Journal of Homotopy and Related Structures, vol. 1(1), 2006, pp.273–283

HOMOTOPY TRANSITION COCYCLES

JAMES WIRTH AND JIM STASHEFF

(*communicated by Hvedri Inassaridze*)

Abstract

For locally homotopy trivial fibrations, one can define transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to H = H(F)$$

where H is the monoid of homotopy equivalences of F to itself but, instead of the cocycle condition, one obtains only that $g_{\alpha\beta}g_{\beta\gamma}$ is homotopic to $g_{\alpha\gamma}$ as a map of $U_\alpha \cap U_\beta \cap U_\gamma$ into H. Moreover, on multiple intersections, higher homotopies arise and are relevant to classifying the fibration.

The full theory was worked out by the first author in his 1965 Notre Dame thesis [**17**]. Here we present it using language that has been developed in the interim. We also show how this points a direction 'on beyond gerbes'.

1. Introduction

In the theory of fibre bundles $E \to B$, a key role is played by transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ with respect to an open cover $\{U_\alpha\}$ of B. Here G is the structural group of the bundle and acts as a group of transformations on the fibre F. One of the striking properties of transition functions is the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$

For fibrations, the situation is more complicated. Assuming the fibration is locally homotopy trivial, one can define transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to H = H(F)$$

where H is the monoid of homotopy equivalences of F to itself but instead of the cocycle condition, one obtains only that $g_{\alpha\beta}g_{\beta\gamma}$ is homotopic to $g_{\alpha\gamma}$ as a map of $U_\alpha \cap U_\beta \cap U_\gamma$ into H. Moreover, on multiple intersections, higher homotopies arise and are relevant to classifying the fibration.

The full theory was worked out by the first author in his Notre Dame thesis [**17**]. The intervening years have provided a language which helps organize the technicalities, though in no way eliminating them. If $\{U_\alpha\}$ is the open covering, the disjoint union $\coprod U_\alpha$ can be given a rather innocuous structure of a topological

Received September 17, 2006, revised December 11, 2006; published on December 26, 2006.
2000 Mathematics Subject Classification: 55R05, 55R35, 55S37, 18B30.
Key words and phrases: fibrations, transition cocycles, higher homotopies.

category U, i.e., Ob $U = \coprod U_\alpha$ and Mor $U = \coprod U_\alpha \cap U_\beta$ that is $x \circ y = x = y$ is defined iff $x \in U_\alpha$, $y \in U_\beta$ and $x = y$. Regarding G or H as a category with one object in the standard way, the one cocycle condition says that the transition functions define a continuous functor. The web of higher homotopies appropriate to a fibration are precisely equivalent to a *functor up to strong homotopy*, also known as a *homotopy coherent functor*, which does arise in other contexts involving topological categories [**4**].

Lest the above give the impression that we have only to translate naturally occurring homotopies into a fancy language, we point out that some powerful topology is necessary to construct fibrations or equivalences of fibrations from the *homotopy cocycle* data. In particular, the first author's patching/glueing/recollement (mapping cylinder) theorem, which is of fundamental importance in more general fibration theories, is essential [**17**].

Recent developments in higher homotopy theory and especially higher gauge theory [**1**] have inspired us to produce this belated and somewhat updated public version of the first author's work. Preliminary versions and a talk at the University of Pennsylvania have led us to work of Breen [**2, 3**] and of Simpson and Hirschowitz [**7**] which have intriguing points of contact with Wirth's much earlier thesis. Breen was well aware at the time of [**2**] of the relation to higher homotopy theory in the context of locally homotopy trivial fibrations. On the other hand, *gerbes* are closely related to a special case of the homotopy transition cocycles we consider. We restrict our point of view to the original topological setting of Wirth's dissertation, leaving to the future further development of the higher homotopy cocycle point of view, especially in relation to algebraic geometry, that is, further 'pursuing stacks'.

In Section 1, we begin with a swift review of standard material about fibrations and see how the higher order transition homotopies occur naturally. In Section 2, we review the realization of a topological category $\mathcal{C}$ as a space, observe that for a numerable cover $\{U_\alpha\}$ of B, the realization $|\mathcal{U}|$ has the homotopy type of B [**9**] and show how a functor up to strong homotopy is sufficient to induce a map of realizations. Thus a fibration $E \to B$ produces a map $B \xrightarrow{\simeq} |\mathcal{U}| \to |H| = BH$, the classifying space of $H = H(F)$ as a topological monoid.

Our emphasis is on the cocycle point of view, although such classifying maps can also be constructed by studying the action of the based loop space ΩB of the base B on the fibre F, that is, in terms of an A_∞-map of ΩB to $H(F)$ [**11, 10**].

In Section 3, we do the topology, showing how to construct a fibration from the higher order transition functions. The usual universal example over BH is only a quasi-fibration. Although this could be improved to a fibration by Fuchs' technique [**6**], Wirth's construction provides a perspicuous alternative.

In Section 4, we confront the full classification theorem: Equivalent fibrations correspond to homotopic functors up to strong homotopy which in turn correspond to homotopic classifying maps. In terms of transition functions, this appears as a direct though complicated generalization of cocycles up to cobounding cocycles.

In Section 5, Wirth's concept of a "fibration theory" is axiomatized. Here too the patching theorem is crucial.

Finally, in section 6, we show the relation of our approach to foliations and Haefliger structures. We also discuss briefly how gerbes provide a particular instantiation of homotopy transition cocycles of a particular 'truncated' type, leaving for further development the relation to the work of Breen and of Simpson and Hirschowitz.

2. Fibrations and transition functions

Since we wish to look at things from a homotopy invariant, not geometric, point of view, a natural class of fibrations to consider is that of Dold fibrations, those with the WCHP (Weak Covering Homotopy Property)[**5**].

Definition 2.1. A map $p : E \to B$ has the WCHP if for every homotopy $H : X \times I \to B$ and $h : X \to E$ such that $ph(x) = H(x, 0)$, there exists a homotopy $\tilde{H} : X \times [-1, 1] \to E$ such that $\tilde{H}(x, -1) = h(x)$ and $p\tilde{H}(x, t) = H(x, t)$ for $t \in [0, 1]$ while $p\tilde{H}(x, t) = H(x, 0)$ for $t \in [-1, 0]$.

In other words, H is covered by a homotopy whose initial position is vertically homotopic to h.

On the other hand, since our emphasis will be on local data such as transition functions, it is more appropriate to consider locally homotopy trivial fibrations.

Definition 2.2. A map $p : E \to B$ is locally homotopy trivial over an open covering $\{U_\alpha\}$ if there exist maps h_α and k_α such that the following diagrams commute:

$$
\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[rr, shift left, "h_\alpha"] \arrow[dr] & & U_\alpha \times F \arrow[ll, shift left, "k_\alpha"] \arrow[dl] \\
& U_\alpha &
\end{tikzcd}
\qquad (1)\ (2)
$$

and h_α and k_α are mutual fibre homotopy inverses.

The two are related by the following:

Theorem 2.3 (Dold). *Let B be a topological space which admits a numerable covering $\{U_\alpha\}$ such that each inclusion $U_\alpha \subset B$ is nullhomotopic, then p has the WCHP if and only if p is fiber homotopy trivial over each U_α.*

Let $H(F)$ be the monoid of all homotopy equivalences of F to itself.

Definition 2.4. The *transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to H(F)$ are defined by the equation

$$h_\alpha k_\beta(x, f) = \big(x, g_{\alpha\beta}(x)(f)\big), \qquad x \in U_\alpha \cap U_\beta,\ f \in F.$$

For nice F, $g_{\alpha\beta}$ will be continuous with respect to the compact-open topology on $H(F)$; otherwise we are content that the adjoint

$$\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \times F \to F$$

is continuous. Now consider the cocycle condition. We have

$$\big(x, g_{\alpha\beta}(x)g_{\beta\gamma}(x)f\big) = h_\alpha k_\beta h_\beta k_\gamma(x, f)$$

which is fiber homotopic to (but not necessarily equal to) $h_\alpha k_\gamma(x, f)$. To go any further, we use specific homotopies $j_\alpha : I \times U_\alpha \times F_\alpha \to U_\alpha \times F$ from $k_\alpha \circ h_\alpha$ at 0 to the identity at 1. Given them, we define

$$g_{\alpha\beta\gamma} : I \times U_\alpha \cap U_\beta \cap U_\gamma \to H(F)$$

by

$$\big(x, g_{\alpha\beta\gamma}(t, x)f\big) = h_\alpha j_\beta\big(t, k_\gamma(x, f)\big).$$

Rather than write down the complicated formulas in general, consider four-fold intersections. Over $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ we have the diagram of homotopies

$$\begin{array}{ccc} g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\delta} & \xrightarrow{g_{\alpha\beta\gamma}g_{\gamma\delta}} & g_{\alpha\gamma}g_{\gamma\delta} \\ {\scriptstyle g_{\alpha\beta}g_{\beta\gamma\delta}}\big\downarrow & & \big\downarrow{\scriptstyle g_{\alpha\gamma\delta}} \\ g_{\alpha\beta}g_{\beta\delta} & \xrightarrow[g_{\alpha\beta\delta}]{} & g_{\alpha\delta}. \end{array}$$

Since they are defined in terms of j_β and j_γ which are independent, it is straightforward to define

$$g_{\alpha\beta\gamma\delta} : I^2 \times U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \to H(F)$$

by

$$\big(x, g_{\alpha\beta\gamma\delta}(t, s, x)f\big) = h_\alpha j_\beta\Big(t, j_\gamma\big(s, k_\delta(x, f)\big)\Big).$$

In general for an $n{+}1$-fold intersection $U_\sigma = U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$ with $\sigma = (\alpha_0, \cdots, \alpha_n)$, we define

$$g_\sigma : I^{n-1} \times U_\sigma \to H(F)$$

by

$$\big(x, g_\sigma(t_1, \cdots, t_{n-1}, x)f\big) = h_{\alpha_0} j_{\alpha_1}\Big(t_1, \cdots, j_{\alpha_{n-1}}\big(t_{n-1}, k_{\alpha_n}(x_1, f)\big) \cdots\Big)$$

and these are compatible in the following way: If $\sigma_i = (\alpha_0, \cdots, \hat{\alpha}_i, \cdots, x_n)$ and $(\varepsilon, i)(I^{n-1})$ is the face $t_i = \varepsilon$ ($\varepsilon = 0$ or 1), then for $i = 1, \cdots, n-1$:

$$g_\sigma|(1, i)(I^{n-1}) \times U_\sigma = g_{\sigma_i}$$
$$g_\sigma|(0, i)(I^{n-1}) \times U_\sigma = g_{(\alpha_0, \cdots, \alpha_i)} g_{(\alpha_i, \cdots, \alpha_n)}.$$

Definition 2.5. The collection $\{g_\sigma\}$ is called a *homotopy transition cocycle* for $p : E \to B$.

These relations are the key to the categorical language we introduce next.

3. Topological categories and realization

One construction of a classifying map for, e.g. G-bundles, [**9, 14**] regards the covering $\mathcal{U} = \{U_\alpha\}$ as a category so that the construction B can be applied to give $B\mathcal{U}$ of the homotopy type of X [except for language, this was known to our ancestors] and to interpret the transition functions as a functor so that they induce $X \simeq B\mathcal{U} \to BG$. The classifying property can be verified directly if we choose the appropriate realization, Milnor's construction [**8**] of a classifying space for a topological group, which has built in a nice "universal" open cover.

Definition 3.1. A category is a *topological category* if the objects and morphisms form topological spaces such that the source, target and composition maps are continuous.

Definition 3.2. The *realization* $|\mathcal{C}|$ of a topological category $\mathcal{C}$ is the following space: Let $\mathcal{C}_0 = \text{Ob } \mathcal{C}$, $\mathcal{C}_1 = \text{Mor } \mathcal{C}$, $\mathcal{C}_p \subset (\text{Mor } \mathcal{C})^p$ consists of all p-tuples $(f_1, \cdots, f_p)$ such that $f_1 \circ \cdots \circ f_p$ is defined. Consider the subset $B\mathcal{C} \subset \Delta^\infty \times (\mathcal{C}_1)^\infty$ consisting of pairs $(\vec{t}, \{g_{ij}\})$ such that
1) $\vec{t} \in \Delta^\infty$
2) i, j runs over all pairs such that $t_i t_j \neq 0$
3) $g_{ij} \in \text{Mor } \mathcal{C}$ and $g_{ii} = Id_i$ and
4) $g_{ij} g_{jk} = g_{ik}$ if $t_i t_j t_k \neq 0$.

(Thus Milnor's reference to "the space of cocyles with values in $\mathcal{C}$.") Topologize this space by the limit of the quotient topologies of the maps

$$\Delta^n \times \mathcal{C}_n \to B\mathcal{C}$$

given by $(s_0, \cdots, s_n, g_1, \cdots, g_n) \to (\vec{t}, \{g_{ij}\})$ with $t_{k_j} = s_j$ for some $k_0 < k_1 < \cdots < k_n$ and $g_{k_i k_j} = g_{i+1} \cdots g_j$.

The universal cover of $B\mathcal{C}$ is given by $U_i = \{t_i^{-1}(0, 1]\}$ and the g_{ij} coordinates regarded as functions $U_i \cap U_j \to \mathcal{C}$ are universal transition functions. (Strictly speaking, the U_i are only point-finite, but following Dold, we can deform the original t_i to functions $\bar{t}_i$ which are locally finite so the associated $\bar{t}_i^{-1}(0, 1]$ are also.)

Now an open covering $\{U_\alpha\}$ can be regarded as a category $\hat{\mathcal{U}}$ in a rather trivial way. For simplicity, well order the index set or, perhaps more naturally, following Segal [**9**], let the objects be the points of the intersections U_σ and let the morphisms be given by the inclusions. This is effectively the barycentric subdivision and hence has a natural partial order. Let $\text{Ob } \mathcal{C} = \coprod U_\alpha$ and $\text{Mor } \mathcal{C} = \coprod_{\alpha > \beta} U_\alpha \cap U_\beta$ with *source*$(x \in U_\alpha \cap U_\beta) = x \in U_\alpha$ and *target*$(x \in U_\alpha \cap U_\beta) = x \in U_\beta$. Thus the composition is defined only for $x \in U_\alpha \cap U_\beta \cap U_\gamma, \alpha > \beta > \gamma$ and $\mathcal{C}_p = \coprod U_{\alpha_0} \cdots U_{\alpha_p} = \coprod U_\sigma$ for ordered simplices σ. We refer to $|\mathcal{U}|$ as the *exploded* X.

There is a natural map $\pi : |\mathcal{U}| \to X$ given by forgetting the simplicial cordinates and the multi-index. If the covering is numerable, i.e., admits a subordinate partition of unity $\{p_{\alpha_0}(x), \cdots, p_{\alpha_n}(x)\}$ for $x \in U_\sigma, \sigma = (\alpha_0, \cdots, \alpha_n)$, there is a corresponding map $\rho : X \to |\mathcal{U}|$. Indeed, this embeds X as a deformation retract of $|\mathcal{U}|$.

Now what is a continuous functor from $\mathcal{U}$ to $\mathcal{C}$? It is a collection of maps

$$f_\alpha : U_\alpha \to \text{Ob } \mathcal{C}$$
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Mor } \mathcal{C}$$

such that

$$f_\alpha = g_{\alpha\beta} f_\beta \quad (3)$$

and

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}. \tag{4}$$

If Ob $\mathcal{C}$ consists of a single point, the first condition is trivial. This is the case for a topological group $G = \mathcal{C}$ and gives rise to the classifying map for a bundle

$$X \xrightarrow{\simeq} |\mathcal{U}| \to |G| = BG.$$

About the same time as Wirth's dissertation, tom Dieck [**14**] proved that Milnor's universal bundle E_G classifies numerable G-bundles over arbitrary spaces by giving explicit formulas. Later, Wellen [**16**] in his diplomarbeit extended the classification theorem to the case of groupoids G.

As we have seen, for a locally trivial fibration, we cannot guarantee the cocycle condition (2) except up to homotopy.

Definition 3.3. Given two topological categories $\mathcal{C}$ and D, a *functor up to strong homotopy* (also known as a *homotopy coherent functor*) is a collection of maps

$$F_0 : \text{Ob } \mathcal{C} \to \text{Ob } D$$
$$F_p : I^{p-1} \times \mathcal{C}_p \to \text{Mor } D$$

such that

$$\begin{aligned} & F_1(c : x \to y) : F_0(x) \to F_0(y) \\ & F_p(t_1, \cdots, t_{p-1}, c_1, \cdots, c_n) = F_{p-1}(\cdots, \hat{t}_i, \cdots, c_i c_{i+1}, \cdots) && \text{if } t_i = 0 \\ & \qquad = F_i(t_1, \cdots, t_{i-1}, c_1, \cdots, c_i) F_{p-i}(t_{i+1}, \cdots, c_{i+1}, \cdots, c_p) && \text{if } t_i = 1. \end{aligned}$$

For the special case in which the categories have one object and their morphisms therefore form a monoid, the equivalent notion is that of a Strongly Homotopy Multiplicative (shm) map due to Sugawara [**13**] as are the formulas for the corresponding map of classifying spaces/realizations. The point is that (F_0, F_1) does not respect the identifications on the nose, but the higher homotopies and the connective tissue in the realization allow one to get around this.

Theorem 3.4. *A functor up to strong homotopy $\mathcal{C}$ to D induces a map of realizations*

$$|\mathcal{C}| \to |D|.$$

In our case, our homotopy transition cocycles can be described as a strong homotopy functor. Thus we have:

Theorem 3.5. *Given $p : E \to B$ locally homotopy trivial with respect to a covering $\{U_\alpha\}$, a choice of coherent transition functions*

$$g_\sigma : I^{n-1} \times U_\sigma \to H(F)$$

determines a map

$$B \to BH(F).$$

To complete the classification, we need the converse and uniqueness up to homotopy. As for bundles, this follows from the construction of a universal fibration.

4. Construction of fibrations

For bundles, the construction from transition functions is very easy: $E : \coprod U_\alpha \times F/g_{\alpha\beta}$, i.e., for $x \in U_\alpha \cap U_\beta$, $(x, f) \in U_\beta \times F$ is identified with $(x, g_{\alpha\beta}(x)f) \in U_\alpha \times F$. For fibrations, not only do the transition functions $\{g_{\alpha\beta}\}$ *not* give an equivalence relation on $\coprod U_\alpha \times F$ because the cocycle condition may fail, but the obvious attempt to use the mapping cylinder $M(g_{\alpha\beta})$ over $U_\alpha \cap U_\beta \times I$ may only produce a quasi-fibration [**15, 18**]. However Wirth has shown:

Theorem 4.1. *Mapping Cylinder Theorem (Wirth [**18**]) Let $\phi : E_0 \to E_1$ be a fibre homotopy equivalence over B, then there is an object $\tilde{M}(\phi)$ over $I \times B$ which serves as a mapping cylinder for ϕ, i.e., $\tilde{M}(\phi)|0 \times B$ is E_0 and $\tilde{M}(\phi)|1 \times B$ is E_1 and, moreover, there is a* characterizing *homotopy equivalence ψ from $\tilde{M}(\phi)$ to $I \times E_1$ such that $\psi|0 \times B = 0 \times \phi$ and $\psi|1 \times B = Id$.*

In fact, the ordinary mapping cylinder will do if ϕ is either a strong deformation retract or is a fibration itself in the category of spaces over B. Finally any map over B factors into a strong deformation retract over B followed by a fibration-over-B, just as an ordinary map factors into a strong deformation retract followed by an (induced from a path space) fibration. That is, ϕ can be factored as

$$E_0 \to E(\phi) \to E_1$$

where

$$E(\phi) = \{(e, \lambda)|e \in E, \lambda : I \to p_1^{-1}(p_0(e)),\ \phi(e) = \lambda(0)\}.$$

Thus given the transition functions $\{g_{\alpha\beta} : U_\alpha \times U_\beta \to H(F)\}$ for $\alpha > \beta$, we let

$$E_0 = \coprod U_\alpha \times F$$

and

$$E_1 = E_0 \cup \tilde{M}(g_{\alpha\beta})$$

with the obvious identifications. Just as we can regard Δ^n as the mapping cylinder of $\dot{\Delta}^n \to *$, so we can regard $|\mathcal{U}|$ as obtained by adding successive mapping cylinders to $\coprod U_\alpha$. Let $|\mathcal{U}|(n) \subset |\mathcal{U}|$ be the subspace represented by points with at most $(n+1)$-simplicial coordinates not equal to zero. Then $|\mathcal{U}|(n) = |\mathcal{U}|(n-1) \cup \coprod_\sigma M(\dot{\Delta}^n \times *) \times U_\sigma$ for all n-simplices σ. Thus to extend E_1 to a fibration over $|\mathcal{U}|(2)$, etc., we need to attach to E_1 something of the homotopy type of $\coprod U_{\alpha\beta\gamma} \times F$. Let $h_{\alpha\beta\gamma} : I \times U_{\alpha\beta\gamma} \times F \to U_{\alpha\beta\gamma} \times F$ be defined by $h_{\alpha\beta\gamma}(t, x, f) = \big(x, g_{\alpha\beta\gamma}(t, x)f\big)$ so that $h_{\alpha\beta\gamma}$ is a fibre homotopy equivalence over $U_{\alpha\beta\gamma}$.

Now let $E_2 = E_1 \cup \coprod_{\alpha>\beta>\gamma} I^2 \times U_{\alpha\beta\gamma} \times F$ attached by

$$\begin{aligned}(s, t, x, f) &\sim (t, x, f) && \text{if } s = 0\\ &\sim (s, x, f) && \text{if } t = 0\\ &\sim (s, x, g_{\beta\gamma}(x)f) && \text{if } t = 1\\ &\sim (x, g_{\alpha\beta\gamma}(t, x)f) && \text{if } s = 1.\end{aligned}$$

Constructed in this way, E_2 will be only a quasi-fibration in general. The proper construction is to regard the above identifications as giving a fibre homotopy equivalence $\phi_{\alpha\beta\gamma}$ of $\partial I^2 \times U_{\alpha\beta\gamma} \times F$ to $E_1|\Delta^2 \times U_{\alpha\beta\gamma}$ and then to attach Wirth's $\tilde{M}(\phi_{\alpha\beta\gamma})$ instead of $I^2 \times U_{\alpha\beta\gamma} \times F$. From Wirth's construction, there is a chracterizing fibre homotopy equivalence $\psi_{\alpha\beta\gamma} : I^2 \times U_{\alpha\beta\gamma} \times F \to \tilde{M}(\phi_{\alpha\beta\gamma}) \subset E_2$.

The general construction should now be clear. Let E_n be constructed inductively over $|\mathcal{U}|(n)$. Define a fibre homotopy equivalence

$$\psi_\sigma : \partial I^n \times U_\sigma \times F \to E_{n-1}|\partial\Delta^n \times U_\sigma$$

by

$$\begin{aligned}
\psi_\sigma(t_1, \cdots, t_n, x, f) = & \\
&= \psi_{\sigma_i}(\cdots, \hat{t}_i, \cdots, x, f) && \text{if } t_i = 0 \\
&= \psi_{\alpha_0\cdots\alpha_{i-1}}(t_1, \cdots, t_{i-1}, x, g_{\alpha_{i-1}\cdots\alpha_n}(t_{i+1}, \cdots, t_n, x, f)) && \text{if } t_i = 1
\end{aligned}$$

where $\psi_{\alpha\beta} = g_{\alpha\beta}$ and $\psi_\alpha = \mathrm{id}$. Define $E_n = E_{n-1} \cup_\sigma \tilde{M}(\phi_\sigma)$ and let

$$\psi_\sigma : I^n \times U_\sigma \times F \to \tilde{M}(\phi_\sigma) \subset E_n$$

be Wirth's characterizing fibre homotopy equivalence.

In particular, this construction applies to the universal cover $\{U_i\}$ of $BH(F)$ with transition functions

$$g_{ij}(\vec{t}, \{c_{\alpha\beta}\}) = c_{ij}$$

where defined. Notice here $g_{ij}g_{jk} = g_{jk}$ (the identity is a true functor), but it is still important to use $\tilde{M}$ in constructing the universal fibration, which is denoted UE since we shall see it is the universal example of a (WCHP) fibration with fibre F.

5. Equivalence of fibrations and transition functions

Given that fibre homotopy equivalence is the appropriate notion, we need to investigate the appropriate equivalence of transition functions. Steenrod [**12**] observes that for bundles $p^i : E^i \to B, i = 1, 2$ with respect to two coverings $\mathcal{U} = \{U^1_\alpha\}$ and $\mathcal{V} = \{V^2_\gamma\}$ with corresponding transition functions $\{g^1_{\alpha\beta}\}$ and $\{g^2_{\gamma\delta}\}$, the bundles are equivalent if and only if there exist maps $\bar{g}_{\alpha\gamma} : U_\alpha \cap V_\gamma \to G$, the group of the bundle, such that $\{g^1_{\alpha\beta}, g^2_{\gamma\delta}, \bar{g}_{\alpha\gamma}\}$ satisfies the cocycle condition. Consider $\mathcal{U} \coprod \mathcal{V} = \{U_\alpha, V_\gamma\}$. The realization $|\mathcal{U} \coprod \mathcal{V}|$ has $|\mathcal{U}|$ and $|\mathcal{V}|$ as deformation retracts, so the cocycle condition above yields a homotopy between the classifying maps between $\{g^1_{\alpha\beta}\}$ and $\{g^2_{\gamma\delta}\}$.

Similarly we define two transition functions $\{g^i_\sigma : I^{n-1} \times U^i_\sigma \to H(F)\}$, $i = 1, 2$, to be equivalent if their union extends to a transition function on $\mathcal{U} \coprod \mathcal{V}$. Thus fibre homotopy equivalent fibrations $p^i : E^i \to X$ with given fibre F yield homotopic maps $X \to BH(F)$. [If $p^i : E^i \to X$ are locally homotopy trivial with typical fibres $F^1 \simeq F^2$, then $BH(F^1) \simeq BH(F^2)$ via an equivalence induced by shm maps $H(F^1) \rightleftarrows H(F^2)$.]

Conversely, it is standard that homotopic maps induce equivalent fibrations, cf. one of the 'Axioms of a fibration theory' (see section 6). Thus we are ready to prove:

Theorem 5.1. *For a space F, fibre homotopy equivalence classes of fibrations $p : E \to X$ locally homotopy trivial with respect to numerable covers of X are in 1-1* *correspondence with homotopy classes of maps $X \to BH(F)$. The correspondence is induced by the classifying map construction above or by the pullback of UE.*

We have left to show that if $f : X \to |\mathcal{U}| \to BH(F)$ is constructed from transition functions for $p : E \to X$, then f^*UE is fibre homotopy equivalent to E and that the classifying construction for UE produces a map homotopic to the identity.

Just as $|\mathcal{U}| \to X$ is induced by forgetting simplicial coordinates, so is $f^*UE \to E$ also and, since it induces a homotopy equivalence on each fibre, is a fibre homotopy equivalence [**5**]. Strictly speaking, for UE the classifying map $BH(F) \to |\mathcal{U}| \to BH(F)$ is only homotopic to the identity since the open sets U_i are $\bar{t}_i^{-1}(0,1]$ rather than $t_i^{-1}(0,1]$.

This discussion should make it clear that a corresponding classification theorem holds for *any fibration theory* as axiomatised by Wirth [**17**].

6. Fibration theories

A *fibration theory* is an assignment of a category $\mathcal{E}(B)$ to each topological space B and of a contravariant functor $f^* : \mathcal{E}(C) \to \mathcal{E}(B)$ to each continuous map $f : B \to C$ such that id^* is the identity functor and satisfying:

Axiom I: For a numerable open cover $\{U_i\}$ of a space B and a system of objects (or morphisms) $\{E_i\}$ over each U_i such that E_i and E_j agree over $U_i \cap U_j$, then there exists a unique common extension of the E_i over B.

Axiom II: If ϕ is a morphism in $\mathcal{E}(B)$ such that each restriction $\phi|U_i$ for a numerable open cover $\{U_i\}$ of B is a homotopy equivalence, the ϕ is a homotopy equivalence.

Axiom III: If $H \in \mathcal{E}(I \times B)$, then the restrictions $H|\{t\} \times B$ are homotopy equivalent (for objects) or homotopic (for morphisms).

Axiom IV (Mapping Cylinder Axiom) If $\phi : E \to E' \in \mathcal{E}(B)$ is a homotopy equivalence, then there is an object $M(\phi) \in \mathcal{E}(I \times B)$ which serves as a *mapping cylinder* for ϕ, that is, $M(\phi)$ restricts to E at $t = 0$ and to E' at $t = 1$ with a charcterising homotopy equivalence $\psi_M : M(\phi) \to I \times E'$ which restricts to $\{0\} \times \phi$, respectively $\{1\} \times id$.

7. Foliations and Gerbes

An approach similar to that for fibrations works for generalized foliations or Haefliger structures. For a topological space X, a Haefliger q-structure $\{U_\alpha, f_\alpha, g_{\alpha\beta}\}$ consists of an open cover $\{U_\alpha\}$, maps $f_\alpha : U_\alpha \to R^q$ and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to$ Diffeo R^q such that $f_\alpha(\alpha) = g_{\alpha\beta}(x) \circ f_\beta(x)$ for $x \in U_\alpha \cap U_\beta$. These satisfy Wirth's Axiom I trivially. The other Axioms are modified by 1) replacing homotopy equivalence by diffeomorphism and 2) defining equivalence of Haefliger structures to mean being induced from a structure on $X \times I$. Thus the above method of classification applies. This emphasizes the central importance of cylinder objects and their relation to equivalence in still more general structure theories.

The essence of all that we have said is the relation of local to global via patching/glueing/recollement. The compatabilities are sometimes referred to as *descent* *data*, whether in our naive topological setting or more generally for e.g. topoi. For example, a *gerbe* can be specified by descent data in terms of an open cover $\{U_\alpha\}$ and a groupoid $\mathcal{G}_\alpha$ for each U_α with 'transition' morphisms

$$g_{\alpha\beta} \in \mathcal{G}_{\alpha\beta}$$

and morphisms

$$c_{\alpha\beta\gamma} \in \mathcal{G}_{\alpha\beta\gamma}$$

acting by conjugation so that

$$g_{\alpha\beta}g_{\beta\gamma} = Ad(c_{\alpha\beta\gamma})g_{\alpha\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma$$

for an inverible element $c_{\alpha\beta\gamma}$. The role of the homotopy $g_{\alpha\beta\gamma}$ is played by conjugation with $c_{\alpha\beta\gamma}$ and such conjugation in a group corresponds to a homotopy at the classifying space level. The higher homotopies on further multiple intersections do not appear for gerbes since the $c_{\alpha\beta\gamma}$ satisfy a strict coherence condition.

However, as Breen suggests, a *2-gerbe* can be specified by descent data requiring coherence at a higher level. There's work to be done 'pursueing stacks' and perhaps more exotic objects.

References

[1] John C. Baez and Urs Schreiber. Higher gauge theory. preprint, 2005. `math.DG/0511710`.

[2] Lawrence Breen. Théorie de Schreier supérieure. *Ann. Sci. École Norm. Sup. (4)*, 25(5):465–514, 1992.

[3] Lawrence Breen. On the classification of 2-gerbes and 2-stacks. *Astérisque*, 225:160, 1994.

[4] Jean-Marc Cordier and Timothy Porter. Homotopy coherent category theory. *Trans. Amer. Math. Soc.*, 349(1):1–54, 1997.

[5] Albrecht Dold. Partitions of unity in the theory of fibrations. *Ann. of Math. (2)*, 78:223–255, 1963.

[6] Martin Fuchs. A modified Dold-Lashof construction that does classify H-principal fibrations. *Math. Ann.*, 192:328–340, 1971.

[7] André Hirschowitz and Carlos Simpson. Descente pour les n-champs (descent for n-stacks). preprint, 1998. `math.AG/9807049` .

[8] John Milnor. Construction of universal bundles. II. *Ann. of Math. (2)*, 63:430–436, 1956.

[9] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, 34:105–112, 1968.

[10] James D. Stasheff. Parallel transport and classification of fibrations. In *Algebraic and geometrical methods in topology (Conf. Topological Methods in Algebraic Topology, State Univ. New York, Binghamton, N.Y., 1973)*, pages 1–17. Lecture Notes in Math., Vol. 428. Springer, Berlin, 1974.

[11] James Dillon Stasheff. "Parallel" transport in fibre spaces. *Bol. Soc. Mat. Mexicana (2)*, 11:68–84, 1966.

[12] Norman Steenrod. *The topology of fibre bundles.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks.

[13] M Sugawara. On the homotopy-commutativity of groups and loop spaces. *Mem. Coll. Sci. Univ. Kyoto, Ser. A Math.*, 33:257–269, 1960/61.

[14] Tammo tom Dieck. Klassifikation numerierbarer Bündel. *Arch. Math. (Basel)*, 17:395–399, 1966.

[15] Patricia Tulley. A strong homotopy equivalence and extensions for Hurewicz fibrations. *Duke Math. J.*, 36:609–619, 1969.

[16] Heinz Wellen. Projektive Diagramme über topologischen kategorien, 1990. Diplomarbeit.

[17] James F. Wirth. *Fiber spaces and the higher homotopy cocycle relations.* PhD thesis, University of Notre Dame, 1965.

[18] James F. Wirth. The mapping cylinder axiom for WCHP fibrations. *Pacific J. Math.*, 54(2):263–279, 1974.

This article may be accessed via WWW at `http://jhrs.rmi.acnet.ge`

James Wirth
`wirthj@mail.ecu.edu`

Computer Science Dept.
East Carolina University
Greenville, NC 27858-4353

Jim Stasheff
`jds@math.upenn.edu`

Math Dept.
University of Pennsylvania,
Philadelphia PA, 19104-6395