tikzcd}$$

where q_1 and q_2 are quotient maps and $\delta q_2 = q_1\overline{\partial_2}, \partial q_1 = \partial_1$, and the quadratic map ω is defined by

$$\omega(\{q_1 x\} \otimes \{q_1 y\}) = q_2(\overline{s_0 x s_1 y s_0 x^{-1}s_1 x s_1 y^{-1}s_1 x^{-1}})$$

for $x, y \in NG_1$, $q_1 x, q_1 y \in M$ and $\{q_1 x\} \otimes \{q_1 y\} \in C \otimes C$ and where $C = ((M)^{cr})^{ab}$.

Proposition 6.4. *The diagram*

$$\begin{tikzcd}
 & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\
L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N
\end{tikzcd}$$

is a quadratic module of groups.

Proof: We show that all the axioms of quadratic module are verified by using the functions $F_{\alpha,\beta}$ in the appendix. $\square$

Alternatively, this proposition can be reproved differently, by making use of the 2-crossed module constructed from a simplicial group by Mutlu and Porter (cf. [**18**]). We now give a sketch of the argument. In [**18**], it is shown that given a simplicial group $\mathbf{G}$, one can construct a 2-crossed module

$$NG_2/\partial_3(NG_3 \cap D_3) \xrightarrow{\overline{\partial_2}} NG_1 \xrightarrow{\partial_1} NG_0 \tag{3}$$

where $\{x, y\} = s_0xs_1ys_0x^{-1}s_1y^{-1}s_1x^{-1}\partial_3(NG_3 \cap D_3)$ for $x, y \in NG_1$.
Clearly we have a commutative diagram

$$\begin{tikzcd}
NG_2/\partial_3(NG_3 \cap D_3) \arrow[r, "\overline{\partial_2}"] \arrow[d, "j"] & NG_1 \arrow[r, "\partial_1"] \arrow[d, equal] & NG_0 \arrow[d, equal] \\
NG_2/\partial_3(NG_3) \arrow[r] & NG_1 \arrow[r, "\partial_1"] & NG_0.
\end{tikzcd}$$

Consider now the quadratic module associated to the 2-crossed module (3), as in Section 5 of this paper.

$$\begin{tikzcd}
 & C \otimes C \arrow[dl, "\omega'"'] \arrow[d] & \\
L' \arrow[r, "\delta'"] & M \arrow[r] & N \arrow[d, equal] \\
\dfrac{NG_2}{\partial_3(NG_3 \cap D_3)} \arrow[u, "q_2'"] \arrow[r, "\overline{\partial_2}"'] & NG_1 \arrow[u] \arrow[r, "\partial_1"'] & NG_0
\end{tikzcd}$$

Then one can see that $L' = \Omega/\partial_3(NG_3 \cap D_3)$, where Ω is the subgroup of NG_2 generated by elements of the form

$$s_0(\langle x, y\rangle)s_1zs_0(\langle x, y\rangle)^{-1}s_1(\langle x, y\rangle)s_1z^{-1}s_1(\langle x, y\rangle)^{-1}$$

and

$$s_0xs_1(\langle y, z\rangle)s_0x^{-1}s_1xs_1(\langle y, z\rangle)^{-1}s_1x^{-1}.$$

On the other hand we have, from Section 5, $L = \Omega/\partial_3(NG_3)$. Hence there is a map $i : L' \to L$ with

$$\omega = i\omega', \quad \delta' = \delta i. \tag{4}$$

Since

$$C \otimes C \xrightarrow{\omega'} L' \xrightarrow{\delta'} M \xrightarrow{\partial} N$$

is, by construction a quadratic module, it is straightforward to check, using (4), that

$$C \otimes C \xrightarrow{\omega} L \xrightarrow{\delta} M \xrightarrow{\partial} N$$

is also a quadratic module.

Proposition 6.5. *Let* $\mathbf{G}$ *be a simplicial group, let* π'_i *be the homotopy groups of its associated quadratic module and let* π_i *be the homotopy groups of the classifying space of* $\mathbf{G}$*; then* $\pi_i \cong \pi'_i$ *for* $i = 0, 1, 2, 3$.

Proof: Let $\mathbf{G}$ be a simplicial group. The nth homotopy groups of $\mathbf{G}$ is the nth homology of the Moore complex of $\mathbf{G}$, i.e.,

$$\pi_n(\mathbf{G}) \cong H_n(\mathbf{NG}) \cong \frac{\ker d_{n-1}^{n-1} \cap NG_{n-1}}{d_n^n(NG_n)}.$$

Thus the homotopy groups $\pi_n(\mathbf{G}) = \pi_n$ of $\mathbf{G}$ are

$$\pi_n = \begin{cases} NG_0/d_1(NG_1) & n = 1, \\ \dfrac{\ker d_1 \cap NG_1}{d_2(NG_2)} & n = 2, \\ \dfrac{\ker d_2 \cap NG_2}{d_3(NG_3)} & n = 3, \\ 0 & n = 0 \text{ or } n > 3. \end{cases}$$

and the homotopy groups π'_n of its associated quadratic module are

$$\pi'_n = \begin{cases} NG_0/\partial(M) & n = 1, \\ \ker \partial/\mathrm{Im}\delta & n = 2, \\ \ker \delta & n = 3, \\ 0 & n = 0 \text{ or } n > 3. \end{cases}$$

We claim that $\pi'_n \cong \pi_n$ for $n = 1, 2, 3$. Since $M = NG_1/P_3(\partial_1)$ and $d_1(P_3(\partial_1)) = 1$, we have

$$\partial(M) = \partial(NG_1/P_3(\partial_1)) = d_1(NG_1)$$

and then

$$\pi'_1 = NG_0/\partial(M) \cong NG_0/d_1(NG_1) = \pi_1.$$

Also $\ker \partial = \dfrac{\ker d_1 \cap NG_1}{P_3(\partial_1)}$ and $\mathrm{Im}\delta = d_2(NG_2)/P_3(\partial_1)$ so that we have

$$\pi'_2 = \frac{\ker \partial}{\mathrm{Im}\delta} = \frac{(\ker d_1 \cap NG_1)/P_3(\partial_1)}{d_2(NG_2)/P_3(\partial_1)} \cong \frac{\ker d_1 \cap NG_1}{d_2(NG_2)} = \pi_2.$$

We know that $P'_3(\partial_1)$ is generated by elements of the form

$$s_0(\langle x, y\rangle)s_1 z s_0(\langle x, y\rangle)^{-1} s_1(\langle x, y\rangle) s_1 z^{-1} s_1(\langle x, y\rangle)^{-1}$$

and

$$s_0xs_1(\langle y,z\rangle)s_0x^{-1}s_1xs_1(\langle y,z\rangle)^{-1}s_1x^{-1}.$$

Since

$$\begin{aligned}&d_2(s_0(\langle x,y\rangle)s_1zs_0(\langle x,y\rangle)^{-1}s_1(\langle x,y\rangle)s_1z^{-1}s_1(\langle x,y\rangle)^{-1})\\&\qquad= {}^{d_1\langle x,y\rangle}z\,\langle x,y\rangle\,(z^{-1})\,\langle x,y\rangle^{-1}\\&\qquad= \langle\langle x,y\rangle\,,z\rangle\in P_3(\partial_1)\end{aligned}$$

and

$$\begin{aligned}&d_2(s_0xs_1(\langle y,z\rangle)s_0x^{-1}s_1xs_1(\langle y,z\rangle)^{-1}s_1x^{-1})\\&\qquad= s_0d_1x\,\langle y,z\rangle\,s_0d_1x^{-1}(x)\,\langle y,z\rangle^{-1}\,x^{-1}\\&\qquad= {}^{d_1x}\,\langle y,z\rangle\,x\,\langle y,z\rangle^{-1}\,x^{-1}\\&\qquad= \langle x,\langle y,z\rangle\rangle\in P_3(\partial_1)\end{aligned}$$

we get $d_2(P_3'(\partial_1)) = P_3(\partial_1)$. The isomorphism between π_3' and π_3 can be proved similarly to the proof of Proposition 5.3. $\square$

7. Quadratic Modules from Crossed Squares

In this section, we will define a functor from the category of crossed squares to that of quadratic modules. Our construction can be briefly explained as:

Given a crossed square, we consider the associated 2-crossed module (from Section 4) and then we build the quadratic module corresponding to this 2-crossed module (from Section 5). In other words, we are just composing two functors. Thus, there is no need to worry in this section about direct proofs, as they hold automatically from the results of Sections 4 and 5. In particular, the homotopy type is clearly preserved, as it is preserved at each step.

Now let

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

be a crossed square of groups. Consider its associated 2-crossed module from Section 4

$$L \xrightarrow{(\lambda^{-1},\lambda')} M \rtimes N \xrightarrow{\mu\nu} P.$$

¿From this 2-crossed module, we can get a quadratic module as in Section 5

$$\begin{tikzcd} & C\otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ C_2 \arrow[r, "\delta"'] & C_1 \arrow[r, "\partial"'] & C_0 \end{tikzcd}$$

where $C_0 = P$, $C_1 = (M \rtimes N)/P_3$, $C_2 = L/P'_3$, $C = ((C_1)^{cr})^{ab}$ and the quadratic map is given by

$$\begin{array}{cccc} \omega : & C \otimes C & \longrightarrow & C_2 \\ & [q_1(m,n)] \otimes [q_1(c,a)] & \longmapsto & q_2(h(m, nan^{-1})) \end{array}$$

for $(m,n),(c,a) \in M \rtimes N$, $q_1(m,n), q_1(c,a) \in C_1$ and $[q_1(m,n)] \otimes [q_1(c,a)] \in C \otimes C$. Furthermore P_3 is the subgroup of $M \rtimes N$ generated by elements of the form

$$\langle \langle (m,n),(c,a) \rangle, (m',n') \rangle \text{ and } \langle (m,n), \langle (c,a),(m',n') \rangle \rangle$$

for $(m,n),(c,a),(m',n') \in M \rtimes N$, and P'_3 is the subgroup of L generated by elements of the form

$$h\left({}^{\nu(nan^{-1})}mm^{-1}, {}^{\upsilon({}^{\mu(m)}(nan^{-1})(na^{-1}n^{-1}))}n'\right)$$

and

$$h\left(m, {}^{\nu(n)}\left({}^{\mu(c)}(an'a^{-1})(an'^{-1}a^{-1})\right)\right)$$

for $(m,n),(c,a),(m',n') \in M \rtimes N$. $\delta : C_2 \longrightarrow C_1$ is defined by $\delta(lP'_3) = (\lambda l^{-1}, \lambda' l)P_3$ and $\partial : C_1 \to C_0$ is defined by $\partial(q_1(m,n)) = \mu(m)\upsilon(n)$.

The proof of the axioms of quadratic module is similar to the proof of the axioms of Proposition 4.1.

8. Appendix

The proof of simplicial identities:

$$\begin{aligned} d_0^2 s_0^1(n,m,p) &= d_0^2((1,(1,m)),(n,(1,p))) \\ &= (n, \lambda 1.{}^{\nu(1)}m, \upsilon(1)\mu(1)p) \\ &= (n,m,p) = id, \end{aligned} \qquad \begin{aligned} d_1^2 s_0^1(n,m,p) &= d_1^2((1,(1,m)),(n,(1,p))) \\ &= (n\lambda' 1.1, m, p) \\ &= (n,m,p) = id \end{aligned}$$

and

$$\begin{aligned} d_1^2 s_1^1(n,m,p) &= d_1^2((1,(n,1)),(1,(m,p))) \\ &= (1.\lambda' 1.n, 1.m, p) \\ &= (n,m,p) = id, \end{aligned} \qquad \begin{aligned} d_2^2 s_1^1(n,m,p) &= d_2^2((1,(n,1)),(1,(m,p))) \\ &= (n,m,p) = id \end{aligned}$$

and

$$d_2^2 s_0^1(n,m,p) = d_2((1,(1,m)),(n,(1,p))) = (1,1,p) = s_0^0(p) = s_0^0 d_1^1(n,m,p).$$

Similarly

$$d_1^1 d_1^2((l,(n,m_1)),(n_1,(m_2,p))) = d_1^1(n_1(\lambda' l)n, m_1 m_2, p) = p$$

and

$$d_1^1 d_2^2((l,(n,m_1)),(n_1,(m_2,p))) = d_1^1(n, m_2, p) = p$$

then we have $d_1^1 d_1^2 = d_1^1 d_2^2$.

$$\begin{aligned} d_0^1 d_1^2((l,(n,m_1)),(n_1,(m_2,p))) &= d_0^1(n_1(\lambda' l)n, m_1 m_2, p) \\ &= \nu(n_1)\nu\lambda'(l)\nu(n)\mu(m_1)\mu(m_2)p \end{aligned}$$

$$\begin{aligned} d_0^1 d_0^2((l,(n,m_1)),(n_1,(m_2,p))) &= d_0^1(n_1, (\lambda l)^{\nu(n)} m_1, \nu(n)\mu(m_2)p) \\ &= \nu(n_1)\mu((\lambda l)^{\nu(n)} m_1)\nu(n)\mu(m_2)p \\ &= \nu(n_1)\mu(\lambda l)\nu(n)\mu(m_1)\nu(n)^{-1}\nu(n)\mu(m_2)p \\ &= \nu(n_1)\nu\lambda'(l)\nu(n)\mu(m_1)\mu(m_2)p \qquad (\nu\lambda' = \mu\lambda) \end{aligned}$$

so $d_0^1 d_1^2 = d_0^1 d_0^2$.

□

The Proof of Axioms (Proposition 4.1):

2CM1)

$$\begin{aligned} \partial_2\{x,y\} &= (\lambda h(m, nan^{-1})^{-1}, \lambda' h(m, nan^{-1})) \\ &= ({}^{\nu(nan^{-1})}mm^{-1}, {}^{\mu(m)}(nan^{-1})(na^{-1}n^{-1})) \\ &= \langle x, y\rangle \end{aligned}$$

by axioms of the crossed square.

2CM2) We will show that $\{\partial_2(l_0), \partial_2(l_1)\} = [l_1, l_0]$. As $\partial_2 l = (\lambda l^{-1}, \lambda' l)$, this need the calculation of $h(\lambda l_0^{-1}, \lambda'(l_0 l_1 l_0^{-1}))$; but the crossed square axioms $h(\lambda l, n) = l^n l^{-1}$ and $h(m, \lambda' l) = ({}^m l)l^{-1}$ together with the fact that the map $\lambda : L \to M$ is a crossed module, give:

$$\begin{aligned} h(\lambda l_0^{-1}, \lambda'(l_0 l_1 l_0^{-1})) &= {}^{\mu\lambda(l_0)^{-1}}(l_0 l_1 l_0^{-1}).l_0 l_1^{-1} l_0^{-1} \\ &= [l_1, l_0]. \end{aligned}$$

2CM3) For the elements of $M \rtimes N$ are; $m = (m_0, n_0), m' = (m_1, n_1), m'' = (m_2, n_2)$ we have

(i)

$$\begin{aligned} \{mm', m''\} &= \{(m_0, n_0)(m_1, n_1), (m_2, n_2)\} \\ &= \{(m_0^{\nu(n_0)}(m_1), n_0 n_1), (m_2, n_2)\} \\ &= h(m_0^{\nu(n_0)}(m_1), n_0 n_1 n_2 n_1^{-1} n_0^{-1}) \\ &= h({}^{\mu(m_0)\nu(n_0)}m_1, {}^{\mu(m_0)\nu(n_0)}(n_1 n_2 n_1^{-1}))h(m_0, n_0 n_1 n_2 n_1^{-1} n_0^{-1}) \\ &= {}^{\mu(m_0)\nu(n_0)}h(m_1, n_1 n_2 n_1^{-1})h(m_0, n_0 n_1 n_2 n_1^{-1} n_0^{-1}) \\ &= {}^{\partial_1(m)}(\{m', m''\})h(m_0, n_0 n_1 n_2 n_1^{-1} n_0^{-1}). \end{aligned}$$

Since

$$\begin{aligned} mm''m'^{-1} &= (m_1,n_1)(m_2,n_2)(m_1,n_1)^{-1} \\ &= (m_1,n_1)(m_2,n_2)({}^{\nu(n_1^{-1})}(m_1^{-1}),n_1^{-1}) \\ &= (m_1^{\nu(n_1)}m_2,n_1n_2)({}^{\nu(n_1^{-1})}(m_1^{-1}),n_1^{-1}) \\ &= (m_1^{\nu(n_1)}m_2{}^{\nu(n_1n_2n_1^{-1})}(m_1)^{-1},n_1n_2n_1^{-1}) \end{aligned}$$

and

$$\begin{aligned} \{m,m'm''m'^{-1}\} &= \{(m_0,n_0),(m_1^{\nu(n_1)}m_2{}^{\nu(n_1n_2n_1^{-1})}(m_1)^{-1},n_1n_2n_1^{-1})\} \\ &= h(m_0,n_0n_1n_2n_1^{-1}n_0^{-1}), \end{aligned}$$

we get

$${}^{\partial_1(m)}(\{m',m''\})h(m_0,n_0n_1n_2n_1^{-1}n_0^{-1}) = {}^{\partial_1(m)}(\{m',m''\})\{m,m'm''m'^{-1}\},$$

and thus

$$\{mm',m''\} = {}^{\partial_1(m)}(\{m',m''\})\{m,m'm''m'^{-1}\}.$$

(ii)

$$\begin{aligned} \{m,m'm''\} &= \{(m_0,n_0),(m_1^{\nu(n_1)}m_2,n_1n_2)\} \\ &= h(m_0,n_0n_1n_2n_0^{-1}) \\ &= h(m_0,n_0n_1n_0^{-1}n_0n_2n_0^{-1}) \\ &= h(m_0,n_0n_1n_0^{-1})h({}^{\nu(n_0n_1n_0^{-1})}m_0,{}^{n_0n_1n_0^{-1}}\,n_0n_2n_0^{-1}) \\ &= \{m,m'\}h({}^{\nu(n_0n_1n_0^{-1})}m_0,{}^{n_0n_1n_0^{-1}}\,n_0n_2n_0^{-1}) \end{aligned}$$

and this gives the following result

$$\{m,m'm''\} = \{m,m'\}^{mm'(m^{-1})}\{m,m''\}.$$

2CM4)

$$\begin{aligned} \{x,\partial_2 l\}\{\partial_2 l,x\} &= \{(m,n),(\lambda l^{-1},\lambda' l)\}\{(\lambda l^{-1},\lambda' l),(m,n)\} \\ &= h(m,n\lambda' l n^{-1})h(\lambda l^{-1},\lambda' l n\lambda' l^{-1}) \\ &= h(m,\lambda'({}^n l))h(\lambda(l^{-1}),\lambda' l n\lambda' l^{-1}) \\ &= {}^{\mu(m)\nu(n)}l^{\nu(n)}(l^{-1})(l^{-1})^{\nu\lambda' l\nu(n)\nu\lambda' l^{-1}}l \end{aligned}$$

and this simplifies as expected to give the correct result.

□

The Proof of Axioms (Proposition 5.2):

QM1) Clearly $\partial : M \to N$ is a nil(2)-module as the Peiffer commutators which in the forms $\langle x, \langle y,z\rangle\rangle$ and $\langle\langle x,y\rangle, z\rangle$ are in $P_3(\partial_1)$.

QM2) It is easy to see that $\delta\partial = 1$. Also

$$\begin{aligned}\delta\omega(\overline{x'}\otimes\overline{y'}) &= \delta q_2(\{x,y\})\\ &= q_1\partial_2\{x,y\}\\ &= q_1({}^{\partial_1 x}y)x(y)^{-1}(x)^{-1}\\ &= ({}^{\partial x'}y')x'(y')^{-1}(x')^{-1}.\end{aligned}$$

for $\overline{x'},\overline{y'}\in C$ and $x',y'\in M$.

QM3) For $x'\in M$ and $[a]\in L$,

$$\begin{aligned}\omega\left(\overline{x'}\otimes\overline{(\partial_2[a])'(\partial_2[a])'}\otimes\overline{x'}\right)[a] &= q_2((\{x,\partial_2 a\}\{\partial_2 a,x\})a) && \text{(by definition)}\\ &= q_2({}^{\partial_1 x}a({}^{x}a^{-1})({}^{x}a)a^{-1}a) && \text{(by } \mathbf{2CM4}\text{)}\\ &= {}^{\partial x'}[a].\end{aligned}$$

QM4)

$$\begin{aligned}\omega\left(\overline{\delta[a]}\otimes\overline{\delta[b]}\right) &= \omega\left(\overline{(\partial_2 a)'}\otimes\overline{(\partial_2 b)'}\right) && \text{(by commutativity)}\\ &= q_2\{\partial_2 a,\partial_2 b\}\\ &= [[b],[a]].\end{aligned}$$

for $[a],[b]\in L$.

□

The Proof of Axioms (Proposition 6.4):

We display the elements omitting the overlines in our calculation to save complication.

QM1) $\partial : M\to N$ is a nil(2)-module as the Peiffer commutators which in the forms $\langle x,\langle y,z\rangle\rangle$ and $\langle\langle x,y\rangle,z\rangle$ are in $P_3(\partial_1)$.

QM2) For all $q_1x, q_1y\in M$,

$$\begin{aligned}\delta\omega(\{q_1x\}\otimes\{q_1y\}) &= \delta q_2(s_0xs_1ys_0x^{-1}s_1xs_1y^{-1}s_1x^{-1})\\ &= q_1\partial_2(s_0xs_1ys_0x^{-1}s_1xs_1y^{-1}s_1x^{-1})\\ &= q_1(d_2(s_0xs_1ys_0x^{-1}s_1xs_1y^{-1}s_1x^{-1}))\\ &= q_1(s_0d_1xys_0d_1x^{-1}xy^{-1}x^{-1})\\ &= \langle q_1x,q_1y\rangle \qquad \text{by } \partial q_1=\partial_1.\end{aligned}$$

QM3) Supposing $D_3 = G_3$, we know from [**18**] that

$$d_3(F_{(2,0)(1)}(x,a)) = [s_0x,s_1d_2a][s_1d_2a,s_1x][s_1x,a][a,s_0x]\in\partial_3(NG_3).$$

¿From this equality we have

$$[s_0x,s_1d_2a][s_1d_2a,s_1x]\equiv[s_0x,a][a,s_1x] \mod \partial_3(NG_3).$$

Thus we get

$$\begin{aligned}\omega(\{q_1x\}\otimes\{\delta q_2a\}) &= \omega\left(\{q_1x\}\otimes\{q_1\partial_2a\}\right)\\ &= q_2\left(s_0(x)s_1d_2(a)s_0(x)^{-1}s_1(x)s_1d_2(a)^{-1}s_1(x)^{-1}\right)\\ &\equiv q_2([s_0x,a][a,s_1x])\\ &= {}^{\partial q_1(x)}q_2(a)q_2({}^x(a^{-1})),\end{aligned}$$

and similarly from

$$d_3(F_{(0)(2,1)}(a,x)) = [s_0d_2a,s_1x][s_1x,s_1d_2a][a,s_1x]\in\partial_3(NG_3\cap D_3)=\partial_3(NG_3),$$

we have

$$\begin{aligned}\omega(\{\delta q_2a\}\otimes\{q_1x\}) &= \omega(\{q_1\partial_2a\}\otimes\{q_1x\})\\ &= q_2(s_0d_2(a)s_1(x)s_0d_2(a)^{-1}s_1d_2(a)s_1(x)^{-1}s_1d_2(a)^{-1})\\ &\equiv q_2([s_1x,a])\\ &= q_2({}^xa)q_2a^{-1}.\end{aligned}$$

Consequently we have,

$$\omega\left(\{q_1x\}\otimes\{\delta q_2a\}\{\delta q_2a\}\otimes\{q_1x\}\right)q_2a = {}^{\partial q_1(x)}q_2a.$$

QM4) From [**18**], we get

$$d_3(F_{(0),(1)}(a,b)) = [s_0d_2a,s_1d_2b][s_1d_2b,s_1d_2a][a,b]\in\partial_3(NG_3\cap D_3)=\partial_3(NG_3).$$

¿From this equality, we can write

$$[s_0d_2a,s_1d_2b][s_1d_2b,s_1d_2a]\equiv[b,a]\mod\partial_3(NG_3).$$

Thus we have

$$\begin{aligned}\omega(\{\delta q_2a\}\otimes\{\delta q_2b\}) &= \omega(\{q_1\partial_2a\}\otimes\{q_1\partial_2b\})\\ &= q_2(s_0d_2(a)s_1d_2(b)s_0d_2(a)^{-1}s_1d_2(a)s_1d_2(b)^{-1}s_1d_2(a)^{-1})\\ &\equiv[q_2b,q_2a].\end{aligned}$$

for $q_2a, q_2b\in L$.

□

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