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CLOSEDNESS PROPERTIES OF INTERNAL RELATIONS IV: EXPRESSING ADDITIVITY OF A CATEGORY VIA SUBTRACTIVITY

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Abstract

The notion of a subtractive category, recently introduced by the author, is a "categorical version" of the notion of a (pointed) subtractive variety of universal algebras, due to A. Ursini. We show that a subtractive variety $\mathcal{C}$, whose theory contains a unique constant, is abelian (i.e. $\mathcal{C}$ is the variety of modules over a fixed ring), if and only if the dual category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$, is subtractive. More generally, we show that $\mathcal{C}$ is additive if and only if both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are subtractive, where $\mathcal{C}$ is an arbitrary finitely complete pointed category, with binary sums, and such that each morphism f in $\mathcal{C}$ can be presented as a composite $f = me$, where m is a monomorphism and e is an epimorphism.

Introduction

A variety $\mathcal{V}$ of universal algebras is *subtractive* in the sense of A. Ursini [**14**], if the theory of $\mathcal{V}$ contains a binary term s (called a *subtraction term*) and a nullary term 0, satisfying the identities $s(x, 0) = x$ and $s(x, x) = 0$. An example of a subtractive variety is the variety of (additive) groups, for which $s(x, y) = x - y$. More generally, any Mal'tsev variety [**12**], whose theory contains a nullary term, is subtractive. An example of a subtractive variety, which is not a Mal'tsev variety, is the variety of implication algebras [**1**].

A *subtraction algebra* is a triple $A = (A, -, 0)$, where A is a set, "–" is a binary operation on A, and 0 is a nullary operation on A, satisfying the axioms $x - 0 = x$ and $x - x = 0$. A *group* can be defined as a subtraction algebra with the additional axiom $(x-y)-(z-y) = x-z$, and an *abelian group* can be defined as a subtraction algebra with the stronger axiom

$$(x - y) - (z - t) = (x - z) - (y - t); \qquad (1)$$

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in both cases, the group addition $+$ is defined by the equality $x + y = x - (0 - y)$, while the unary operation of taking the inverse $-x$ of an element x, is defined by the equality $-x = 0 - x$. Note that, to require the axiom (1), is the same as to require that the binary operation

$$- : A \times A \longrightarrow A$$

is a homomorphism of subtraction algebras. Thus, an abelian group is a subtraction algebra whose algebraic structure is at the same time an internal subtraction structure in the category $\mathcal{S}$ of all subtraction algebras. Further, any internal subtraction structure on an object A in $\mathcal{S}$, coincides with the underlying subtraction structure of A (see [**5**], [**3**]*), and so an internal subtraction algebra in $\mathcal{S}$ is always an internal abelian group.

The variety $\mathcal{S}$ is clearly a subtractive variety. Further, the algebraic theory of $\mathcal{S}$ contains a unique nullary term; equivalently, $\mathcal{S}$ is a *pointed category*, i.e. the terminal object in $\mathcal{S}$ coincides with the initial object. The notion of a subtractive category, introduced in [**8**] (see also [**9**]), extends the notion of a pointed subtractive variety to abstract pointed categories — a pointed variety is subtractive in the sense of Ursini, if and only if it is a subtractive category. The purpose of this paper is to show that for a finitely complete pointed category, with binary sums, and such that each morphism f in $\mathcal{C}$ can be presented as a composite $f = me$, where m is a monomorphism and e is an epimorphism, we have the following result: $\mathcal{C}$ is additive if and only if both $\mathcal{C}$ and its dual $\mathcal{C}^{\mathrm{op}}$ are subtractive (Theorem 2.2). This implies that a Barr exact [**2**] pointed category $\mathcal{C}$, having binary sums, is an abelian category, if and only if both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are subtractive. In particular, for varieties of universal algebras (which are always Barr exact and have binary sums), we obtain: a pointed subtractive variety $\mathcal{C}$ is abelian (i.e. $\mathcal{C}$ is the variety of modules over a fixed ring) if and only if $\mathcal{C}^{\mathrm{op}}$ is also subtractive.

1. The definition of a subtractive category

Let $\mathcal{C}$ be a pointed category with binary products. $\mathcal{C}$ is said to be subtractive, if for any object A in $\mathcal{C}$, any subobject $r : R \longrightarrow A \times A$ of $A \times A$ satisfies the following condition: if there exist morphisms $f_1, f_2 : A \longrightarrow R$, such that both diagrams

$$\begin{tikzcd} & R \arrow[d, "r"] \\ A \arrow[ur, "f_1"] \arrow[r, "(1_A,1_A)"'] & A \times A \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[d, "r"] \\ A \arrow[ur, "f_2"] \arrow[r, "(1_A,0)"'] & A \times A \end{tikzcd} \tag{2}$$

*In [**3**], the term "protosubtraction" is used for the operation of subtraction of a subtraction algebra A, and "subtraction" is used only when A is a group; in [**5**] (and also in most cases in [**3**] too), "$x - y$" is written as "$y \setminus x$", and accordingly, a "(proto)subtraction" is called a "(proto)devision".

commute, then there exists a morphism $f_3 : A \longrightarrow R$, such that the diagram

$$\begin{tikzcd} & R \arrow[d, "r"] \\ A \arrow[ur, "f_3"] \arrow[r, "(0,1_A)"'] & A \times A \end{tikzcd} \tag{3}$$

commutes. The above condition on the subobject R can be reformulated, using generalized elements, as follows: if $(a, a) \in R$ and $(a, 0) \in R$ for all $a \in A$, then $(0, a) \in R$ for all $a \in A$. As we know from [**9**], if $\mathcal{C}$ has finite limits, then $\mathcal{C}$ is subtractive if and only if the following condition is satisfied for any object A in $\mathcal{C}$, and any subobject R of $A \times A$: for all $a \in A$,

$$(a, a) \in R \ \wedge \ (a, 0) \in R \ \Rightarrow \ (0, a) \in R. \tag{4}$$

The condition (4) on R states precisely that R is (regarded as a binary relation on A) *M-closed* in the sense of [**9**], where M is the extended matrix

$$M = \left(\begin{array}{cc|c} x & x & 0 \\ x & 0 & x \end{array} \right)$$

of terms in the algebraic theory $\mathcal{T}$ of pointed sets — 0 is the unique constant of $\mathcal{T}$, while x is an arbitrary variable of $\mathcal{T}$. Note that this matrix directly arises from the system of equations

$$\begin{cases} x - x = 0, \\ x - 0 = 0, \end{cases}$$

which defines an operation of subtraction. In the present paper we write such matrices mostly in the transposed form

$$M = \left(\begin{array}{cc} x & x \\ x & 0 \\ \hline 0 & x \end{array} \right).$$

R is M-closed means that for any *interpretation*

$$M^a = \left(\begin{array}{cc} a & a \\ a & 0 \\ \hline 0 & a \end{array} \right)$$

of M, where $a \in A$ and 0 is the base point of A, if the first two rows of M^a belong to R (considered as generalized elements of $A \times A$), then also the third row (i.e. the row below the horizontal line) belongs to R. Thus, R is M-closed if and only if the implication (4) is satisfied for all $a \in A$, as already said above. Recall from [**9**], that if we take

$$M = \left(\begin{array}{cc} x & y \\ x & 0 \\ \hline 0 & y \end{array} \right),$$

then, again, a (finitely complete pointed) category with M-closed relations is the same as a (finitely complete) subtractive category. If we further modify M by adding to it columns of the form

$$\begin{pmatrix} u \\ u \\ \hline 0 \end{pmatrix}, \text{ or of the form } \begin{pmatrix} v \\ 0 \\ \hline v \end{pmatrix},$$

then again (by Proposition 1.7 of [**10**]), a subtractive category is the same as a pointed category in which every n-ary relation $R \longrightarrow A^n$ is M-closed, where now M is the new matrix, and n denotes the number of columns of M.

2. *Additive* is the same as *subtractive and cosubtractive*

A *semi-abelian category* in the sense of G. Janelidze, L. Márki, and W. Tholen [**7**], is a pointed category $\mathcal{C}$ satisfying the following conditions:

- $\mathcal{C}$ is Barr exact [**2**],
- $\mathcal{C}$ has binary sums,
- $\mathcal{C}$ is protomodular in the sense of D. Bourn [**4**].

As shown in [**6**], a pointed variety of universal algebras is semi-abelian if and only if it is *classically ideal determined* in the sense of A. Ursini [**14**] (this is the same as *BIT speciale* in the sense of [**13**]).

As observed in [**8**], any semi-abelian category is subtractive.

The following equation for categories was obtained in [**7**]:

$$\text{semi-abelian} \ + \ \text{semi-abelian}^{\text{op}} = \text{abelian}.$$

This equation says: a category $\mathcal{C}$ is an abelian category if and only if both $\mathcal{C}$ and its dual category $\mathcal{C}^{\text{op}}$ are semi-abelian. From Theorem 2.2 below, it follows that, more generally, for any Barr exact category with binary sums, we have (see Corollary 2.3, and see also Remark 2.4):

$$\text{subtractive} \ + \ \text{subtractive}^{\text{op}} = \text{abelian}.$$

Recall that a Barr exact category $\mathcal{C}$ is an abelian category if and only if it is additive, i.e. if and only if $\mathcal{C}$ is enriched in the category of abelian groups — each object in $\mathcal{C}$ is equipped with an internal abelian group structure, so that each morphism in $\mathcal{C}$ is a homomorphism of internal abelian groups. More generally, we have:

Lemma 2.1. *A category with finite products is additive if and only if each object in it is equipped with an internal subtraction structure, so that each morphism is a homomorphism of internal subtraction algebras.*

Proof. This follows immediately from the definition of an additive category, and the known fact that if the operation of subtraction of a subtraction algebra is homomorphic, then the subtraction algebra becomes an abelian group. □

Theorem 2.2. *Let $\mathcal{C}$ be a finitely complete pointed category with binary sums, in which*

(*) any morphism $f : X \longrightarrow Y$ can be decomposed $f = me$ into a monomorphism m and an epimorphism e.

Then, $\mathcal{C}$ is an additive category if and only if both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are subtractive.

Proof. If $\mathcal{C}$ is additive, then it is subtractive. Indeed, then for any pair of commutative diagrams (2), we can form the third one (3), by taking in it $f_3 = f_1 - f_2$, where "$-$" denotes the operation of subtraction on $\hom(A, R)$, induced by the operation of subtraction of the abelian group structure of R. We will then have

$$rf_3 = r(f_1 - f_2) = rf_1 - rf_2 = (1_A, 1_A) - (1_A, 0) = (1_A - 1_A, 1_A - 0) = (0, 1_A),$$

so the diagram (3) indeed commutes. On the other hand, if $\mathcal{C}$ is additive, then so is $\mathcal{C}^{\mathrm{op}}$, and hence $\mathcal{C}^{\mathrm{op}}$ is subtractive. This proves the first part of the theorem.

Assume now that both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are subtractive. According to Lemma 2.1, to show that $\mathcal{C}$ is additive, it suffices to show that each object A in $\mathcal{C}$ is equipped with an internal subtraction structure, $A = (A, -, 0)$, so that each morphism $f : A \longrightarrow A'$ in $\mathcal{C}$ is a homomorphism of (internal) subtraction algebras. Let A be an object in $\mathcal{C}$. Then we can form a commutative diagram

$$\begin{tikzcd} & R \arrow[dr, "r=(r_1,r_2)"] & \\ A+A \arrow[ur, "q=\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}"] \arrow[rr, "\begin{pmatrix} 1_A & 1_A \\ 1_A & 0 \end{pmatrix}"'] & & A \times A \end{tikzcd}$$

where r is a monomorphism and q is an epimorphism. This commutative diagram yields the following two commutative diagrams:

$$\begin{tikzcd} R \arrow[dr, "r_1"] & \\ A+A \arrow[u, "q"] \arrow[r, "\begin{pmatrix} 1_A \\ 1_A \end{pmatrix}"'] & A \end{tikzcd} \qquad \begin{tikzcd} R \arrow[dr, "r_2"] & \\ A+A \arrow[u, "q"] \arrow[r, "\begin{pmatrix} 1_A \\ 0 \end{pmatrix}"'] & A \end{tikzcd}$$

Since $\mathcal{C}^{\mathrm{op}}$ is subtractive, there exists a morphism $s : R \longrightarrow A$ such that the diagram

$$\begin{tikzcd} R \arrow[dr, "s"] & \\ A+A \arrow[u, "q"] \arrow[r, "\begin{pmatrix} 0 \\ 1_A \end{pmatrix}"'] & A \end{tikzcd}$$

commutes. Next, we regard R as the subobject $r : R \longrightarrow A \times A$ of $A \times A$. Then $s : R \longrightarrow A$ becomes a partial binary operation on A. Commutativity of the two diagrams

$$\begin{tikzcd} & R \arrow[d, "r"] \\ A \arrow[ur, "q_1"] \arrow[r, "(1_A,1_A)"'] & A \times A \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[d, "r"] \\ A \arrow[ur, "q_2"] \arrow[r, "(1_A,0)"'] & A \times A \end{tikzcd}$$

and the fact that $\mathcal{C}$ is subtractive, yield that there exists a morphism $p : A \longrightarrow R$ such that the diagram

$$\begin{tikzcd} & R \arrow[d, "r"] \\ A \arrow[ur, "p"] \arrow[r, "(0,1_A)"'] & A \times A \end{tikzcd}$$

commutes. From the commutativity of the last four diagrams we obtain: For all $a \in A$,

$$(a, a), (a, 0), (0, a) \in R$$

and, further, $s(a, a) = 0$ and $s(a, 0) = a$. We now show $R = A \times A$. Consider the ternary relation $R' \longrightarrow A^3$ defined as follows:

$$(a, b, c) \in R' \quad \Leftrightarrow \quad (b, c) \in R \ \wedge \ (a, s(b, c)) \in R.$$

Since $\mathcal{C}$ is subtractive, R' is closed with respect to the matrix

$$\left(\begin{array}{ccc} x & y & y \\ 0 & 0 & y \\ \hline x & y & 0 \end{array} \right).$$

For each $a, b \in A$ we then have:

$$\begin{array}{rcl} (b, b) \in R \ \wedge \ (a, s(b, b)) = (a, 0) \in R & \Rightarrow & (a, b, b) \in R', \\ (0, b) \in R \ \wedge \ (0, s(0, b)) \in R & \Rightarrow & (0, 0, b) \in R', \\ & & \Downarrow \\ (a, b) = (a, s(b, 0)) \in R & \Leftarrow & (a, b, 0) \in R'. \end{array}$$

Thus, for all $a, b \in A$, $(a, b) \in R$, which shows $R = A \times A$. So s is a full binary operation on A, and since we already know that it satisfies the identities $s(a, a) = 0$ and $s(a, 0) = a$, we get that $(A, s, 0)$ is a subtraction algebra. We have thus shown

- that the morphism r is an isomorphism; this implies that the morphism

$$A+A \xrightarrow[\left(\begin{smallmatrix} 1_A & 1_A \\ 1_A & 0 \end{smallmatrix}\right)]{} A \times A$$

  is an epimorphism;
- that there exists a subtraction algebra $(A, -, 0)$; note that the subtraction axioms for the operation $- : A \times A \longrightarrow A$ state precisely that the diagram

$$\begin{tikzcd} A+A \arrow[r, "\left(\begin{smallmatrix} 1_A & 1_A \\ 1_A & 0 \end{smallmatrix}\right)"'] \arrow[rr, bend left, "\left(\begin{smallmatrix} 0 \\ 1_A \end{smallmatrix}\right)"] & A \times A \arrow[r, "-"'] & A \end{tikzcd}$$

  commutes.

Now let $(A', -', 0')$ be another subtraction algebra, obtained from an object A', like $(A, -, 0)$ is obtained from A. Let f be any morphism $f : A \longrightarrow A'$. In the diagram

$$\begin{tikzcd} A+A \arrow[r, "\left(\begin{smallmatrix} 1_A & 1_A \\ 1_A & 0 \end{smallmatrix}\right)"] \arrow[d, "f+f"'] & A \times A \arrow[r, "-"] \arrow[d, "f\times f"'] & A \arrow[d, "f"] \\ A'+A' \arrow[r, "\left(\begin{smallmatrix} 1_{A'} & 1_{A'} \\ 1_{A'} & 0' \end{smallmatrix}\right)"'] & A' \times A' \arrow[r, "-'"'] & A' \end{tikzcd}$$

the left inner rectangle commutes and the outer rectangle commutes. Since the first morphism in the top row is an epimorphism, this implies that the right inner rectangle commutes. We thus obtain that f is a homomorphism $f : (A, -, 0) \longrightarrow (A', -', 0')$ of subtraction algebras. This concludes the proof. $\square$

Corollary 2.3. *A Barr exact category $\mathcal{C}$ with binary sums is an abelian category, if and only if both $\mathcal{C}$ and its dual $\mathcal{C}^{\mathrm{op}}$ are subtractive.*

Proof. An ablian category is the same as an additive Barr exact category. On the other hand, a Barr exact category, being a regular category, has finite limits and any morphism f in it can be decomposed $f = me$ into a monomorphism m and an epimorphism e (moreover, any morphism can be decomposed into a monomorphism and a regular epimorphism). After these observations, it is only left to apply Theorem 2.2. $\square$

Remark 2.4. *I do not know if Theorem 2.2 remains true if $\mathcal{C}$ in it does not have the property (*). However, it does remain true if in addition we replace "subtractive" in the theorem with "strongly unital" in the sense of D. Bourn [**5**]; this is an immediate consequence of the following known facts (see [3] and [**8**]):*

(i) *a finitely complete category is additive if and only if it is subtractive and enriched in the category of commutative monoids,*

(ii) *a category $\mathcal{C}$ with binary sums and products is enriched in the category of commutative monoids if and only if both $\mathcal{C}$ and its dual $\mathcal{C}^{\mathrm{op}}$ are unital in the sense of D. Bourn [**5**],*

(iii) *a finitely complete category is strongly unital if and only if it is subtractive and unital.*

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