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HOMOTOPY TYPES OF ORBIT SPACES AND THEIR SELF-EQUIVALENCES FOR THE PERIODIC GROUPS $\mathbb{Z}/a \rtimes (\mathbb{Z}/b \times T_n^\star)$ AND $\mathbb{Z}/a \rtimes (\mathbb{Z}/b \times O_n^\star)$

MAREK GOLASIŃSKI AND DACIBERG LIMA GONÇALVES

(*communicated by Lionel Schwartz*)

Abstract

Let G be a finite group given in one of the forms listed in the title with period $2d$ and $X(n)$ an n-dimensional CW-complex with the homotopy type of an n-sphere.

We study the automorphism group $\mathrm{Aut}\,(G)$ to compute the number of distinct homotopy types of orbit spaces $X(2dn-1)/\mu$ with respect to free and cellular G-actions μ on all CW-complexes $X(2dn-1)$. At the end, the groups $\mathcal{E}(X(2dn-1)/\mu)$ of self homotopy equivalences of orbit spaces $X(2dn-1)/\mu$ associated with free and cellular G-actions μ on $X(2dn-1)$ are determined.

Introduction.

Given a free and cellular action μ of a finite group G with order $|G|$ on a CW-complex X, write X/μ for the corresponding orbit space. The problem of determining all possible homotopy types of X/μ among all free and cellular actions μ on X, as well the group $\mathcal{E}(X/\mu)$ of self homotopy equivalences of X/μ has been extensively studied for a number of spaces e.g., in [**9**]. Notoriously, for an odd dimensional sphere $\mathbb{S}^{2n-1}$ with a free action of a finite cyclic group $\mathbb{Z}/k$ this corresponds to the classification of lens spaces and the calculation of the groups of it self homotopy equivalences studied in [**4**]. A larger family of interesting examples are given by a free and cellular action of a finite group G with order $|G|$ on a CW-complex $X(2n-1)$ with the homotopy type of a $(2n-1)$-sphere. Write $X(2n-1)/\mu$ for the corresponding orbit space called a $(2n-1)$-*spherical space form* or a *Swan*

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$(2n-1)$-*complex* (see e.g., [**2**]). Taking into account [**10**], the case of spherical space forms presents a special interest. Furthermore, Swan [**11**] has shown that any finite group with periodic cohomology of period $2d$ acts freely and cellularly on a $(2d-1)$-dimensional CW-complex of the homotopy type of a $(2d-1)$-sphere. It is worth to mention that useful cohomological and geometric aspects associated to group actions are presented in [**2**] and a list of basic conjectures is provided.

Backing to the case of a $(2n-1)$-dimensional CW-complex $X(2n-1)$ with the homotopy type of a $(2n-1)$-sphere, by means of results in [**11**], it is shown in [**12**, Theorem 1.8] that the set of homotopy types of spherical space forms of all free cellular G-actions on $X(2n-1)$ is in one-to-one correspondence with the orbits, which contain a generator of the cyclic group $H^{2n}(G) = \mathbb{Z}/|G|$ under the action of $\pm\mathrm{Aut}\,(G)$ (see [**4**] for another approach). This plays also a fundamental role in the calculation of the group $\mathcal{E}(X(2n-1)/\mu)$ of self homotopy equivalences of the orbit space $X(2n-1)/\mu$.

All finite periodic groups has been completely described by Suzuki-Zassenhaus and their classification can be found in the table [**1**, Chapter IV; Theorem 6.15]. The present paper is part of the project to describe the homotopy types of the orbit spaces and the group of self homotopy equivalences for all periodic groups. It continues the works of [**4, 5, 6, 8**], where the cases corresponding to the families I and II from the table [**1**, Chapter IV; Theorem 6.15] with the Suzuki-Zassenhaus classification of finite periodic groups have been solved. Here we have two goals. The first one is to calculate the numbers of homotopy types of spherical spaces forms for the groups $\mathbb{Z}/a \rtimes (\mathbb{Z} \times T_n^\star)$ and $\mathbb{Z}/a \rtimes (\mathbb{Z} \times O_n^\star)$ corresponding to the families III and IV from the table mentioned above. The second one is to determine the group of homotopy classes of self-equivalences for space forms given by free actions of those both families of finite periodic groups. The results of [**4, 5, 6, 8**], taking care for the groups from families I and II of that table, are essential to make crucial calculations to develop the main results stated in Theorem 2.2 and Theorem 3.2.

In order to obtain these results, we divide the paper into two parts. The first part consists of some algebraic results. The automorphism group $\mathrm{Aut}(A \rtimes_\alpha G)$ of a semi-direct product $A \rtimes_\alpha G$ of some finite groups A, G leads in [**6**] to a splitting short exact sequence

$$0 \to \mathrm{Der}_\alpha\,(G, A) \longrightarrow \mathrm{Aut}\,(A \rtimes_\alpha G) \longrightarrow \mathrm{Aut}\,(A) \times \mathrm{Aut}_\alpha(G) \to 1.$$

Section 1 makes use of this to achieve automorphisms of the groups in question. This is the approach to develop in Proposition 1.1 and Proposition 1.2 the groups $\mathrm{Der}_\alpha\,(G, A)$ and $\mathrm{Aut}\,(A) \times \mathrm{Aut}_\alpha(G)$, respectively.

Then, in the second part, we present geometric interpretations of those algebraic results in terms of G-actions. Section 2 uses the group $\mathrm{Aut}\,(A) \times \mathrm{Aut}_\alpha(G)$ established in Proposition 1.2 and Lyndon-Hochschild-Serre spectral sequence to deal with the number of homotopy types of spherical space forms for actions of the groups $\mathbb{Z}/a \rtimes (\mathbb{Z}/b \times T_n^\star)$ and $\mathbb{Z}/a \rtimes (\mathbb{Z}/b \times O_n^\star)$. The main results of this section are stated in

Theorem 2.2. *Let $\gamma = (\gamma_1, \gamma_2) : \mathbb{Z}/b \times T_n^\star \to (\mathbb{Z}/a)^\star$ and $\tau = (\tau_1, \tau_2) : \mathbb{Z}/b \times O_n^\star \to (\mathbb{Z}/a)^\star$ be actions with $(a, b) = (ab, 6) = 1$ and $n \geqslant 3$, where $\gamma_1 : \mathbb{Z}/b \to (\mathbb{Z}/a)^\star$, $\gamma_2 : T_n^\star \to (\mathbb{Z}/a)^\star$ and $\tau_1 : \mathbb{Z}/b \to (\mathbb{Z}/a)^\star$, $\tau_2 : O_n^\star \to (\mathbb{Z}/a)^\star$ are appropriate* *restrictions of γ and τ, respectively. Then*:

(1) card $\mathcal{K}^{2k[\ell(\gamma),2]-1}_{\mathbb{Z}/a\rtimes_\gamma(\mathbb{Z}/b\times T^\star_n)}/\simeq \;=\; 2^{t+t'+1}3^{n_0}O(a,k[\ell(\gamma),2])O_{Aut_{\gamma_1}(\mathbb{Z}/b)}(b,k[\ell(\gamma),2])$ $O(3^{n-n_0},\, k[\ell(\gamma),2])^{-1}$ *for some* $0\leqslant t\leqslant 2$ *and* $0\leqslant t'\leqslant 1$;

(2) card $\mathcal{K}^{2k[\ell(\tau),2]-1}_{\mathbb{Z}/a\rtimes_\tau(\mathbb{Z}/b\times O^\star_n)}/\simeq \;=\; 2^{t+1}\times 3^{n-1}O(a,k[\ell(\tau),2])O_{Aut_{\tau_1}(\mathbb{Z}/b)}(b,k[\ell(\tau),2])$ *for some* $0\leqslant t\leqslant 1$.

Then, Corollary 2.3 says that the number of such homotopy types of those space forms coincides with that of $(4n-1)$-lens spaces studied in [**4**] provided the least period of the groups in question is $\leqslant 4$.

The group of crossed homomorphisms $\mathrm{Der}_\alpha(G,A)$ studied in Proposition 1.1 plays a key role in Section 3 dealing with the structure of groups $\mathcal{E}(X(2dn-1)/\mu)$ of self homotopy equivalences for spherical space forms $X(2dn-1)/\mu$ with respect to free and cellular $\mathbb{Z}/a\rtimes(\mathbb{Z}/b\times T^\star_n)$– and $\mathbb{Z}/a\rtimes(\mathbb{Z}/b\times O^\star_n)$–actions μ, respectively. We point out that by means of [**4**, Proposition 3.1] (see also [**10**, Theorem 1.4]), the group $\mathcal{E}(X(2k-1)/\mu)$ is independent of the action μ on $X(2k-1)$. Writing $X(2k-1)/G$ for the corresponding orbit space, we close the paper with

Theorem 3.2. *Let* $\gamma=(\gamma_1,\gamma_2):\mathbb{Z}/b\times T^\star_n\to(\mathbb{Z}/a)^\star$ *(resp.* $\tau=(\tau_1,\tau_2):\mathbb{Z}/b\times O^\star_n\to(\mathbb{Z}/a)^\star$*) be an action with* $(a,b)=(ab,6)=1$ *for* $n\geqslant 3$*. If the group* $\mathbb{Z}/a\rtimes_\gamma(\mathbb{Z}/b\times T^\star_n)$ *(resp.* $\mathbb{Z}/a\rtimes_\tau(\mathbb{Z}/b\times O^\star_n)$*) acts freely and cellularly on a CW-complex* $X(2k[\ell(\gamma),2]-1)$ *(resp.* $X(2k[\ell(\tau),2]-1)$*) then*

$$\begin{gathered}\mathcal{E}(X(2k[\ell(\gamma),2]-1)/(\mathbb{Z}/a\rtimes_\gamma(\mathbb{Z}/b\times T^\star_n)))\cong \mathrm{Der}_\gamma(\mathbb{Z}/b\times T^\star_n,\mathbb{Z}/a)\rtimes\\ (\mathcal{E}(X(2k[\ell(\gamma_1),2]-1)/(\mathbb{Z}/a))\times\mathcal{E}_{\gamma_1}(X(2k[\ell(\beta),2]-1)/(\mathbb{Z}/b))\times S_4\times\\ \mathbb{Z}/\left(\frac{3^{n-n_0}}{(3^{n-n_0},k[\ell(\gamma),2])}\right)\end{gathered}$$

$$\begin{gathered}(\mathit{resp.}\ \mathcal{E}(X(2k[\ell(\tau),2]-1)/(\mathbb{Z}/a\rtimes_\tau(\mathbb{Z}/b\times O^\star_n)))\cong \mathrm{Der}_\tau(\mathbb{Z}/b\times O^\star_n,\mathbb{Z}/a)\rtimes\\ (\mathcal{E}(X(2k[\ell(\tau_1),2]-1)/(\mathbb{Z}/a))\times\mathcal{E}_{\tau_1}(X(2k[\ell(\beta),2]-1)/(\mathbb{Z}/b))\times O_n\rtimes\\ \mathbb{Z}/\left(\frac{3^{n-1}}{(3^{n-1},k[\ell(\tau),2])}\right))\end{gathered}$$

which deals with explicit formulae for those groups of self homotopy equivalences.

Approaching of homotopy types of spherical space forms and their self homotopy equivalences for the rest of the groups from the table in [**1**, Chapter IV; Theorem 6.15], or more precisely for the family of VI of this table, is in progress.

1. Algebraic backgrounds.

Let a finite group G be given by an extension

$$1\to G_1\to G\to G_2\to 1,$$

where the orders of groups G_1 and G_2 are relatively prime. We recall that by [**7**] any automorphism of G leaves the subgroup G_1 invariant and consequently, there is a map $\psi:\mathrm{Aut}(G)\to\mathrm{Aut}(G_1)\times\mathrm{Aut}(G_2)$ of automorphism groups. Given an

H-action $\alpha : H \to \mathrm{Aut}\,(A)$ on an abelian group A write $\mathrm{Der}_\alpha\,(H, A)$ for the abelian group of crossed homomorphisms. For H-actions $\alpha_1 : H \to \mathrm{Aut}\,(A_1)$ and $\alpha_2 : H \to \mathrm{Aut}\,(A_2)$ consider the obvious induced action $(\alpha_1, \alpha_2) : H \to \mathrm{Aut}\,(A_1 \times A_2)$. Then, an isomorphism

$$(\star) \qquad \mathrm{Der}_{(\alpha_1,\alpha_2)}\,(H, A_1 \times A_2) \xrightarrow{\cong} \mathrm{Der}_{\alpha_1}\,(H, A_1) \times \mathrm{Der}_{\alpha_2}\,(H, A_2)$$

follows.

Now, let $0 \to A \to G \to H \to 1$ be a short exact sequence, with A an abelian group. Then, there is an obvious H-action $\alpha : H \to \mathrm{Aut}\,(A)$. If groups A and H are finite with relatively prime orders then the cohomology group $H^1(H, A)$ vanishes (see e.g., [**1**, Corollary 5.4]) and consequently, $\mathrm{Der}_\alpha\,(H, A) = A/A^H$, where A^H is the subgroup of A consisting of all elements fixed under the action of H. Furthermore, by [**6**, Lemma 1.2] this sequence $0 \to A \to G \to H \to 1$ of finite groups yields the exact sequence

$$0 \to \mathrm{Der}_\alpha\,(H, A) \to \mathrm{Aut}(G) \xrightarrow{\psi} \mathrm{Aut}\,(A) \times \mathrm{Aut}\,(H).$$

For an action $\alpha : G \to \mathrm{Aut}\,(A)$, let $A \rtimes_\alpha G$ denote the semi-direct product of A and G with respect to the action α. Let the orders of A and G be relatively primes and $\psi : \mathrm{Aut}(A \rtimes_\alpha G) \to \mathrm{Aut}\,(A) \times \mathrm{Aut}(G)$ be the obvious map. Then, by [**6**], $\mathrm{Im}\,\psi = \mathrm{Aut}\,(A) \times \mathrm{Aut}_\alpha(G)$, where $\varphi \in \mathrm{Aut}_\alpha(G)$ if and only if $\alpha = \alpha\varphi$ or equivalently

$$\mathrm{Aut}_\alpha(G) = \{\varphi \in \mathrm{Aut}\,(G);\ \varphi(\mathrm{Ker}\alpha) = \mathrm{Ker}\,\alpha \text{ and } \bar{\varphi} = \mathrm{id}_{G/\mathrm{Ker}\,\alpha}\},$$

where $\bar{\varphi}$ denotes the map induced by φ on the quotient group $G/\mathrm{Ker}\,\alpha$.

Now, let Q_8 be the classical quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order 8, where $1, i, j$ and k are generators of the quaternion algebra over reals. Consider the action $\alpha : \mathbb{Z}/3 \to \mathrm{Aut}\,(Q_8)$ such that a generator of $\mathbb{Z}/3$ is sent to the automorphism $\tau \in \mathrm{Aut}\,(Q_8)$ defined by: $\tau(i) = j$, $\tau(j) = k$ and $\tau(k) = i$. Since $\mathrm{Aut}\,(Q_8) \cong S_4$, the symmetric group on four letters (see e.g., [**1**, Lemma 6.9]) any two faithful representations of $\mathbb{Z}/3$ in the group Q_8 are conjugated. Whence, without losing generality, we can choose the action α given above. Then, we consider the semi-direct product $Q_8 \rtimes_\alpha \mathbb{Z}/3 = T^\star$, the *binary tetrahedral* group. More generally, for $n \geqslant 1$ consider the action $\alpha_n : \mathbb{Z}/3^n \to \mathrm{Aut}\,(Q_8)$ as the composition of the quotient map $\mathbb{Z}/3^n \to \mathbb{Z}/3$ with the action $\alpha : \mathbb{Z}/3 \to \mathrm{Aut}\,(Q_8)$. Then, for the group

$$T^\star_n = Q_8 \rtimes_{\alpha_n} \mathbb{Z}/3^n,$$

by means of [**13**, p. 198], it holds

$$T^\star_n : \begin{cases} X^{3^n} = P^4 = 1,\ P^2 = Q^2,\ XPX^{-1} = Q, \\ XQX^{-1} = PQ,\ PQP^{-1} = Q^{-1} \end{cases}$$

in virtue of generators and relations. In particular, the cyclic group $\mathbb{Z}/3^n$ is the abelianization of $T^\star_n$ for any $n \geqslant 1$ and the center $\mathcal{Z}(T^\star_n) = \mathbb{Z}/2 \oplus \mathbb{Z}/3^{n-1}$.

The symmetric group S_3 has two distinct extensions by Q_8, with respect to the outer action $\alpha : S_3 \to \mathrm{Out}\,(Q_8) = \mathrm{Aut}\,(Q_8)/\mathrm{Inn}\,(Q_8)$ which is the composition of the inclusion $S_3 \subseteq \mathrm{Aut}\,(Q_8)$ with the projection $\mathrm{Aut}\,(Q_8) \to \mathrm{Out}(Q_8)$. This follows from the facts that $\mathcal{Z}(Q_8) = \mathbb{Z}/2$ and $H^2(S_3, \mathbb{Z}/2) = \mathbb{Z}/2$. These extensions are the semi-direct product $Q_8 \rtimes S_3$ and

$$1 \to Q_8 \to O^\star \xrightarrow{\varphi} S_3 \to 1,$$

where $O^\star$ is the *binary octahedral* group. Because $\varphi^{-1}(A_3) = T^\star$ for the alternating subgroup $A_3 \subseteq S_3$, so we achieve the extension

$$1 \to T^\star \to O^\star \to \mathbb{Z}/2 \to 1.$$

In general, since $\mathcal{Z}(T_n^\star) = \mathbb{Z}/2 \oplus \mathbb{Z}/3^{n-1}$ and $H^2(S_3, \mathcal{Z}(T_n^\star)) = H^2(S_3, \mathbb{Z}/2 \oplus \mathbb{Z}/3^{n-1}) = H^2(S_3, \mathbb{Z}/2) = \mathbb{Z}/2$, we achieve the non-trivial extension

$$1 \to T_{n-1}^\star \to O_n^\star \xrightarrow{\varphi_n} S_3 \to 1$$

for $n \geqslant 1$, where $T_0^\star = Q_8$. Because $\varphi_n^{-1}(A_3) = T_n^\star$ a *fortiori* the new extension

$$1 \to T_n^\star \to O_n^\star \to \mathbb{Z}/2 \to 1$$

is obtained.

In the light of [**13**, p. 198] the group $O_n^\star$ is given by

$$O_n^\star : \begin{cases} X^{3^n} = P^4 = 1,\ P^2 = Q^2 = R^2,\ PQP^{-1} = Q^{-1}, \\ XPX^{-1} = Q,\ XQX^{-1} = PQ,\ RXR^{-1} = X^{-1}, \\ RPR^{-1} = QP,\ RQR^{-1} = Q^{-1} \end{cases}$$

in virtue of generators and relations. It follows that the cyclic group $\mathbb{Z}/2$ is isomorphic to the abelianization of $O_n^\star$ and the center $\mathcal{Z}(O_n^\star)$ as well.

Now, consider the periodic groups $\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)$ and $\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)$ corresponding to the families III and IV [**1**, Theorem 6.15] with $(a, b) = (ab, 6) = 1$ and $n \geqslant 1$, where $\gamma : \mathbb{Z}/b \times T_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ and $\tau : \mathbb{Z}/b \times O_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ are actions of $\mathbb{Z}/b \times T_n^\star$ and $\mathbb{Z}/b \times O_n^\star$, respectively, on the cyclic group $\mathbb{Z}/a$. The group $\mathrm{Aut}\,(\mathbb{Z}/a)$ is abelian, a *fortiori* the actions γ and τ are uniquely determined by their restrictions $\gamma_1 : \mathbb{Z}/b \to \mathrm{Aut}\,(\mathbb{Z}/a)$, $\gamma_2 : T_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ and $\tau_1 : \mathbb{Z}/b \to \mathrm{Aut}\,(\mathbb{Z}/a)$, $\tau_2 : O_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$. But the abelianizations of $T_n^\star$ and $O_n^\star$ are isomorphic to the groups $\mathbb{Z}/3^n$ and $\mathbb{Z}/2$, respectively. Whence, the actions γ_2 and τ_2 are uniquely determined by $\gamma_2(X)$ and $\tau_2(R)$, respectively, with $\gamma_2(X)^{3^n} = \mathrm{id}_{\mathbb{Z}/a}$ and $\tau_2(R)^2 = \mathrm{id}_{\mathbb{Z}/a}$.

To study the groups $\mathrm{Aut}\,(\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star))$ and $\mathrm{Aut}\,(\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times O_n^\star))$, we need, in the light of [**6**, Proposition 1.3], to describe the groups $\mathrm{Der}_\gamma\,(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/a)$, $\mathrm{Aut}_\gamma\,(\mathbb{Z}/b \times T_n^\star)$ and $\mathrm{Der}_\tau\,(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/a)$, $\mathrm{Aut}_\tau\,(\mathbb{Z}/b \times O_n^\star)$, respectively.

First, let $a = p^k$ with $p \neq 2, 3$ prime and $k \geqslant 0$. Because the actions γ_2 and τ_2 factor through the abelianizations of $T_n^\star$ and $O_n^\star$, which are isomorphic to the groups $\mathbb{Z}/3^n$ and $\mathbb{Z}/2$, respectively, whence $\mathrm{Ker}\,\gamma_2$ is trivial or $\mathrm{Ker}\,\gamma_2 = Q_8 \rtimes_{\alpha_n} \mathbb{Z}/3^{n_0}$ for some $n_0 \leqslant n$ and $\mathrm{Ker}\,\tau_2$ is trivial or equals $T_n^\star$ (as a subgroup with index two in $O_n^\star$ and containing $T_n^\star$). Consequently, by [**6**], we achieve that $\mathrm{Der}_\gamma\,(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p^n) = \mathrm{Der}_{\gamma_1}\,(\mathbb{Z}/b, \mathbb{Z}/p^n)$ provided γ_2 is trivial and $\mathrm{Der}_\gamma\,(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p^k) = \mathrm{Der}_{\bar\gamma}\,(\mathbb{Z}/b \times \mathbb{Z}/3^{n-n_0}, \mathbb{Z}/p^n)$ provided $\mathrm{Ker}\,\gamma_2 = Q_8 \rtimes_{\alpha_n} \mathbb{Z}/3^{n_0}$, where $\bar\gamma : \mathbb{Z}/b \times \mathbb{Z}/3^{n-n_0} = \mathbb{Z}/b \times (T_n^\star/\mathrm{Ker}\,\gamma_2) \to \mathrm{Aut}\,(\mathbb{Z}/a)$ is the action induced by γ. Furthermore, $\mathrm{Der}_\tau\,(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/p^n) = \mathrm{Der}_{\tau_1}\,(\mathbb{Z}/b, \mathbb{Z}/p^n)$ provided τ_2 is trivial and $\mathrm{Der}_\tau\,(\mathbb{Z}/b \times$

$O_n^\star, \mathbb{Z}/p^k) = \mathrm{Der}_{\bar\tau}(\mathbb{Z}/b \times \mathbb{Z}/2, \mathbb{Z}/p^n)$ provided $\mathrm{Ker}\,\tau_2 = T_n^\star$, where $\bar\tau : \mathbb{Z}/b \times \mathbb{Z}/2 = \mathbb{Z}/b \times O_n^\star/T_n^\star \to \mathrm{Aut}(\mathbb{Z}/a)$ is the action induced by τ. Because $(b, 6) = 1$, the groups $\mathbb{Z}/b \times \mathbb{Z}/3^{n-n_0}$ and $\mathbb{Z}/b \times \mathbb{Z}/2$ are cyclic whence, as in [**6**, Corollary 1.5], elements of $\mathrm{Der}_{\bar\tau}(\mathbb{Z}/b \times \mathbb{Z}/3^{n-n_0}, \mathbb{Z}/p^n)$ and $\mathrm{Der}_{\bar\tau}(\mathbb{Z}/b \times \mathbb{Z}/2, \mathbb{Z}/p^n)$ might be described by means of some elements in $\mathbb{Z}/p^n$.

Now, if a is a positive integer with $(a, 6) = 1$ and $a = p_1^{k_1} \cdots p_s^{k_s}$ its prime factorization with $k_i \geqslant 1$ then $p_i \neq 2, 3$ for all $i = 1, \dots, s$. Obviously, any isomorphism $\mathbb{Z}/a \xrightarrow{\cong} \mathbb{Z}/p_1^{n_1} \times \cdots \times \mathbb{Z}/p_s^{n_s}$ yields isomorphisms $\alpha : \mathrm{Aut}(\mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Aut}(\mathbb{Z}/p_1^{n_1} \times \cdots \times \mathbb{Z}/p_s^{n_s})$ and

$$\mathrm{Der}_\gamma(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Der}_{\alpha\gamma}(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p_1^{k_1} \times \cdots \times \mathbb{Z}/p_s^{k_s})$$

for an action $\gamma : \mathbb{Z}/b \times T_n^\star \to \mathrm{Aut}(\mathbb{Z}/a)$, and

$$\mathrm{Der}_\tau(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Der}_{\alpha\tau}(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/p_1^{k_1} \times \cdots \times \mathbb{Z}/p_s^{k_s})$$

for an action $\tau : \mathbb{Z}/b \times O_n^\star \to \mathrm{Aut}(\mathbb{Z}/a)$. Then, the well-known (see e.g. [**6**, Lemma 1.1]) isomorphism $\mathrm{Aut}(\mathbb{Z}/p_1^{k_1} \times \cdots \times \mathbb{Z}/p_s^{k_s}) \xrightarrow{\cong} \mathrm{Aut}(\mathbb{Z}/p_1^{k_1}) \times \cdots \times \mathrm{Aut}(\mathbb{Z}/p_s^{k_s})$ and $(\star)$ lead to isomorphisms

$$\mathrm{Der}_\gamma(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Der}_{\alpha_1\gamma}(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p_1^{k_1}) \times \cdots \times \mathrm{Der}_{\alpha_s\gamma}(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p_s^{n_s})$$

and

$$\mathrm{Der}_\tau(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Der}_{\alpha_1\tau}(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/p_1^{k_1}) \times \cdots \times \mathrm{Der}_{\alpha_s\tau}(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/p_s^{n_s}),$$

where α_i is the composition of α with an appropriate projection map $\mathrm{Aut}(\mathbb{Z}/p_1^{k_1}) \times \cdots \times \mathrm{Aut}(\mathbb{Z}/p_s^{k_s}) \to \mathrm{Aut}(\mathbb{Z}/p_i^{k_i})$ for $i = 1, \dots, s$. Thus, we may summarize the discussion above as follows.

Proposition 1.1. *Let $\mathbb{Z}/b$ and $\mathbb{Z}/p^k$ be cyclic groups with p prime and $k \geqslant 1$, $(bp^k, 6) = (b, p^k) = 1$ and let $\gamma : \mathbb{Z}/b \times T_n^\star \to \mathrm{Aut}(\mathbb{Z}/p^k)$, $\tau : \mathbb{Z}/b \times O_n^\star \to \mathrm{Aut}(\mathbb{Z}/p^k)$ be actions. Write $\gamma_1 : \mathbb{Z}/b \to \mathrm{Aut}(\mathbb{Z}/p^k)$, $\gamma_2 : T_n^\star \to \mathrm{Aut}(\mathbb{Z}/p^k)$ and $\tau_1 : \mathbb{Z}/b \to \mathrm{Aut}(\mathbb{Z}/p^k)$, $\tau_2 : O_n^\star \to \mathrm{Aut}(\mathbb{Z}/p^k)$ for the appropriate restrictions of γ and τ, respectively. Then:*

(1)
$$\mathrm{Der}_\gamma(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p^k) \cong \mathrm{Der}_{\gamma_1}(\mathbb{Z}/b, \mathbb{Z}/p^k)$$

and

$$\mathrm{Der}_\tau(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/p^k) \cong \mathrm{Der}_{\tau_1}(\mathbb{Z}/b, \mathbb{Z}/p^k)$$

if γ_2 and τ_2 are trivial;

(2) $\mathrm{Der}_\gamma(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p^k) \cong \mathrm{Der}_{\bar\gamma}(\mathbb{Z}/b \times \mathbb{Z}/3^{n-n_0}, \mathbb{Z}/p^k)$ *provided* $\mathrm{Ker}\,\gamma_2 = Q_8 \rtimes_{\alpha_n} \mathbb{Z}/3^{n_0}$, *where $\bar\gamma : \mathbb{Z}/b \times \mathbb{Z}/3^{n-n_0} \cong \mathbb{Z}/b \times (T_n^\star/\mathrm{Ker}\,\gamma_2) \to \mathrm{Aut}(\mathbb{Z}/p^k)$ is the action induced by γ*
and
$\mathrm{Der}_\tau(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/p^k) \cong \mathrm{Der}_{\bar\tau}(\mathbb{Z}/b \times \mathbb{Z}/2, \mathbb{Z}/p^k)$ *provided* $\mathrm{Ker}\,\tau_2 = T_n^\star$, *where $\bar\tau : \mathbb{Z}/b \times \mathbb{Z}/2 \cong \mathbb{Z}/b \times (O_n^\star/T_n^\star) \to \mathrm{Aut}(\mathbb{Z}/p^k)$ is the action induced by τ.*

If $\gamma : \mathbb{Z}/b \times T_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ *and* $\tau : \mathbb{Z}/b \times O_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ *are actions with* $(a,b) = (ab,6) = 1$ *and* $a = p_1^{k_1}\cdots p_s^{k_s}$ *is the prime factorization of* a *with* $k_i \geqslant 1$ *for* $i = 1,\dots,s$ *then*

$$\mathrm{Der}_\gamma\,(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Der}_{\alpha_1\gamma}(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p_1^{k_1}) \times \cdots$$
$$\times \mathrm{Der}_{\alpha_s\gamma}\,(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p_s^{n_s})$$

and

$$\mathrm{Der}_\tau\,(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Der}_{\alpha_1\tau}(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/p_1^{k_1}) \times \cdots$$
$$\times \mathrm{Der}_{\alpha_s\tau}\,(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/p_s^{n_s}),$$

where α_i *is the composition of an isomorphism* $\alpha : \mathrm{Aut}\,(\mathbb{Z}/a) \xrightarrow{\cong} \mathrm{Aut}\,(\mathbb{Z}/p_1^{k_1} \times \cdots \times \mathbb{Z}/p_s^{n_s})$ *with an appropriate projection map* $\mathrm{Aut}\,(\mathbb{Z}/p_1^{k_1}) \times \cdots \times \mathrm{Aut}\,(\mathbb{Z}/p_s^{n_s}) \to \mathrm{Aut}\,(\mathbb{Z}/p_i^{k_i})$ *for* $i = 1,\dots,s$.

Now, move to the groups $\mathrm{Aut}_\gamma(\mathbb{Z}/b \times T_n^\star)$ and $\mathrm{Aut}_\tau(\mathbb{Z}/b \times O_n^\star)$, where $\gamma = (\gamma_1,\gamma_2) : \mathbb{Z}/b \times T_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ and $\tau = (\tau_1,\tau_2) : \mathbb{Z}/b \times O_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$. Because $(ab,6) = 1$, [**6**, Lemma 1.1] yields $\mathrm{Aut}\,(\mathbb{Z}/b \times T_n^\star) \cong \mathrm{Aut}\,(\mathbb{Z}/b) \times \mathrm{Aut}\,(T_n^\star)$ and $\mathrm{Aut}\,(\mathbb{Z}/b \times O_n^\star) \cong \mathrm{Aut}\,(\mathbb{Z}/b) \times \mathrm{Aut}\,(O_n^\star)$. Furthermore, the groups $\mathrm{Aut}\,(T_n^\star)$ and $\mathrm{Aut}\,(O_n^\star)$ have been fully described in [**7**] for all $n \geqslant 1$. In the light of [**6**, Corollary 1.4] we achieve isomorphisms

$$\mathrm{Aut}_\gamma\,(\mathbb{Z}/b \times T_n^\star) \xrightarrow{\cong} \mathrm{Aut}_{\gamma_1}\,(\mathbb{Z}/b) \times \mathrm{Aut}_{\gamma_2}\,(T_n^\star)$$

and

$$\mathrm{Aut}_\tau\,(\mathbb{Z}/b \times O_n^\star) \xrightarrow{\cong} \mathrm{Aut}_{\tau_1}\,(\mathbb{Z}/b) \times \mathrm{Aut}_{\tau_2}\,(O_n^\star).$$

But $\varphi \in \mathrm{Aut}_{\gamma_2}(T_n^\star)$ (resp. $\varphi \in \mathrm{Aut}_{\tau_2}(O_n^\star)$) if and only if $\gamma_2(X) = (\gamma_2\varphi)(X)$ (resp. $\tau_2(R) = (\tau_2\varphi)(R)$). Now, if $\mathrm{Ker}\,\gamma_2 = Q_8 \rtimes_{\alpha_n} \mathbb{Z}/3^{n_0}$ then, from the list of elements in $\mathrm{Aut}\,(T_n^\star)$ presented in [**7**], it follows that

$$\mathrm{Aut}_{\gamma_2}(T_n^\star) = \{\varphi \in \mathrm{Aut}\,(T_n^\star);\ \varphi(X) = X^{l(1+3^{n_0+1})} \text{ for } l = 0,\dots,3^{n-n_0-1}\}.$$

By means of [**7**], any $\varphi \in \mathrm{Aut}\,(O_n^\star)$ restricts to an automorphism of $T_n^\star$ with the identity on the quotient $O_n^\star/T_n^\star = \mathbb{Z}/2$ *a fortiori* $\tau_2(R) = (\tau_2\varphi)(R)$ holds for all $\varphi \in \mathrm{Aut}\,(O_n^\star)$. Now, in virtue of [**7**, Proposition 3.2], we are ready to close this section with

Proposition 1.2. *Let* $\mathbb{Z}/a$ *and* $\mathbb{Z}/b$ *with* $(a,b) = (ab,6) = 1$ *and* $\gamma : \mathbb{Z}/b \times T_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$, $\tau : \mathbb{Z}/b \times O_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ *be actions. Write* $\gamma_1 : \mathbb{Z}/b \to \mathrm{Aut}\,(\mathbb{Z}/a)$, $\gamma_2 : T_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ *for the restrictions of* γ *and* $\tau_1 : \mathbb{Z}/b \to \mathrm{Aut}\,(\mathbb{Z}/a)$, $\tau_2 : O_n^\star \to \mathrm{Aut}\,(\mathbb{Z}/a)$ *for the restrictions of* τ. *Then*:

(1) $\mathrm{Aut}_{\gamma_2}\,(T_n^\star) \cong S_4 \times \mathbb{Z}/3^{n-n_0}$ *provided* $\mathrm{Ker}\,\gamma_2 = Q_8 \rtimes_{\alpha_n} \mathbb{Z}/3^{n_0}$*;*

(2) $\mathrm{Aut}_{\tau_2}\,(O_n^\star) \cong \mathrm{Aut}\,(O_n^\star)$.

Certainly, the groups $\mathrm{Aut}_{\gamma_1}\,(\mathbb{Z}/b)$ and $\mathrm{Aut}_{\tau_1}\,(\mathbb{Z}/b)$ could be described by [**6**, Proposition 1.5 and Corollary 1.6]. Observe that $\ell(\gamma_1), \ell(\tau_1) \leqslant 2$ implies $\ell(\gamma_1), \ell(\tau_1) = 1$ because b is odd and consequently, $\mathrm{Aut}_{\gamma_1}\,(\mathbb{Z}/b) = \mathrm{Aut}_{\tau_1}\,(\mathbb{Z}/b) = \mathrm{Aut}\,(\mathbb{Z}/b)$, where

$\ell(\gamma_1)$ (resp. $\ell(\tau_1)$) denotes the order of $\gamma_1(1_b)$ (resp. $\tau_1(1_b)$) in $\mathrm{Aut}\,(\mathbb{Z}/b)$ for a generator 1_b of the cyclic group $\mathbb{Z}/b$.

2. Homotopy types of space forms.

Given a group G, write $H^k(G)$ for its kth cohomology group with constant coefficients in the integers $\mathbb{Z}$ for $k \geqslant 0$. Then, any automorphism $\varphi \in \mathrm{Aut}\,(G)$ yields the induced automorphism $\varphi^* \in \mathrm{Aut}\,(H^n(G))$ and we write $\eta : \mathrm{Aut}\,(G) \to \mathrm{Aut}\,(H^k(G))$ for the corresponding anti-homomorphism. By a *period* of a group G we mean an integer d such that $H^k(G) = H^{k+d}(G)$ for all $k > 0$, and a group G with this property is called *periodic*. Among all periods of a group G there is the least one; and all others are multiple of that one. That least one period we call *the period* of the group and by [**3**, Section 11] the period of any periodic group is even.

Throughout the rest of the paper, $X(k)$ denotes a k-dimensional CW-complex with the homotopy type of a k-sphere and the group $\mathrm{Aut}\,(\mathbb{Z}/a)$ is identified with the unit group $(\mathbb{Z}/a)^\star$ of the $\mathrm{mod}\, a$ ring $\mathbb{Z}/a$. Given a free cellular action μ of a finite group G with order $|G|$ on a CW-complex $X(2k-1)$ write $X(2k-1)/\mu$ for the corresponding orbit space called a $(2k-1)$-*spherical space form* or a *Swan* $(2k-1)$-*complex* (see e.g., [**2**]). Then, the group G is periodic with period $2d$ dividing $2k$ and by [**3**, Chap. XVI, §9] there is an isomorphism $H^{2n}(G) \cong \mathbb{Z}/|G|$. Two spherical space forms $X(2k-1)/\mu$ and $X'(2k-1)/\mu'$ are called equivalent if they are homeomorphic and let $\mathcal{K}_G^{2k-1}$ denote the set of all such classes. We say that two such classes $[X(2k-1)/\mu]$ and $[X'(2k-1)/\mu']$ are homotopic if the space forms $X(2k-1)/\mu$ and $X'(2k-1)/\mu'$ are homotopy equivalent. Write $\mathcal{K}_G^{2k-1}/_{\simeq}$ for the associated quotient set of $\mathcal{K}_G^{2k-1}$ and $\mathrm{card}\,\mathcal{K}_G^{2k-1}/_{\simeq}$ for its cardinality, respectively. By means of [**11**], it is shown in [**12**, Theorem 1.8] that elements of the set $\mathcal{K}_G^{2k-1}/_{\simeq}$ are in one-to-one correspondence with the orbits, which contain a generator of $H^{2k}(G) = \mathbb{Z}/|G|$ under an action of $\pm\mathrm{Aut}\,(G)$ (see also [**4**] for another approach). But generators of the group $\mathbb{Z}/|G|$ are given by the unit group $(\mathbb{Z}/|G|)^\star$ of the ring $\mathbb{Z}/|G|$. Thus, those homotopy types are in one-to-one correspondence with the quotient $(\mathbb{Z}/|G|)^\star/\{\pm\varphi^*;\ \varphi \in \mathrm{Aut}\,(G)\}$, where φ^* is the induced automorphism on the cohomology $H^{2k}(G) = \mathbb{Z}/|G|$ by $\varphi \in \mathrm{Aut}\,(G)$.

Now, let G_1 and G_2 be finite groups with relatively prime orders $|G_1|$ and $|G_2|$, respectively. If G_1 and G_2 are also periodic with periods $2d_1$ and $2d_2$, respectively then by [**1**] the least common multiple $[2d_1, 2d_2]$ of $2d_1$ and $2d_2$ is the least period of the product $G_1 \times G_2$. Furthermore, given a finite group G with an action $\alpha : G \to (\mathbb{Z}/a)^\star$ write $|\alpha(g)|$ for the order of $\alpha(g)$ with $g \in G$. Let $\ell(\alpha) = [|\alpha(g)|;\ \text{for } g \in G]$ be the least common multiple of those orders. Then, for a semi-direct product $\mathbb{Z}/a \rtimes_\alpha G$, we have shown in [**6**], by means of the Lyndon-Hochschild-Serre spectral sequence, the following result.

Proposition 2.1. *Let $\mathbb{Z}/a$ be a cyclic group of order a, G a finite group, $\alpha : G \to \mathbb{Z}/a$ an action and $(|G|, a) = 1$. If G is periodic with the period $2d$ then the semi-direct product $\mathbb{Z}/a \rtimes_\alpha G$ is also a period finite group with the least period $2[\ell(\alpha), d]$.*

Thus, we are in a position to investigate the periodic groups $\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)$ and $\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)$. First, we find the least periods of the groups $T_n^\star$ and $O_n^\star$.

Because $T_n^\star = Q_8 \rtimes_\alpha \mathbb{Z}/3^n$ whence the Lyndon-Hochschild-Serre spectral sequence applied to the short one

$$0 \to Q_8 \longrightarrow T_n^\star \longrightarrow \mathbb{Z}/3^n \to 0$$

yields

$$E_2^{p,q}(T_n^\star) = H^p(\mathbb{Z}/3^n, H^q(Q_8)) = \begin{cases} 0, \text{ if } p, q > 0; \\ \mathbb{Z}, \text{ if} p, q = 0; \\ 0, \text{ if } q = 0 \text{ and } p \text{ odd}; \\ \mathbb{Z}/3^n, \text{ if } q = 0 \text{and } p \text{ even with } \neq 0; \\ H^0(\mathbb{Z}/3^n, H^q(Q_8)) = (H^q(Q_8))^{\mathbb{Z}/3^n}, \text{ if } q > 0. \end{cases}$$

Using the cohomology

$$H^k(Q_8) = \begin{cases} \mathbb{Z}, \ k = 0; \\ 0, \text{ if } k = 1 + 4l; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, \text{ if } k = 2 + 4l; \\ 0, \text{ if } k = 3 + 4l; \\ \mathbb{Z}/8, \text{ if } k = 4 + 4l \end{cases}$$

with $l \geqslant 0$, we can easily get

$$H^k(T_n^\star) = \begin{cases} \mathbb{Z}, \text{ if } k = 0; \\ 0, \text{ if } k = 1 + 4l; \\ \mathbb{Z}/3^n, \text{ if } k = 2 + 4l; \\ 0, \text{ if } k = 3 + 4l; \\ \mathbb{Z}/(8 \times 3^n), \text{ if } k = 4 + 4l \end{cases}$$

with $l \geqslant 0$ and consequently, 4 is the least period of the group $T_n^\star$. Whence, by Proposition 2.1, the number $2[\ell(\gamma), 2]$ is the least period of the group $\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)$.

To find the least period of the group $O_n^\star$, we apply Lyndon-Hochschild-Serre spectral sequence to the short one

$$0 \to T_n^\star \longrightarrow O_n^\star \longrightarrow \mathbb{Z}/2 \to 0.$$

Then, $E_2^{p,q}(O_n^\star) = H^p(\mathbb{Z}/2, H^q(T_n^\star))$. Next, observe that $E_2^{p,4} = H^p(\mathbb{Z}/2, \mathbb{Z}/(8 \times 3^n)) = H^p(\mathbb{Z}/2, \mathbb{Z}/8) \oplus H^p(\mathbb{Z}/2, \mathbb{Z}/3^n)$. Because $H^p(\mathbb{Z}/2, \mathbb{Z}/3^n) = 0$ for $p > 0$ and by [**11**] the action of $\mathbb{Z}/2$ on $\mathbb{Z}/8$ is trivial $H^p(\mathbb{Z}/2, \mathbb{Z}/(8 \times 3^n)) = \mathbb{Z}/2$ for $p > 0$. Then, we can easily find that

$$E_2^{p,q}(O_n^\star) = H^p(\mathbb{Z}/2, H^q(T_n^\star)) = \begin{cases} \mathbb{Z}, \text{ if } p = q = 0; \\ 0, \text{ if } p \text{ odd}, \ q = 0; \\ \mathbb{Z}/2, \text{ if } p \text{ even}, \ q = 0; \\ 0, \text{ if } p \geqslant 0, \ q = 1 + 4l, 2 + 4l, 3 + 4l; \\ \mathbb{Z}/(8 \times 3^n), \text{ if } p = 0, \ q = 4 + 4l; \\ \mathbb{Z}/2, \text{ if } p > 0, \ q = 4 + 4l \end{cases}$$

with $l \geqslant 0$.

To find the cohomology $H^*(O_n^\star)$ consider the generalized quaternion group Q_{16} as the subgroup of $O_n^\star$ generated by P, Q, R (according to the presentation of $O_n^\star$ given in Section 1) and its subgroup Q_8 generated by P, Q. The exact sequence $0 \to Q_8 \longrightarrow Q_{16} \longrightarrow \mathbb{Z}/2 \to 0$ leads to Lyndon-Hochschild-Serre spectral sequence with $E_2^{p,q}(Q_{16}) = H^p(\mathbb{Z}/2, H^q(Q_8))$. Because the action of $\mathbb{Z}/2$ on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is given by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and by means of [**11**], the action of $\mathbb{Z}/2$ on $\mathbb{Z}/8$ is trivial, applying $H^\star(Q_8)$, we derive:

$$E_2^{p,0}(Q_{16}) = H^p(\mathbb{Z}/2) = \begin{cases} \mathbb{Z}, \text{ if } p = 0; \\ 0, \text{ if } p \text{ odd}; \\ \mathbb{Z}/2, \text{ if } p \text{ even}, \end{cases}$$

$$E_2^{p,1}(Q_{16}) = 0, \quad E_2^{p,2}(Q_{16}) = H^p(\mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, \text{ if } p = 0; \\ 0, \text{ if } p > 0, \end{cases}$$

$E_2^{p,3}(Q_{16}) = 0$ and $E_2^{p,4}(Q_{16}) = H^p(\mathbb{Z}/2, \mathbb{Z}/8) = \mathbb{Z}/2$. Writing $E_k(Q_{16})$ for the k-term of that spectral sequence, we can deduce that $E_2(Q_{16}) \cong E_3(Q_{16}) \cong E_4(Q_{16}) \cong E_5(Q_{16})$ and $d_5(E_5^{1,4}(Q_{16})) = E_5^{6,0}(Q_{16})$, $d_k(E_k^{0,q}(Q_{16})) = 0$ for $k \geqslant 2$. Then, using the multiplicative structure of that spectral sequence and the periodicity of the groups Q_8 and $\mathbb{Z}/2$, we get further isomorphisms $E_6(Q_{16}) = H(E_5(Q_{16}), d_5) \cong E_7(Q_{16}) \cong \cdots \cong E_\infty(Q_{16}) = \mathcal{G}\left(H^*(Q_{16})\right)$, where by [**3**, Chapter XII] it holds:

$$H^k(Q_{16}) = \begin{cases} \mathbb{Z}, \text{ if } k = 0; \\ 0, \text{ if } k = 1 + 4l; \\ \mathbb{Z} \oplus \mathbb{Z}/2, \text{ if } k = 2 + 4l; \\ 0, \text{ if } k = 3 + 4l; \\ \mathbb{Z}/16, \text{ if } k = 4 + 4l \end{cases}$$

with $l \geqslant 0$. The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Q_8 \arrow[r] \arrow[d] & Q_{16} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T_n^\star \arrow[r] & O_n^\star \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

leads to a map

$$E_k(O_n^\star) \longrightarrow E_k(Q_{16})$$

for $k \geqslant 2$. Because of the isomorphism $E_2^{p,q}(O_n^\star) \xrightarrow{\cong} E_2^{p,q}(Q_{16})$ for $p > 0$, we can get $E_6(O_n^\star)$ and then $E_\infty(O_n^\star)$ as well. Therefore, we can read that

$$H^k(O_n^\star) = \begin{cases} \mathbb{Z}, \text{ if } k = 0; \\ 0, \text{ if } k = 1 + 4l; \\ \mathbb{Z}/2, \text{ if } k = 2 + 4l; \\ 0, \text{ if } k = 3 + 4l; \\ A \oplus \mathbb{Z}/3^n, \text{ if } k = 4 + 4l \end{cases}$$

with $l \geqslant 0$, where A is an abelian group of order 16. Because of the monomorphism $H^{4l}(O_n^\star)_{(2)} \to H^{4l}(Q_{16})$ on the 2-primary component of $H^{4l}(O_n^\star)$ for $l > 0$, we deduce an isomorphism $A \cong \mathbb{Z}/16$. Thus,

$$H^k(O_n^\star) = \begin{cases} \mathbb{Z}, \text{ if } k = 0; \\ 0, \text{ if } k = 1 + 4l; \\ \mathbb{Z}/2, \text{ if } k = 2 + 4l; \\ 0, \text{ if } k = 3 + 4l; \\ \mathbb{Z}/(16 \times 3^n), \text{ if } k = 4 + 4l \end{cases}$$

with $l \geqslant 0$ and consequently, 4 is the least period of the group $O_n^\star$. Whence, by means of Proposition 2.1, the number $2[\ell(\tau), 2]$ is the least period of the group $\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)$.

By [**6**, Lemma 1.1] any automorphism $\varphi \in \operatorname{Aut}(\mathbb{Z}/a \rtimes_\alpha G)$ for $(a, |G|) = 1$ determines a pair $(\varphi_1, \varphi_2) \in (\mathbb{Z}/a)^\star \times \operatorname{Aut}(G)$ with the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/a \arrow[r] \arrow[d, "\varphi_1"] & \mathbb{Z}/a \rtimes_\alpha G \arrow[r] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, "\varphi_2"] & 0 \\ 0 \arrow[r] & \mathbb{Z}/a \arrow[r] & \mathbb{Z}/a \rtimes_\alpha G \arrow[r] & G \arrow[r] & 0. \end{tikzcd}$$

Then, Lyndon-Hochshild-Serre spectral sequence and its naturality lead to the commutative diagram of cyclic groups with exact and splitting rows

$$\begin{tikzcd} 0 \arrow[r] & H^{2k[d,\ell(\alpha)]}(G) \arrow[r] \arrow[d, "\varphi_2^*"] & H^{2k[d,\ell(\alpha)]}(\mathbb{Z}/a \rtimes_\alpha G) \arrow[r] \arrow[d, "\varphi^*"] & H^{2k[d,\ell(\alpha)]}(\mathbb{Z}/a) \arrow[r] \arrow[d, "\varphi_1^*"] & 0 \\ 0 \arrow[r] & H^{2k[d,\ell(\alpha)]}(G) \arrow[r] & H^{2k[d,\ell(\alpha)]}(\mathbb{Z}/a \rtimes_\alpha G) \arrow[r] & H^{2k[d,\ell(\alpha)]}(\mathbb{Z}/a) \arrow[r] & 0 \end{tikzcd}$$

for $k > 0$, where $2d$ is the least period of G. Whence, φ^* is uniquely determined by the corresponding pair $(\varphi_2^*, \varphi_1^*)$ and consequently, there is the factorization

$$\begin{tikzcd} \operatorname{Aut}(\mathbb{Z}/a \rtimes_\alpha G) \arrow[r, "\eta"] \arrow[d, "\psi"'] & \operatorname{Aut}(H^{2k[\ell(\alpha),d]}(\mathbb{Z}/a \rtimes_\alpha G)) \\ (\mathbb{Z}/a)^\star \times \operatorname{Aut}_\alpha(G) \arrow[ur, "\eta'"'] & \end{tikzcd}$$

for all $k > 0$, where $\mathrm{Aut}_\alpha(G)$ is the subgroup of $\mathrm{Aut}(G)$ defined in Section 1. But $H^{2k[\ell(\alpha),d]}(\mathbb{Z}/a \rtimes_\alpha G) \cong \mathbb{Z}/a|G|$, so in the light of the above, to describe the number $\operatorname{card} \mathcal{K}^{2k[\ell(\alpha),d]-1}_{\mathbb{Z}/a\rtimes_\alpha G}/_{\simeq}$ of homotopy types of spherical space forms for $\mathbb{Z}/a \rtimes_\alpha G$ we are led to compute the order of the quotient $(\mathbb{Z}/a|G|)^\star/\{\pm\varphi^*;\ \varphi \in (\mathbb{Z}/a)^\star \times \mathrm{Aut}_\alpha(G)\}$ where φ^* is the induced automorphism on the cohomology $H^{2k[\ell(\alpha),d]}(\mathbb{Z}/a \rtimes G)$ for $\varphi \in (\mathbb{Z}/a)^\star \times \mathrm{Aut}_\alpha(G)$.

Now, for a periodic group G_1 with the least period $2d_1$ and an action $\omega : G_2 \to \mathrm{Aut}(G_1)$, we achieve anti-homomorphism $G_2 \to \mathrm{Aut}(H^{2kd_1}(G_1)) = \mathrm{Aut}(\mathbb{Z}/|G_1|)$. Write $(\mathbb{Z}/|G_1|)^\star/\pm G_2$ for the quotient group $(\mathbb{Z}/|G_1|)^\star/\{\pm\omega(g_2)^*;\ g_2 \in G_2\}$ and $O_{G_2}(|G_1|, 2kd_1)$ for its order, where $\omega(g_2)^*$ denotes the induced map on the cohomology $H^{2kd_1}(G_1)$. Furthermore, we set $O(m,n)$ for the order of the quotient group $(\mathbb{Z}/m)^\star/\{\pm l^n;\ l \in (\mathbb{Z}/m)^\star\}$.

Given $\varphi \in \mathrm{Aut}(T^\star_n)$ for the group $T^\star_n = Q_8 \rtimes_\alpha \mathbb{Z}/3^n$ there is the corresponding pair $(\varphi_1, \varphi_2) \in \mathrm{Aut}(Q_8) \times (\mathbb{Z}/3^n)^\star$ and by means of [**7**] maps φ_2 exhaust all automorphisms of the group $\mathbb{Z}/3^n$. The periodicity of $T^\star_n$, Lyndon-Hochshild-Serre spectral sequence and its naturality lead to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{4k}(\mathbb{Z}/3^n) \arrow[r] \arrow[d, "\varphi_2^*"] & H^{4k}(T^\star_n) \arrow[r] \arrow[d, "\varphi^*"] & H^{4k}(Q_8) \arrow[r] \arrow[d, "\varphi_1^*"] & 0 \\
0 \arrow[r] & H^{4k}(\mathbb{Z}/3^n) \arrow[r] & H^{4n}(T^\star_n) \arrow[r] & H^{4k}(Q_8) \arrow[r] & 0
\end{tikzcd}$$

of cyclic groups with exact rows for $k \geqslant 0$. But, by means of [**11**], φ_1^* is the identity map and a *fortiori* φ^* is uniquely determined by φ_2^*.

By [**7**], any $\varphi \in \mathrm{Aut}(O^\star_n)$ yields also a pair $(\varphi_1, \varphi_2) \in \mathrm{Aut}(T^\star_n) \times (\mathbb{Z}/2)^\star$. Again, the periodicity of $O^\star_n$, Lyndon-Hochshild-Serre spectral sequence and its naturality lead to the commutative diagram of cyclic groups

$$\begin{tikzcd}
0 \arrow[r] & H^{4k}(\mathbb{Z}/2) \arrow[r] \arrow[d, "\varphi_2^*"] & H^{4k}(O^\star_n) \arrow[r] \arrow[d, "\varphi^*"] & H^{4k}(T^\star_n) \arrow[r] \arrow[d, "\varphi_1^*"] & 0 \\
0 \arrow[r] & H^{4k}(\mathbb{Z}/2) \arrow[r] & H^{4k}(O^\star_n) \arrow[r] & H^{4k}(T^\star_n) \arrow[r] & 0
\end{tikzcd}$$

with exact and splitting rows for $k \geqslant 0$. Because the restriction of φ_1^* to Q_8, denoted by $\varphi_|$ induces the identity on $H^{4n}(Q_8) = \mathbb{Z}/8$, by means of the description of $H^{4k}(O^\star_n)$ and $H^{4k}(T^\star_n)$, we derive from the above the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \mathbb{Z}/16 \arrow[r] \arrow[d, "\varphi_|^*"] & \mathbb{Z}/8 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/8 \arrow[r] & 0.
\end{tikzcd}$$

Therefore, the restriction $\varphi_|^* : \mathbb{Z}/16 \to \mathbb{Z}/16$ is the identity map or the multipication by 9. Certainly, both cases might hold. Namely, consider the automorphism $\varphi : O^\star_n \to O^\star_n$ given by $\varphi(P) = P$, $\varphi(Q) = Q$, $\varphi(R) = -R$ and $\varphi(X) = X$, where P, Q, R, X are generators of $O^\star_n$. But the subgroup of $O^\star_n$ generated by P, Q, R

is the generalized quaternion group Q_{16} with the relation $(RP)^4 = R^2$. Then, $\varphi(RP) = (RP)^5$ and by [**11**] we achieve that $\varphi^* : H^{4k}(O_n^\star) \to H^{4k}(O_n^\star)$ restricts on $H^{4n}(Q_{16}) = \mathbb{Z}/16$ to the multiplication by 9.

Now, the group $\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)$ with an action $\gamma = (\gamma_1, \gamma_2) : \mathbb{Z}/b \times T_n^\star \to (\mathbb{Z}/a)^\star$ yields the factorization

$$\begin{tikzcd}
\operatorname{Aut}(\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)) \arrow[r, "\eta"] \arrow[d, "\psi"'] & \operatorname{Aut}(H^{2k[\ell(\gamma),2]}(\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)) \\
(\mathbb{Z}/a)^\star \times \operatorname{Aut}_{\gamma_1}(\mathbb{Z}/b) \times \operatorname{Aut}_{\gamma_2}(T_n^\star) \arrow[ur, "\eta'"'] &
\end{tikzcd}$$

for all $k > 0$. By [**6**, Proposition 2.2] we obtain the short exact sequence

$$0 \to (\mathbb{Z}/2)^t \longrightarrow (\mathbb{Z}/(8 \times 3^n ab)/ \pm ((\mathbb{Z}/a)^\star \times \operatorname{Aut}_{\gamma_1}(\mathbb{Z}/b) \times \operatorname{Aut}_{\gamma_2}(T_n^\star)) \longrightarrow$$

$$((\mathbb{Z}/a)^\star / \pm (\mathbb{Z}/a)^\star) \times ((\mathbb{Z}/b)^\star / \pm \operatorname{Aut}_{\gamma_1}(\mathbb{Z}/b)) \times ((\mathbb{Z}/(8 \times 3^n))^\star / \pm \operatorname{Aut}_{\gamma_2}(T_n^\star)) \to 0$$

for some $0 \leqslant t \leqslant 2$. But, by means of Proposition 1.2, $\operatorname{Aut}_{\gamma_2}(T_n^\star) \cong S_4 \times \mathbb{Z}/3^{n-n_0}$ with $\operatorname{Ker}\gamma_2 = Q_8 \rtimes \mathbb{Z}/3^{n-n_0}$. Because the action of S_4 on $H^{4k}(Q_8)$ is trivial, the canonical imbedding $(\mathbb{Z}/3^{n-n_0})^\star \hookrightarrow (\mathbb{Z}/3^n)^\star$ leads to the other short exact sequence

$$0 \to \mathbb{Z}/2)^{t'} \to (\mathbb{Z}/(8 \times 3^n))^\star / \pm \operatorname{Aut}_{\gamma_2}(T_n^\star) \longrightarrow$$

$$((\mathbb{Z}/8)^\star / \{\pm 1\}) \times ((\mathbb{Z}/3^n)^\star / (\mathbb{Z}/3^{n-n_0})^\star) \to 0$$

for some $0 \leqslant t' \leqslant 1$.

Now, we move to the group $\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)$ with an action $\tau : \mathbb{Z}/b \times O_n^\star \to (\mathbb{Z}/a)^\star$. By Proposition 1.2, $\operatorname{Aut}_{\tau_2}(O_n^\star) = \operatorname{Aut}(O_n^\star)$, so we achieve the factorization

$$\begin{tikzcd}
\operatorname{Aut}(\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)) \arrow[r, "\eta"] \arrow[d, "\psi"'] & \operatorname{Aut}(H^{2k[\ell(\tau),2]}(\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)) \\
(\mathbb{Z}/a)^\star \times \operatorname{Aut}_{\tau_1}(\mathbb{Z}/b) \times \operatorname{Aut}(O_n^\star) \arrow[ur, "\eta'"'] &
\end{tikzcd}$$

for all $k > 0$. Because

$$(\mathbb{Z}/(16 \times 3^n))^\star / \pm \operatorname{Aut}(O_n^\star) =$$

$$((\mathbb{Z}/16)^\star / \{\pm 1, \pm 9\}) \times ((\mathbb{Z}/3^n)^\star / \{\pm l^{k[\ell(\tau),2]};\ l \in (\mathbb{Z}/3^n)^\star\}),$$

we derive that $O_{\operatorname{Aut}(O_n^\star)}(16 \times 3^n, 2k[\ell(\tau), 2]) = 2 \times 3^{n-1}$. Then, the discussion above yields the main result.

Theorem 2.2. *Let* $\gamma = (\gamma_1, \gamma_2) : \mathbb{Z}/b \times T_n^\star \to (\mathbb{Z}/a)^\star$ *and* $\tau = (\tau_1, \tau_2) : \mathbb{Z}/b \times O_n^\star \to (\mathbb{Z}/a)^\star$ *be actions with* $(a, b) = (ab, 6) = 1$ *and* $n \geqslant 3$*, where* $\gamma_1 : \mathbb{Z}/b \to (\mathbb{Z}/a)^\star$,

$\gamma_2 : T_n^\star \to (\mathbb{Z}/a)^\star$ and $\tau_1 : \mathbb{Z}/b \to (\mathbb{Z}/a)^\star$, $\tau_2 : O_n^\star \to (\mathbb{Z}/a)^\star$ are appropriate restrictions of γ and τ, respectively. Then:

(1) $\mathrm{card}\,\mathcal{K}^{2k[\ell(\gamma),2]-1}_{\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)}/_{\simeq} \;=\; 2^{t+t'+1}3^{n_0}O(a,k[\ell(\gamma),2])O_{\mathrm{Aut}_{\gamma_1}(\mathbb{Z}/b)}(b,k[\ell(\gamma),2])$ $O(3^{n-n_0},\, k[\ell(\gamma),2])^{-1}$ *for some* $0 \leqslant t \leqslant 2$ *and* $0 \leqslant t' \leqslant 1$;

(2) $\mathrm{card}\,\mathcal{K}^{2k[\ell(\tau),2]-1}_{\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)}/_{\simeq} \;=\; 2^{t+1} \times 3^{n-1}O(a,k[\ell(\tau),2])O_{\mathrm{Aut}_{\tau_1}(\mathbb{Z}/b)}(b,k[\ell(\tau),2])$ *for some* $0 \leqslant t \leqslant 1$.

We point out that the numbers t, t' above are given by [**6**, Proposition 2.2] and the orders $O_{\mathrm{Aut}_{\gamma_1}(\mathbb{Z}/b)}(b,k[\ell(\gamma),2])$, $O_{\mathrm{Aut}_{\tau_1}(\mathbb{Z}/b)}(b,k[\ell(\tau),2])$ are determined by [**6**, Corollary 2.3].

Now, let $\gamma = (\gamma_1,\gamma_2) : \mathbb{Z}/b \times T_n^\star \to (\mathbb{Z}/a)^\star$ and $\tau = (\tau_1,\tau_2) : \mathbb{Z}/b \times O_n^\star \to (\mathbb{Z}/a)^\star$ be actions with $\ell(\gamma), \ell(\tau) \leqslant 2$. Then, of course $2\ell(\gamma), 2[\ell(\tau),2] \leqslant 4$, a *fortiori* γ_2 is trivial and the groups $\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)$, and $\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)$ act on a CW-complex $X(4k-1)$ for any $k \geqslant 1$. Furthermore, as it was observed in [**6**], $\mathrm{Aut}_{\gamma_1}(\mathbb{Z}/b) = \mathrm{Aut}_{\tau_1}(\mathbb{Z}/b) = (\mathbb{Z}/b)^\star$. Hence, $\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star) \cong \mathbb{Z}/ab \times T_n^\star$ and $\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star) \cong \mathbb{Z}/ab \rtimes_{\tau'} O_n^\star$ with the action $\tau' : O_n^\star \xrightarrow{\tau_2} (\mathbb{Z}/a)^\star \hookrightarrow (\mathbb{Z}/ab)^\star$, respectively. Then, in the light of [**6**, Proposition 2.2], we are in a position to deduce

Corollary 2.3. *Let $\gamma : \mathbb{Z}/b \times T_n^\star \to (\mathbb{Z}/a)^\star$ and $\tau : \mathbb{Z}/b \times O_n^\star \to (\mathbb{Z}/a)^\star$ be actions with $(a,b) = (ab,6) = 1$ and $\ell(\gamma), \ell(\tau) \leqslant 2$. Then:*

(1) $\mathrm{card}\,\mathcal{K}^{4k-1}_{\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)} = 2 \times 3^n \mathrm{card}\,\mathcal{K}^{4n-1}_{\mathbb{Z}/ab}/_{\simeq}$

and

(2) $\mathrm{card}\,\mathcal{K}^{4k-1}_{\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)} = 2 \times 3^{n-1} \mathrm{card}\,\mathcal{K}^{4n-1}_{\mathbb{Z}/ab}/_{\simeq}$.

We point out that $\mathrm{card}\,\mathcal{K}^{4n-1}_{\mathbb{Z}/ab}/_{\simeq}$ as the number of homotopy types of $(4n-1)$-lens spaces has been fully described in [**4**].

3. Groups of self homotopy equivalences.

Let μ be a free and cellular action of a finite group G on a CW-complex $X(2k-1)$. Write $\tilde{\eta} : \mathrm{Aut}\,(G) \to (\mathbb{Z}/|G|)^\star/\{\pm 1\}$ for the composition of the anti-homomorphism $\eta : \mathrm{Aut}\,(G) \to H^{2k}(G) = (\mathbb{Z}/|G|)^\star$ considered in the previous section with the quotient map $(\mathbb{Z}/|G|)^\star \to (\mathbb{Z}/|G|)^\star/\{\pm 1\}$. Then, by means of [**4**, Proposition 3.1] (see also [**10**, Theorem 1.4]), the group $\mathcal{E}(X(2k-1)/\mu)$ of homotopy classes of self homotopy equivalences for the space form $X(2k-1)/\mu$ is independent of the action μ of the group G as isomorphic to the kernel of the map $\tilde{\eta} : \mathrm{Aut}\,(G) \to (\mathbb{Z}/|G|)^\star/\{\pm 1\}$ for all $n \geqslant 1$ provided $|G| > 2$. Whence, we simply write $\mathcal{E}(X(2k-1)/G)$ for this group.

Let $\alpha : G \to (\mathbb{Z}/a)^\star$ be an action, $(a,|G|) = 1$ and $2d$ a period of G. Then, in virtue of [**10**, Theorem 1.4], one gets

$$\mathcal{E}\left(X(2kd-1)/(\mathbb{Z}/a \rtimes_\alpha G)\right) \cong \begin{cases} \mathbb{Z}/2, & \text{if } a|G| \leqslant 2; \\ \mathcal{E}\left(X(2kd-1)/G\right), & \text{if } a|G| > 2 \text{ and } a \leqslant 2. \end{cases}$$

By [**6**], the following generalization of [**10**, Theorem 1.8] holds.

Proposition 3.1. *Let the group $\mathbb{Z}/a \rtimes_\alpha G$ with $(a,|G|)=1$ acts freely and celullary on a CW-complex $X(2k[\ell(\alpha),d]-1)$ for $n \geqslant 1$, where $2d$ is a period of G. Then, there are isomorphisms*:

$$\mathcal{E}\left(X(2k[\ell(\alpha),d]-1)/(\mathbb{Z}/a \rtimes_\alpha G)\right) \cong \begin{cases} \mathbb{Z}/2, & \text{if } a|G| \leqslant 2;\\ \mathcal{E}\left(X(2k[\ell(\alpha),d]-1)/G\right), & \text{if } a|G|>2 \text{ and } a \leqslant 2; \end{cases}$$

however for $a|G|>2$ with $a>2$ it holds

$$\mathcal{E}(X(2k[\ell(\alpha),d]-1)/(\mathbb{Z}/a \rtimes_\alpha G)) \cong$$

$$\begin{cases} \operatorname{Der}_\alpha(G,\mathbb{Z}/a) \rtimes \left(\mathcal{E}\left(X(2k[\ell(\alpha),d]-1)/(\mathbb{Z}/a)\right) \times \mathcal{E}_\alpha\left(X(2n[\ell(\alpha),d]-1)/G\right)\right), & \text{if } |G|>2;\\ \mathbb{Z}/a \rtimes \mathcal{E}\left(X(2k[\ell(\alpha),d]-1)/(\mathbb{Z}/a)\right), & \text{if } |G| \leqslant 2, \end{cases}$$

where $\mathcal{E}_\alpha\left(X(2n[\ell(\alpha),d]-1)/G\right)$ is the subgroup of $\mathcal{E}\left(X(2n[\ell(\alpha),d]-1)/G\right)$ determined by the subgroup $\operatorname{Aut}_\alpha(G) \subseteq \operatorname{Aut}(G)$.

The paper [**5**, Section 3] deals with the group $\mathcal{E}\left(X(2k[\ell(\alpha),d]-1)/(\mathbb{Z}/a)\right)$, however the group $\mathcal{E}_\alpha\left(X(2k[\ell(\alpha),d]-1)/G\right)$ consists of all automorphisms $\varphi \in \operatorname{Aut}_\alpha(G)$ with $\varphi^* = \pm\mathrm{id}_{(\mathbb{Z}/|G|)^\star}$ provided $|G|>2$.

If now $\gamma=(\gamma_1,\gamma_2): \mathbb{Z}/b \times T_n^\star \to (\mathbb{Z}/a)^\star$ and $\tau=(\tau_1,\tau_2): \mathbb{Z}/b \times O_n^\star \to (\mathbb{Z}/a)^\star$ are actions considered in the previous section then it holds $|\mathbb{Z}/b \times T_n^\star| = 8 \times 3^n b > 2$ and $|\mathbb{Z}/b \times O_n^\star| = 16 \times 3^n b > 2$ for the order of those groups. Furthermore,

$$\mathcal{E}_\gamma\left(X(2k[\ell(\gamma),2]-1)/(\mathbb{Z}/b) \times T_n^\star\right) \cong$$

$$\mathcal{E}_{\gamma_1}\left(X(2k[\ell(\tau),2]-1)/(\mathbb{Z}/b)\right) \times \mathcal{E}_{\gamma_2}\left(X(2k[\ell(\gamma),2]-1)/T_n^\star\right)$$

and

$$\mathcal{E}_\tau\left(X(2k[\ell(\tau),2]-1)/(\mathbb{Z}/b) \times O_n^\star\right) \cong$$

$$\mathcal{E}_{\tau_1}\left(X(2k[\ell(\tau),2]-1)/(\mathbb{Z}/b)\right) \times \mathcal{E}_{\tau_2}\left(X(2k[\ell(\tau),2]-1)/O_n^\star\right).$$

But, the group $\mathcal{E}_{\gamma_1}\left(X(2k[\ell(\gamma),2]-1)/(\mathbb{Z}/b)\right)$ and $\mathcal{E}_{\tau_1}\left(X(2k[\ell(\tau),2]-1)/(\mathbb{Z}/b)\right)$ has been fully described in [**6**, Theorem 3.2].

To study the group $\mathcal{E}_{\gamma_2}\left(X(2k[\ell(\gamma),2]-1)/T_n^\star\right)$, we recall that by [**11**] any automorphism $\varphi \in \operatorname{Aut}(Q_8) \cong S_4$ induces the identity map on the cohomology group $H^{k[\ell(\gamma),2]}(Q_8)$ and by Proposition 1.2, $\operatorname{Aut}_{\gamma_2}(T_n^\star) \cong S_4 \times \mathbb{Z}/3^{n-n_0}$ provided $\operatorname{Ker}\gamma_2 = Q_8 \rtimes_{\alpha_n} \mathbb{Z}/3^{n_0}$. Then, we easily derive an isomorphism

$$\mathcal{E}_{\gamma_2}\left(X(2k[\ell(\gamma),2]-1)/T_n^\star\right) \cong S_4 \times \mathbb{Z}/\left(\frac{3^{n-n_0}}{(3^{n-n_0},k[\ell(\gamma),2])}\right).$$

Now, to move to the group $\mathcal{E}_{\tau_2}\left(X(2k[\ell(\tau),2]-1)/O_n^\star\right)$, we first recall that by Proposition 1.2, $\operatorname{Aut}_{\tau_2}(O_n^\star) = \operatorname{Aut}(O_n^\star)$. Because of the isomorphism $H^{2k[\ell(\gamma),2]}(O_n^\star) \cong \mathbb{Z}/(16 \times 3^n ab)$ from Section 1, we must study all automorphisms $\varphi \in \operatorname{Aut}(O_n^\star)$ with $\varphi^* = \mp\mathrm{id}_{\mathbb{Z}/(16\times 3^n ab)}$.

Let $\operatorname{Aut}^0(O_n^\star) = \{\varphi \in \operatorname{Aut}(O_n^\star);\ \varphi^* = \mathrm{id}_{\mathbb{Z}/(16\times 3^n ab)}\}$. Consider the automorphisms $\varphi, \psi \in \operatorname{Aut}(O_n^\star)$ defined on generators (according to the presentation of $O_n^\star$ given in Section 1) by: $\varphi(P)=P$, $\varphi(Q)=Q$, $\varphi(X)=X$, $\varphi(R)=R^{-1}$ and

$\psi(P) = P$, $\psi(Q) = Q$, $\psi(X) = X^4$, $\varphi(R) = R$ and write $\langle \varphi, \psi \rangle$ for the subgroup of $\mathrm{Aut}\,(O_n^\star)$ generated by φ and ψ. It is easy to check that there is an isomorphism $\langle \varphi, \psi \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3^{n-1}$ and $\langle \varphi, \psi \rangle \cap \mathrm{Inn}\,(O_n^\star) = E$, the trivial subgroup of $O_n^\star$. Then, by the results of [**7**] and order arguments, there is the splitting short exact sequence

$$1 \to \mathrm{Inn}\,(O_n^\star) \longrightarrow \mathrm{Aut}(O_n^\star) \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/3^{n-1} \to 1.$$

Consequently, a simple calculation, by means of the list of elements in $\mathrm{Aut}\,(O_n^\star)$ presented in [**7**] and considerations in the first paragraph on page 14 provides an isomorphism

$$\mathrm{Aut}^0(O_n^\star) \cong \mathrm{Inn}\,(O_n^\star) \rtimes \mathbb{Z}/\Big(\frac{3^{n-1}}{(3^{n-1}, k[\ell(\tau), 2])}\Big).$$

Because the restriction of any automorphism of $\mathrm{Aut}\,(O_n^\star)$ to the subgroup Q_8 induces the identity in cohomology at dimension multiple of 4, there is no element of $\mathrm{Aut}\,(O_n^\star)$ which induces the minus identity in cohomology at dimension $2k[\ell(\tau), 2]$. Since $\mathcal{Z}(O_n^\star) = \mathbb{Z}/2$ and so $\mathrm{Inn}\,(O_n^\star) \cong O_n^\star/(\mathbb{Z}/2) = O_n$ (the group considered in [**7**]) and consequently, we derive an isomorphism

$$\mathcal{E}_{\tau_2}\,(X(2k[\ell(\tau), 2] - 1)/O_n^\star) \cong O_n \rtimes \mathbb{Z}/\Big(\frac{3^{n-1}}{(3^{n-1}, k[\ell(\tau), 2])}\Big).$$

Finally, by Proposition 3.1 and the consideration above, we can close the paper with

Theorem 3.2. *Let* $\gamma = (\gamma_1, \gamma_2) : \mathbb{Z}/b \times T_n^\star \to (\mathbb{Z}/a)^\star$ *(resp.* $\tau = (\tau_1, \tau_2) : \mathbb{Z}/b \times O_n^\star \to (\mathbb{Z}/a)^\star$*) be an action with* $(a, b) = (ab, 6) = 1$ *for* $n \geqslant 3$*. If the group* $\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)$ *(resp.* $\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)$*) acts freely and cellularly on a CW-complex* $X(2k[\ell(\gamma), 2] - 1)$ *(resp.* $X(2k[\ell(\tau), 2] - 1)$*) then*

$$\mathcal{E}(X(2k[\ell(\gamma), 2] - 1)/(\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star))) \cong \mathrm{Der}_\gamma\,(\mathbb{Z}/b \times T_n^\star, \mathbb{Z}/a) \rtimes (\mathcal{E}(X(2k[\ell(\gamma_1), 2] - 1)/(\mathbb{Z}/a)) \times \mathcal{E}_{\gamma_1}\,(X(2k[\ell(\beta), 2] - 1)/(\mathbb{Z}/b)) \times S_4 \times \mathbb{Z}/\Big(\frac{3^{n-n_0}}{(3^{n-n_0}, k[\ell(\gamma), 2])}\Big)$$

(resp. $\mathcal{E}(X(2k[\ell(\tau), 2] - 1)/(\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star))) \cong \mathrm{Der}_\tau\,(\mathbb{Z}/b \times O_n^\star, \mathbb{Z}/a) \rtimes (\mathcal{E}(X(2k[\ell(\tau_1), 2] - 1)/(\mathbb{Z}/a)) \times \mathcal{E}_{\tau_1}\,(X(2k[\ell(\beta), 2] - 1)/(\mathbb{Z}/b)) \times O_n \rtimes \mathbb{Z}/\Big(\frac{3^{n-1}}{(3^{n-1}, k[\ell(\tau), 2])}\Big)$*).*

Thus, in the light of Proposition 1.1, the groups $\mathcal{E}(X(2k[\ell(\gamma), 2] - 1)/(\mathbb{Z}/a \rtimes_\gamma (\mathbb{Z}/b \times T_n^\star)))$ and $\mathcal{E}(X(2k[\ell(\tau), 2] - 1)/(\mathbb{Z}/a \rtimes_\tau (\mathbb{Z}/b \times O_n^\star)))$ have been fully described.

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Marek Golasiński `marek@mat.uni.torun.pl`

Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18, 87-100 Toruń, Poland

Daciberg Lima Gonçalves `dlgoncal@ime.usp.br`

Department of Mathematics-IME
University of São Paulo
Caixa Postal 66.281-AG. Cidade de São Paulo
05311-970 São Paulo, Brasil