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RELATIVE DIRECTED HOMOTOPY THEORY OF PARTIALLY ORDERED SPACES

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(*communicated by Martin Raussen*)

Abstract

Algebraic topological methods have been used successfully in concurrency theory, the domain of theoretical computer science that deals with parallel computing. L. Fajstrup, E. Goubault, and M. Raussen have introduced partially ordered spaces (pospaces) as a model for concurrent systems. In this paper it is shown that the category of pospaces under a fixed pospace is both a fibration and a cofibration category in the sense of H. Baues. The homotopy notion in this fibration and cofibration category is relative directed homotopy. It is also shown that the category of pospaces is a closed model category such that the homotopy notion is directed homotopy.

1. Introduction

It has turned out in the recent past that homotopy theoretical methods can be employed efficiently to study problems in concurrency theory. This is the domain of theoretical computer science that deals with parallel computing and distributed databases. Various topological models have been introduced in order to describe concurrent systems. Examples are partially ordered spaces [**4**], flows [**5**], globular CW-complexes [**6**], and d-spaces [**9**]. The reader is referred to E. Goubault [**8**] for a recent introduction to different topological models for concurrency.

In this paper we shall study the homotopy theory of partially ordered spaces which have been introduced as a model for concurrent systems by L. Fajstrup, E. Goubault, and M. Raussen in [**4**]. A partially ordered space (or pospace) is a topological space X equipped with a partial order $\leqslant$. The space X is interpreted as the state space of a concurrent system. The partial order $\leqslant$ represents the time flow. The idea here is that the execution of a system is a process in time so that a system in each state x can only proceed to subsequent states $y \geqslant x$ and not go back to preceding states $y < x$. A natural question is whether a system in a given state x can reach another state y or, in other words, whether there is an "execution path" from x to y. Such problems can be formalized appropriately using the following notion of maps between pospaces. A dimap (short for directed map) from a pospace

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$(X, \leqslant)$ to a pospace $(Y, \leqslant)$ is a continuous map $f : X \to Y$ such that $x \leqslant y$ implies $f(x) \leqslant f(y)$. An execution path from a state x of a pospace $(X, \leqslant)$ to a state y can now formally be defined to be a dimap f from the unit interval $I = [0, 1]$ with the natural order to $(X, \leqslant)$ such that $f(0) = x$ and $f(1) = y$.

Consider a very simple concurrent system where two processes A and B modify one at a time a shared resource. This situation can be modeled by the pospace $(X, \leqslant)$ where $X = (I \times I) \setminus (]\frac{1}{3}, \frac{2}{3}[\times]\frac{1}{3}, \frac{2}{3}[)$ and $\leqslant$ is the componentwise natural order. If in a state $(x, y) \in X$, $x < \frac{1}{3}$ then A has not yet accessed the resource; if $x = \frac{1}{3}$, A has accessed the resource and is ready to modify it, if $\frac{1}{3} < x < \frac{2}{3}$ then A is modifying the resource, and if $x \geqslant \frac{2}{3}$ then A has modified the resource. Similarly, B has not yet accessed, has accessed, modifies, and has modified the resource if $y \in [0, \frac{1}{3}[$, $y = \frac{1}{3}$, $y \in]\frac{1}{3}, \frac{2}{3}[$, and $y \in [\frac{2}{3}, 1]$ respectively. Since the processes cannot modify the resource simultaneously, there are no possible states in $]\frac{1}{3}, \frac{2}{3}[\times]\frac{1}{3}, \frac{2}{3}[$. The system has an initial state $(0, 0)$ and a final state $(1, 1)$ and there are infinitely many execution paths from $(0, 0)$ to $(1, 1)$. There are two kinds of such paths: those whose second coordinate is in $[0, \frac{1}{3}]$ when the first coordinate is in $]\frac{1}{3}, \frac{2}{3}[$ and those whose second coordinate is in $[\frac{2}{3}, 1]$ when the first coordinate is in $]\frac{1}{3}, \frac{2}{3}[$. The execution paths of the first kind correspond to executions where A modifies the resource before B and the execution paths of the second kind correspond to executions where B modifies the resource before A. From a computer scientific point of view it therefore makes sense to regard execution paths of the same kind as equivalent. The equivalence relation behind this is dihomotopy (short for directed homotopy) relative to the initial and final states. As the name suggests, this is a kind of homotopy and so homotopy theory becomes relevant to concurrency theory.

Before we define dihomotopy we note that for every topological space X the diagonal $\Delta \subset X \times X$ is a partial order. We also note that the product of two pospaces exists in the category-theoretical sense and is the topological product with the componentwise order. Two dimaps $f, g : (X \leqslant) \to (Y, \leqslant)$ are said to be dihomotopic if there exists a dimap $H : (X, \leqslant) \times (I, \Delta) \to (Y, \leqslant)$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The example above shows that one also needs a relative notion of dihomotopy. Indeed, in the absolute sense, any execution path is dihomotopic to a constant dimap. As P. Bubenik [**2**] has pointed out, another reason for considering a relative notion of dihomotopy is the fact that it depends a lot on the context whether two pospaces can be interpreted as models of the same concurrent system. In order to define relative dihomotopy we work in the comma category of pospaces under a fixed pospace $(C, \leqslant)$. A $(C, \leqslant)$-pospace is a triple $(X, \leqslant, \xi)$ consisting of a pospace $(X, \leqslant)$ and a dimap $\xi : (C, \leqslant) \to (X, \leqslant)$. A $(C, \leqslant)$-dimap $f : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ is a dimap $f : (X, \leqslant) \to (Y, \leqslant)$ such that $f \circ \xi = \theta$. Two $(C, \leqslant)$-dimaps $f, g : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ are said to be dihomotopic relative to $(C, \leqslant)$ if there exists a dimap $H : (X, \leqslant) \times (I, \Delta) \to (Y, \leqslant)$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ $(x \in X)$, and $H(\xi(c), t) = \theta(c)$ $(c \in C,\ t \in I)$. In the above example let $(C, \leqslant)$ be the discrete space $\{0, 1\}$ with the natural order and consider the inclusion $\iota : \{0, 1\} \hookrightarrow I$ and the dimap $\xi : (\{0, 1\}, \leqslant) \to (X, \leqslant)$ given by $\xi(0) = (0, 0)$ and $\xi(1) = (1, 1)$. Then two execution paths from $(0, 0)$ to $(1, 1)$, i.e., two $(\{0, 1\}, \leqslant)$-dimaps $f, g : (I, \leqslant, \iota) \to (X, \leqslant, \xi)$, are of the same kind if and only if they are dihomotopic relative to $(\{0, 1\}, \leqslant)$.

The best known framework for homotopy theory is certainly the one of closed model categories in the sense of D. Quillen [**10**]. A closed model category is a category with three classes of morphisms, called weak equivalences, fibrations, and cofibrations, which are subject to certain axioms. The structure of a closed model category splits up into two dual structures which are essentially the structure of a cofibration category and the structure of a fibration category. Cofibration and fibration categories have been introduced by H. Baues [**1**] who has developed an extensive homotopy theory for these categories. In this paper we show that the category of $(C, \leqslant)$-pospaces is both a fibration and a cofibration category (Theorems 5.5 and 7.2). We also show that the category of absolute pospaces (i.e., pospaces) is a closed model category (Theorem 8.2). The main ingredient of the homotopy theory of a cofibration, fibration, or closed model category is of course a notion of homotopy. We show that this notion of homotopy in the cofibration and fibration category of $(C, \leqslant)$-pospaces is dihomotopy relative to $(C, \leqslant)$ (cf. 5.9 and 7.2). Similarly, the homotopy notion of the closed model category of pospaces is dihomotopy (cf. 8.2).

L. Fajstrup, E. Goubault, and M. Raussen [**4**] also introduce locally partially ordered spaces, or local pospaces, which consitute a more advanced model for concurrency than the "global" pospaces we consider here. There are dimaps of local pospaces and there is a concept of (relative) dihomotopy. One can show that the category of local pospaces (under a fixed local pospace) is a fibration category such that the homotopy notion is (relative) dihomotopy. Unfortunately, it is not known whether there are enough colimits for a cofibration or a closed model category structure. Note, however, that P. Bubenik and K. Worytkiewicz [**3**] have constructed a closed model category containing the category of local pospaces under a fixed local pospace as a subcategory. Another interesting model category for concurrency theory is the one of flows introduced by P. Gaucher [**5**]. Some authors, as for instance M. Grandis [**9**] and P. Bubenik and K. Worytkiewicz [**3**], work with a stronger notion of dihomotopy. They use the directed interval $(I, \leqslant)$ (where $\leqslant$ is the natural order) instead of the free interval (I, Δ) in the definition of dihomotopy. Obviously, dihomotopy type in the sense of the present paper is an invariant of that stronger dihomotopy type. From the point of view of that stronger dihomotopy theory, dihomotopy theory in the sense of this paper may be considered as an approximation, in the same way as for instance rational homotopy theory can be regarded as an approximation of homotopy theory.

The paper is organized as follows. In section 2 we show that the category of $(C, \leqslant)$-pospaces is complete and cocomplete. Section 3 contains the fundamental material about dihomotopy. In particular, we define dihomotopy equivalences relative to $(C, \leqslant)$ and the adjoint cylinder and path $(C, \leqslant)$-pospace functors. In section 4 we define $(C, \leqslant)$-difibrations and prove some fundamental facts about them. The main result of section 5 is Theorem 5.5 which states that the category of $(C, \leqslant)$-pospaces is a fibration category where fibrations are $(C, \leqslant)$-difibrations and weak equivalences are dihomotopy equivalences relative to $(C, \leqslant)$. This result is a consequence of the fact that the the category of $(C, \leqslant)$-pospaces is a P-category in the sense of [**1**] which is proved in 5.2. Proposition 5.9 contains the result that two

$(C, \leqslant)$-dimaps are homotopic in the fibration category of $(C, \leqslant)$-pospaces if and only if they are dihomotopic relative to $(C, \leqslant)$. In section 6 we study cofibrations in a fibration category and show in Theorem 6.8 that they induce under certain conditions the structure of a cofibration category. We show that the homotopy notions of the cofibration and fibration category structures coincide (cf. 6.10). The internal cofibrations of the fibration category of $(C, \leqslant)$-pospaces are called $(C, \leqslant)$-dicofibrations. In 7.2 we show that the conditions of 6.8 are satisfied so that the category of $(C, \leqslant)$-pospaces is a cofibration category in which the homotopy notion is dihomotopy relative to $(C, \leqslant)$. In the last section it is shown that the category of absolute pospaces is a closed model category such that the homotopy notion is dihomotopy.

2. Pospaces

Definition 2.1. A *pospace* (short for *partially ordered space*) is a pair $(X, \leqslant)$ consisting of a space X and a partial order $\leqslant$ on X. A *dimap* (short for *directed map*) $f : (X, \leqslant) \to (Y, \leqslant)$ is a continuous map $f : X \to Y$ such that for all $x, x' \in X$, $x \leqslant x'$ implies $f(x) \leqslant f(x')$. The category of pospaces will be denoted by **poTop**.

In the original definition (cf. [**4**]) the partial order of a pospace $(X, \leqslant)$ is required to be closed as a subspace of $X \times X$. It is shown in [**4**] that a space X can be equipped with such a closed partial order if and only if it is a Hausdorff space, and in some sense pospaces with a closed partial order are for general pospaces what Hausdorff spaces are for general topological spaces. An interesting topological space will of course in general be a Hausdorff space. From the homotopy theoretical point of view, however, a restriction to Hausdorff spaces is not necessary and it is indeed easier to develop ordinary homotopy theory in the category of all topological spaces than in the category of Hausdorff spaces. For the same reason of simplicity we shall work with general pospaces rather than with pospaces having a closed partial order.

For every topological space X the diagonal $\Delta \subset X \times X$ is a partial order (and this partial order is closed if and only if X is a Hausdorff space). The functor $X \mapsto (X, \Delta)$ from the category **Top** of topological spaces to **poTop** is left adjoint to the forgetful functor **poTop** $\to$ **Top**.

Proposition 2.2. *The category* **poTop** *is complete and cocomplete.*

Proof. Products, coproducts, and equalizers are constructed as in **Top**. In order to construct the coequalizer of two dimaps $f, g \colon (X, \leqslant) \to (Y, \leqslant)$, consider the coequalizer of f and g in in **Top**, i.e., the quotient space $Y/\sim$ where $\sim$ is the equivalence relation given by $f(x) \sim g(x)$. Define a reflexive and transitive relation $\lhd$ on $Y/\sim$ by

$$\alpha \lhd \beta \Leftrightarrow \exists\, y_1, \dots, y_n \in Y : y_1 \in \alpha, y_n \in \beta, \text{ and } y_1 \leqslant y_2 \sim y_3 \leqslant \cdots \sim y_{n-1} \leqslant y_n.$$

The coequalizer of f and g is the pospace $((Y/\sim)/\lhd\rhd, \leqslant)$ where $\lhd\rhd$ is the equivalence relation on $Y/\sim$ defined by $\alpha \lhd\rhd \beta \Leftrightarrow \alpha \lhd \beta$ and $\beta \lhd \alpha$ and $\leqslant$ is the partial order defined by $A \leqslant B \Leftrightarrow \forall\, \alpha \in A, \beta \in B : \alpha \lhd \beta$. $\square$

Definition 2.3. Let $(C, \leqslant)$ be a pospace. A $(C, \leqslant)$-*pospace* is a triple $(X, \leqslant, \xi)$ consisting of a pospace $(X, \leqslant)$ and a dimap $\xi : (C, \leqslant) \to (X, \leqslant)$. A $(C, \leqslant)$-*dimap* from $(X, \leqslant, \xi)$ to $(Y, \leqslant, \theta)$ is a dimap $f : (X, \leqslant) \to (Y, \leqslant)$ such that $f \circ \xi = \theta$. The category of $(C, \leqslant)$-pospaces is denoted by $(C, \leqslant)$-**poTop**.

Proposition 2.4. *For any pospace $(C, \leqslant)$ the category $(C, \leqslant)$-**poTop** is complete and cocomplete.*

Proof. This follows from 2.2. □

Remark 2.5. An absolute pospace is the same as a $(\emptyset, \Delta)$-pospace.

3. Relative dihomotopy

Throughout this section we work under a fixed pospace $(C, \leqslant)$. We define dihomotopy relative to $(C, \leqslant)$, introduce the adjoint cylinder and path $(C, \leqslant)$-pospace functors, and give characterizations of relative dihomotopy by means of these constructions.

Definition 3.1. Two $(C, \leqslant)$-dimaps $f, g : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ are said to be *dihomotopic relative to* $(C, \leqslant)$, $f \simeq g$ *rel.* $(C, \leqslant)$, if there exists a *dihomotopy relative to* $(C, \leqslant)$ from f to g, i.e., a dimap $H : (X, \leqslant) \times (I, \Delta) \to (Y, \leqslant)$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ $(x \in X)$, and $H(\xi(c), t) = \theta(c)$ $(c \in C,\ t \in I)$. If $C = \emptyset$ we simply talk of dihomotopies and dihomotopic dimaps and we simply write $f \simeq g$.

Proposition 3.2. *Dihomotopy relative to $(C, \leqslant)$ is a natural equivalence relation.*

Proof. This is an easy exercise. □

Definition 3.3. The equivalence class of a $(C, \leqslant)$-dimap with respect to dihomotopy relative to $(C, \leqslant)$ is called its *dihomotopy class relative to* $(C, \leqslant)$. The quotient category $(C, \leqslant)$-**poTop**$/ \simeq$ *rel.* $(C, \leqslant)$ is the *dihomotopy category relative to* $(C, \leqslant)$. A *dihomotopy equivalence relative to* $(C, \leqslant)$ is a $(C, \leqslant)$-dimap $f : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ such that there exists a *dihomotopy inverse relative to* $(C, \leqslant)$, i.e., a $(C, \leqslant)$-dimap $g : (Y, \leqslant, \theta) \to (X, \leqslant, \xi)$ satisfying $f \circ g \simeq id_{(Y, \leqslant, \theta)}$ *rel.* $(C, \leqslant)$ and $g \circ f \simeq id_{(X, \leqslant, \xi)}$ *rel.* $(C, \leqslant)$. Two $(C, \leqslant)$-pospaces $(X, \leqslant, \xi)$ and $(Y, \leqslant, \theta)$ are said to be *dihomotopy equivalent relative to* $(C, \leqslant)$ or of the *same dihomotopy type relative to* $(C, \leqslant)$ if there exists a dihomotopy equivalence relative to $(C, \leqslant)$ from $(X, \leqslant, \xi)$ to $(Y, \leqslant, \theta)$. If $C = \emptyset$ we simply talk of dihomotopy classes, the dihomotopy category, dihomotopy equivalences, and dihomotopy equivalent pospaces.

Note that a $(C, \leqslant)$-dimap is a dihomotopy equivalence relative to $(C, \leqslant)$ if and only if its dihomotopy class relative to $(C, \leqslant)$ is an isomorphism in the dihomotopy category relative to $(C, \leqslant)$. Similarly, two $(C, \leqslant)$-pospaces are dihomotopy equivalent relative to $(C, \leqslant)$ if and only if they are isomorphic in the dihomotopy category relative to $(C, \leqslant)$.

Proposition 3.4. *Any isomorphism of $(C, \leqslant)$-pospaces is a dihomotopy equivalence relative to $(C, \leqslant)$. Let $f : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ and $g : (Y, \leqslant, \theta) \to (Z, \leqslant, \zeta)$ be two $(C, \leqslant)$-dimaps. If two of f, g, and $g \circ f$ are dihomotopy equivalences relative to $(C, \leqslant)$, so is the third. Any retract of a dihomotopy equivalence relative to $(C, \leqslant)$ is a dihomotopy equivalence relative to $(C, \leqslant)$.*

Proof. The first statement is obvious and the others follow from the corresponding facts for isomorphisms. □

Let $(X, \leqslant, \xi)$ be a $(C, \leqslant)$-pospace and S be a space. Form the pushout diagram of pospaces

$$\begin{tikzcd} (C,\leqslant) \times (S,\Delta) \arrow[r, "pr_C"] \arrow[d, "\xi \times id_S"'] & (C,\leqslant) \arrow[d, "\bar{\xi}"] \\ (X,\leqslant) \times (S,\Delta) \arrow[r] & (X \square_C S, \leqslant). \end{tikzcd}$$

The space $X \square_C S$ is the pushout of the underlying diagram of spaces. If $S = \emptyset$, $(X \square_C S, \leqslant) = (C, \leqslant)$. If $S \neq \emptyset$, we may construct $X \square_C S$ as the quotient space $(X \times S)/\sim$ where

$$(x, s) \sim (y, t) \Leftrightarrow (x, s) = (y, t) \text{ or } \exists\, c \in C : x = y = \xi(c).$$

The partial order on $X \square_C S$ is then given by

$$[x, s] \leqslant [x', s'] \Leftrightarrow (x, s) \leqslant (x', s') \text{ or } \exists\, c \in C : x \leqslant \xi(c) \leqslant x'.$$

We define a pospace under $(C, \leqslant)$ by setting

$$(X, \leqslant, \xi) \square_{(C,\leqslant)} S = (X \square_C S, \leqslant, \bar{\xi}).$$

It is clear that this construction is natural and defines a functor

$$\square_{(C,\leqslant)} : (C, \leqslant)\text{-}\mathbf{poTop} \times \mathbf{Top} \to (C, \leqslant)\text{-}\mathbf{poTop}.$$

Definition 3.5. The *cylinder* on a $(C, \leqslant)$-pospace $(X, \leqslant, \xi)$ is the $(C, \leqslant)$-pospace $(X, \leqslant, \xi) \square_{(C,\leqslant)} I$.

Note that if $C = \emptyset$ then the cylinder on a pospace $(X, \leqslant)$ is just the product pospace $(X, \leqslant) \times (I, \Delta)$.

Proposition 3.6. *Two $(C, \leqslant)$-dimaps $f, g : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ are dihomotopic relative to $(C, \leqslant)$ if and only if there exists a $(C, \leqslant)$-dimap $H : (X, \leqslant, \xi) \square_{(C,\leqslant)} I \to (Y, \leqslant, \theta)$ such that $H([x, 0]) = f(x)$ and $H([x, 1]) = g(x)$.*

Proof. This is straightforward. □

Recall that the path space X^I of a topological space X is the set of all continuous maps $\omega : I \to X$ with the compact-open topology.

Definition 3.7. Let $(X, \leqslant, \xi)$ be a $(C, \leqslant)$-pospace. The *path $(C, \leqslant)$-pospace* of $(X, \leqslant, \xi)$ is the $(C, \leqslant)$-pospace $(X^I, \leqslant, \mathsf{c}_\xi)$ where the partial order is given by

$$\omega \leqslant \nu \Leftrightarrow \forall\, t \in I : \omega(t) \leqslant \nu(t)$$

and the dimap $\mathsf{c}_\xi : (C, \leqslant) \to (X^I, \leqslant)$ is given by $\mathsf{c}_{\xi(c)}(t) = \xi(c)$ $(c \in C,\ t \in I)$.

The path $(C, \leqslant)$-pospace is obviously functorial. Note also that for each $t \in I$ the evaluation map $ev_t : X^I \to X$, $\omega \mapsto \omega(t)$ is a $(C, \leqslant)$-dimap $(X^I, \leqslant, \mathsf{c}_\xi) \to (X, \leqslant, \xi)$.

Proposition 3.8. *The path $(C, \leqslant)$-pospace functor is right adjoint to the cylinder functor $-\Box_{(C,\leqslant)} I$.*

Proof. The natural correspondence between the $(C, \leqslant)$-dimaps $h : (X, \leqslant, \xi) \to (Y^I, \leqslant, \mathsf{c}_\theta)$ and the $(C, \leqslant)$-dimaps $H : (X, \leqslant, \xi)\Box_{(C,\leqslant)} I \to (Y, \leqslant, \theta)$ is given by the formula $h(x)(t) = H([x, t])$. $\Box$

Using this adjunction one easily establishes the following characterization of dihomotopy relative to $(C, \leqslant)$:

Proposition 3.9. *Two $(C, \leqslant)$-dimaps $f, g : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ are dihomotopic relative to $(C, \leqslant)$ if and only if there exists a $(C, \leqslant)$-dimap $h : (X, \leqslant, \xi) \to (Y^I, \leqslant, \mathsf{c}_\theta)$ such that $f = ev_0 \circ h$ and $g = ev_1 \circ h$.*

4. $(C, \leqslant)$-difibrations

As in the preceding section we work under a fixed pospace $(C, \leqslant)$. Recall the following terminology:

Definition 4.1. Let $\mathbf{C}$ be a category and $\mathcal{A}$ be a class of morphisms. A morphism $f : X \to Y$ is said to have the *right lifting property* with respect to $\mathcal{A}$ if for every morphism $a : A \to B$ of $\mathcal{A}$ and for all morphisms $g : A \to X$ and $h : B \to Y$ satisfying $f \circ g = h \circ a$ there exists a morphism $\lambda : B \to X$ such that $f \circ \lambda = h$ and $\lambda \circ a = g$. Similarly, a morphism $f : X \to Y$ is said to have the *left lifting property* with respect to $\mathcal{A}$ if for every morphism $a : A \to B$ of $\mathcal{A}$ and for all morphisms $g : X \to A$ and $h : Y \to B$ satisfying $a \circ g = h \circ f$ there exists a morphism $\lambda : Y \to A$ such that $a \circ \lambda = h$ and $\lambda \circ f = g$.

Definition 4.2. A $(C, \leqslant)$-*difibration* is a $(C, \leqslant)$-dimap having the right lifting property with respect to the $(C, \leqslant)$-dimaps of the form

$$i_0 : (X, \leqslant, \xi) \to (X, \leqslant, \xi)\Box_{(C,\leqslant)} I, \quad i_0(x) = [x, 0].$$

If $C = \emptyset$ we simply talk of difibrations.

It is a general fact that any class of morphisms in a category which is defined by having the right (resp. left) lifting property with respect to another class of morphisms contains all isomorphisms and is closed under base change (resp. cobase change), composition, and retracts. We therefore have

Proposition 4.3. *The class of $(C, \leqslant)$-difibrations is closed under composition, retracts, and base change. Every isomorphism of $(C, \leqslant)$-pospaces is a $(C, \leqslant)$-difibration.*

We leave it to the reader to check the following $\Box$-free characterization of $(C, \leqslant)$-difibrations:

Proposition 4.4. *A $(C,\leqslant)$-dimap $p:(E,\leqslant,\varepsilon)\to(B,\leqslant,\beta)$ is a $(C,\leqslant)$-difibration if and only if for every $(C,\leqslant)$-dimap $f:(X,\leqslant,\xi)\to(E,\leqslant,\varepsilon)$ and every dimap $H:(X,\leqslant)\times(I,\Delta)\to(B,\leqslant)$ satisfying $H(x,0)=(p\circ f)(x)$ $(x\in X)$ and $H(\xi(c),t)=\beta(c)$ $(c\in C)$ there exists a dimap $G:(X,\leqslant)\times(I,\Delta)\to(E,\leqslant)$ such that $G(x,0)=f(x)$ $(x\in X)$, $p\circ G=H$, and $G(\xi(c),t)=\varepsilon(c)$ $(c\in C,\ t\in I)$.*

Proposition 4.5. *For every $(C,\leqslant)$-pospace $(X,\leqslant,\xi)$ the final $(C,\leqslant)$-dimap $*:(X,\leqslant,\xi)\to(*,\Delta,*)$ is a $(C,\leqslant)$-difibration.*

Proof. Let $f:(W,\leqslant,\psi)\to(X,\leqslant,\xi)$ be a $(C,\leqslant)$-dimap and $F:(W,\leqslant)\times(I,\Delta)\to(*,\Delta)$ be a (the only) dimap. Define a dimap $H:(W,\leqslant)\times(I,\Delta)\to(X,\leqslant)$ by $H(w,t)=f(w)$. Then $H(w,0)=f(w)$, $*\circ H=F$, and $H(\psi(c),t)=\xi(c)$. □

It is a very useful fact in ordinary homotopy theory (due to A. Strøm [**11**]) that fibrations have a much stronger lifting property than the defining homotopy lifting property. The last point of this section is an adaptation of this result to $(C,\leqslant)$-difibrations. We shall need the following lemma:

Lemma 4.6. *Let $(X,\leqslant,\xi)$ be a pospace and S be a space. Then*

$$(X,\leqslant,\xi)\square_{(C,\leqslant)}(S\times I)=((X,\leqslant,\xi)\square_{(C,\leqslant)}S)\square_{(C,\leqslant)}I.$$

Proof. Consider the defining pushout of $(X,\leqslant,\xi)\square_{(C,\leqslant)}S$:

$$\begin{array}{ccc}(C,\leqslant)\times(S,\Delta) & \xrightarrow{pr_C} & (C,\leqslant)\\ \downarrow{\scriptstyle \xi\times id_S} & & \downarrow{\scriptstyle \bar{\xi}}\\ (X,\leqslant)\times(S,\Delta) & \longrightarrow & (X\square_C S,\leqslant).\end{array}$$

Since the functor $-\times(I,\Delta):\mathbf{poTop}\to\mathbf{poTop}$ is a left adjoint, it preserves colimits. It follows that both squares in the following diagram of pospaces are pushouts:

$$\begin{array}{ccccc}(C,\leqslant)\times(S,\Delta)\times(I,\Delta) & \xrightarrow{pr_C\times id_I} & (C,\leqslant)\times(I,\Delta) & \xrightarrow{pr_C} & (C,\leqslant)\\ \downarrow{\scriptstyle \xi\times id_S\times id_I} & & \downarrow{\scriptstyle \bar{\xi}\times id_I} & & \downarrow{\scriptstyle \bar{\bar{\xi}}}\\ (X,\leqslant)\times(S,\Delta)\times(I,\Delta) & \longrightarrow & (X\square_C S,\leqslant)\times(I,\Delta) & \longrightarrow & ((X\square_C S)\square_C I,\leqslant).\end{array}$$

This implies that the whole diagram is the defining pushout of $(X,\leqslant,\xi)\square_{(C,\leqslant)}(S\times I)$ and thus that $(X,\leqslant,\xi)\square_{(C,\leqslant)}(S\times I)=((X,\leqslant,\xi)\square_{(C,\leqslant)}S)\square_{(C,\leqslant)}I$. □

By a *trivial cofibration* of spaces we mean a closed cofibration which also is a homotopy equivalence. The following characterization of $(C,\leqslant)$-difibrations is of fundamental importance:

Proposition 4.7. *A $(C,\leqslant)$-dimap $p:(E,\leqslant,\varepsilon)\to(B,\leqslant,\beta)$ is a $(C,\leqslant)$-difibration if and only if for every $(C,\leqslant)$-pospace $(Z,\leqslant,\zeta)$, every trivial cofibration of spaces $i:A\to X$, every dimap $f:(Z,\leqslant)\times(A,\Delta)\to(E,\leqslant)$ satisfying $f(\zeta(c),a)=\varepsilon(c)$ $(c\in C,\ a\in A)$, and every dimap $g:(Z,\leqslant)\times(X,\Delta)\to(B,\leqslant)$ satisfying $g(z,i(a))=$*

$p(f(z,a))$ ($z \in Z$, $a \in A$) and $g(\zeta(c),x) = \beta(c)$ ($c \in C$, $x \in X$) there exists a dimap $\lambda : (Z,\leqslant)\times(X,\Delta) \to (E,\leqslant)$ such that $\lambda(z,i(a)) = f(z,a)$ ($z \in Z$, $a \in A$), $p\circ\lambda = g$, and $\lambda(\zeta(c),x) = \varepsilon(c)$ ($c \in C$, $x \in X$).

Proof. If p has this lifting property, it is a $(C,\leqslant)$-difibration: it suffices to consider the trivial cofibration $\{0\} \hookrightarrow I$. Suppose that p is a $(C,\leqslant)$-difibration and consider a $(C,\leqslant)$-pospace $(Z,\leqslant,\zeta)$, a trivial cofibration of spaces $i : A \to X$, a dimap $f : (Z,\leqslant)\times(A,\Delta) \to (E,\leqslant)$ satisfying $f(\zeta(c),a) = \varepsilon(c)$, and a dimap $g : (Z,\leqslant)\times(X,\Delta) \to (B,\leqslant)$ satisfying $g(z,i(a)) = p(f(z,a))$ and $g(\zeta(c),x) = \beta(c)$. Since i is a trivial cofibration, i is a closed inclusion and A is a strong deformation retract of X. There hence exist a retraction $r : X \to A$ of i and a homotopy $H : X\times I \to X$ such that $H(x,0) = r(x)$, $H(x,1) = x$ ($x \in X$), and $H(a,t) = a$ ($a \in A$, $t \in I$). There also exists a continuous map $\phi : X \to I$ such that $A = \phi^{-1}(0)$. Consider the map $G : X\times I \to X$ defined by

$$G(x,t) = \begin{cases} H(x,\frac{t}{\phi(x)}) & t < \phi(x), \\ x & t \geqslant \phi(x). \end{cases}$$

As in [**13**, I.7.15] one can show that G is continuous. We have $G(x,0) = (i\circ r)(x)$ for all $x \in X$. Consider the following commutative diagram of $(C,\leqslant)$-pospaces where $\bar{f}$ and $\bar{g}$ are given by $\bar{f}([z,x]) = f(z,r(x))$ and $\bar{g}([z,x,t]) = g(z,G(x,t))$:

$$\begin{tikzcd}
(Z,\leqslant,\zeta)\Box_{(C,\leqslant)}X \arrow[r, "\bar{f}"] \arrow[d, "id_Z\Box_C i_0"'] & (E,\leqslant,\varepsilon) \arrow[d, "p"] \\
(Z,\leqslant,\zeta)\Box_{(C,\leqslant)}(X\times I) \arrow[r, "\bar{g}"'] & (B,\leqslant,\beta).
\end{tikzcd}$$

By 4.6, we may identify the $(C,\leqslant)$-dimap $id_Z\Box_C i_0$ with the $(C,\leqslant)$-dimap

$$(Z,\leqslant,\zeta)\Box_{(C,\leqslant)}X \to ((Z,\leqslant,\zeta)\Box_{(C,\leqslant)}X)\Box_{(C,\leqslant)}I, \quad [z,x] \mapsto [[z,x],0].$$

Since p is a $(C,\leqslant)$-difibration, there exists a $(C,\leqslant)$-dimap

$$F : (Z,\leqslant,\zeta)\Box_{(C,\leqslant)}(X\times I) \to (E,\leqslant,\varepsilon)$$

such that $F\circ(id_Z\Box_C i_0) = \bar{f}$ and $p\circ F = \bar{g}$. Consider the dimap $\lambda : (Z,\leqslant)\times(X,\Delta) \to (E,\leqslant)$ defined by $\lambda(z,x) = F([z,x,\phi(x)])$. We have

$$(p\circ\lambda)(z,x) = p(F([z,x,\phi(x)])) = \bar{g}([z,x,\phi(x)]) = g(z,G(x,\phi(x))) = g(z,x),$$

$$\lambda(z,i(a)) = \lambda(z,a) = F([z,a,\phi(a)]) = F([z,a,0]) = \bar{f}([z,a]) = f(z,r(a)) = f(z,a),$$

and $\lambda(\zeta(c),x) = F([\zeta(c),x,\phi(x)]) = \varepsilon(c)$. This shows that p has the required lifting property. □

5. The fibration category structure

The first result of this section is the fact that the category of $(C,\leqslant)$-pospaces is a P-category in the sense of the following definition:

Definition 5.1. [**1**, I.3a] A category $\mathbf{C}$ equipped with a class of morphisms, called *fibrations* (indicated by $\twoheadrightarrow$), and a *path object functor* $P : \mathbf{C} \to \mathbf{C}$, $X \mapsto X^I$, $f \mapsto f^I$ is said to be a *P-category* if it has a final object $*$ and if the following axioms are satisfied:

P1 There are natural transformations $q_0, q_1 : P \to id_{\mathbf{C}}$, $c : id_{\mathbf{C}} \to P$ such that $q_0 \circ c = q_1 \circ c = id$.

P2 The pullback of two morphisms one of which is a fibration exists. The functor P carries such a pullback into a pullback and preserves the final object. The fibrations are closed under base change.

P3 The composite of two fibrations is a fibration. Every isomorphism is a fibration and every final morphism $X \to *$ is a fibration. Every fibration $p : E \to B$ has the *homotopy lifting property*, i.e., given morphisms $h : X \to B^I$ and $f : X \to E$ such that $p \circ f = q_\tau \circ h$ ($\tau = 0$ or $\tau = 1$), there exists a morphism $H : X \to E^I$ such that $q_\tau \circ H = f$ and $p^I \circ H = h$.

P4 For every fibration $p : E \to B$ the morphism

$$(q_0, q_1, p^I) : E^I \to (E \times E) \times_{B \times B} B^I$$

is a fibration. Here, the target object is the fibered product of the morphisms $p \times p$ and $(q_0, q_1) : B^I \to B \times B$.

P5 For each object X there exists a morphism $T : (X^I)^I \to (X^I)^I$ such that $q_\tau^I \circ T = q_\tau$ and $q_\tau \circ T = q_\tau^I$ ($\tau = 0, 1$).

Theorem 5.2. *Let $(C, \leqslant)$ be a pospace. The category $(C, \leqslant)$-**poTop** is a P-category. The fibrations are the $(C, \leqslant)$-difibrations and the functor P is the path $(C, \leqslant)$-pospace functor.*

Proof. The natural transformations q_0 and q_1 are the evaluation maps ev_0 and ev_1. The natural transformation $c : (X, \leqslant, \xi) \to (X^I, \leqslant, \mathsf{c}_\xi)$ is given by $c(x) = \mathsf{c}_x$. By 2.4, $(C, \leqslant)$-**poTop** is complete. Since P has a left adjoint (cf. 3.8), it preserves all limits. By 4.3, the class of $(C, \leqslant)$-difibrations contains all isomorphisms and is closed under base change and composition. By 4.5, every final morphism is a $(C, \leqslant)$-difibration. Using the adjunction between P and the cylinder functor (cf. 3.8) one easily sees that the homotopy lifting property is equivalent to the defining property of $(C, \leqslant)$-difibrations. For a $(C, \leqslant)$-pospace $(X, \leqslant, \xi)$ the $(C, \leqslant)$-dimap $T : ((X^I)^I, \leqslant, \mathsf{c}\mathsf{c}_\xi) \to ((X^I)^I, \leqslant, \mathsf{c}\mathsf{c}_\xi)$ is given by $T(\omega)(s)(t) = \omega(t)(s)$.

It remains to check P4. Let $p : (E, \leqslant, \varepsilon) \to (B, \leqslant, \beta)$ be a $(C, \leqslant)$-difibration. We have to show that the $(C, \leqslant)$-dimap

$$(ev_0, ev_1, p^I) : (E^I, \leqslant, \mathsf{c}_\varepsilon) \to ((E \times E) \times_{B \times B} B^I, \leqslant, (\varepsilon, \varepsilon, \mathsf{c}_\beta))$$

is a $(C, \leqslant)$-difibration. Consider a $(C, \leqslant)$-dimap $f : (X, \leqslant, \xi) \to (E^I, \leqslant, \mathsf{c}_\varepsilon)$ and a dimap $F : (X, \leqslant) \times (I, \Delta) \to ((E \times E) \times_{B \times B} B^I, \leqslant)$ such that $((ev_0, ev_1, p^I) \circ f)(x) = F(x, 0)$ and $F(\xi(c), t) = (\varepsilon(c), \varepsilon(c), \mathsf{c}_{\beta(c)})$. Write $F = (F_0, F_1, F_2)$ and consider the following commutative diagram of spaces where j is the obvious inclusion and ϕ and G are given by $\phi(x, t, 0) = f(x)(t)$, $\phi(x, 0, s) = F_0(x, s)$, $\phi(x, 1, s) = F_1(x, s)$, and $G(x,t,s) = F_2(x,s)(t)$:

$$\begin{tikzcd} X \times (I \times \{0\} \cup \{0,1\} \times I) \arrow[r, "\phi"] \arrow[d, "id_X \times j"'] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "G"'] & B. \end{tikzcd}$$

Let $(x,t,s),(x',t',s') \in X \times (I \times \{0\} \cup \{0,1\} \times I)$ such that $(x,t,s) \leqslant (x',t',s')$ in $(X,\leqslant) \times (I \times \{0\} \cup \{0,1\} \times I, \Delta)$. Then $x \leqslant x'$, $t = t'$, and $s = s'$. It follows that $s = 0 \Rightarrow s' = 0$, $t = 0 \Rightarrow t' = 0$, and $t = 1 \Rightarrow t' = 1$. Since f and F are dimaps, we obtain that $\phi(x,t,s) \leqslant \phi(x',t',s')$ and hence that ϕ is a dimap $(X,\leqslant) \times (I \times \{0\} \cup \{0,1\} \times I, \Delta) \to (E,\leqslant)$. Moreover, $\phi(\xi(c),t,s) = \varepsilon(c)$. Since F_2 is a dimap, G is a dimap $(X,\leqslant) \times (I \times I, \Delta) \to (B,\leqslant)$. Moreover, $G(\xi(c),t,s) = \beta(c)$. Since j is a trivial cofibration in **Top**, there exists, by 4.7, a dimap $H : (X,\leqslant) \times (I \times I, \Delta) \to (E,\leqslant)$ such that $p \circ H = G$, $H \circ (id_X \times j) = \phi$, and $H(\xi(c),t,s) = \varepsilon(c)$. Consider the dimap $\lambda : (X,\leqslant) \times (I,\Delta) \to (E^I,\leqslant)$ defined by $\lambda(x,s)(t) = H(x,t,s)$. We have $(ev_0, ev_1, p^I) \circ \lambda = F$, $\lambda(x,0) = f(x)$, and $\lambda(\xi(c),s) = \mathsf{c}_{\varepsilon(c)}$. This shows that (ev_0, ev_1, p^I) is a $(C,\leqslant)$-difibration. □

Definition 5.3. [**1**, I.3a] Let **C** be a P-category. Two morphisms $f, g : X \to Y$ are said to be *homotopic*, $f \simeq g$, if there exists a morphism $h : X \to Y^I$ such that $q_0 \circ h = f$ and $q_1 \circ h = g$. A morphism $f : X \to Y$ is called a *homotopy equivalence* if there exists a morphism $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

By 3.9, two $(C,\leqslant)$-dimaps are homotopic in the P-category $(C,\leqslant)$-**poTop** if and only if they are dihomotopic relative to $(C,\leqslant)$. A $(C,\leqslant)$-dimap is a homotopy equivalence in the P-category $(C,\leqslant)$-**poTop** if and only if it is a dihomotopy equivalence relative to $(C,\leqslant)$.

The main result of the homotopy theory of a P-category is that it is a fibration category (cf. [**1**, I.3a.4]). There is an extensive homotopy theory available for fibration categories (cf. [**1**]).

Definition 5.4. [**1**, I.1a] A category **F** equipped with two classes of morphisms, *weak equivalences* (denoted by $\xrightarrow{\sim}$) and *fibrations* ($\twoheadrightarrow$), is a *fibration category* if it has a final object $*$ and if the following axioms are satisfied:

- F1 An isomorphism is a *trivial fibration*, i.e., a morphism which is both a fibration and a weak equivalence. The composite of two fibrations is a fibration. If two of the morphisms $f : X \to Y$, $g : Y \to Z$, and $g \circ f : X \to Z$ are weak equivalences, so is the third.
- F2 The pullback of two morphisms one of which is a fibration exists. The fibrations and trivial fibrations are stable under base change. The base extension of a weak equivalence along a fibration is a weak equivalence.
- F3 Every morphism f admits a factorization $f = p \circ j$ where p is a fibration and j is weak equivalence.

F4 For each object X there exists a trivial fibration $Y \to X$ such that Y is *cofibrant*, i.e., every trivial fibration $E \to Y$ admits a section.

An object X is said to be $*$-*fibrant* if the final morphism $X \to *$ is a fibration.

Note that in [**1**] a fibration category is not required to have a final object.

Theorem 5.5. *Let $(C, \leqslant)$ be a pospace. The category $(C, \leqslant)$-**poTop** of $(C, \leqslant)$-pospaces is a fibration category. The weak equivalences are the dihomotopy equivalences relative to $(C, \leqslant)$ and the fibrations are the $(C, \leqslant)$-difibrations. All objects are $(*, \Delta, *)$-fibrant and cofibrant.*

Proof. This follows from [**1**, I.3a.4] and 5.2. □

Remark 5.6. Let **C** be a P-category. By F3, every morphism $f : X \to Y$ admits a factorization $f = p \circ j$ where $p : W \to Y$ is a fibration and $j : X \to W$ is a homotopy equivalence. By [**1**, I.3a], the object W can be chosen to be the *mapping path object* of f, i.e., the fibered product $W = X \times_Y Y^I$ of the morphisms f and q_0. The homotopy equivalence j is then the morphism $(id_X, c \circ f)$ and the fibration p is the composite $q_1 \circ pr_{Y^I}$.

Definition 5.7. [**1**, I.1a] Let **F** be a fibration category, $p : E \twoheadrightarrow B$ be a fibration, and X be a cofibrant object. Two morphisms $f, g : X \to E$ satisfying $p \circ f = p \circ g$ are said to be *homotopic over* B if for some factorization of the morphism $(id_E, id_E) \colon E \to E \times_B E$ into a weak equivalence $E \xrightarrow{\sim} P$ and a fibration $q : P \twoheadrightarrow E \times_B E$ there exists a morphism $h : X \to P$ such that $q \circ h = (f, g)$. If $B = *$, one simply speaks of *homotopic* morphisms, and *homotopy equivalences* are defined in the obvious way.

Proposition 5.8. *Let **C** be a P-category and $p : E \twoheadrightarrow B$ be a fibration. Two morphisms $f, g : X \to E$ satisfying $p \circ f = p \circ g$ are homotopic over B in the fibration category **C** if and only if there exists a morphism $h : X \to E^I$ such that $q_0 \circ h = f$, $q_1 \circ h = g$ and $p^I \circ h = c \circ p \circ f = c \circ p \circ g$.*

Proof. We first construct a factorization of the morphism $(id_E, id_E) \colon E \to E \times_B E$ into a weak equivalence and a fibration. Consider the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "c"] \arrow[d, "id_B"'] & B^I \arrow[d, "(q_0,q_1)"] & E^I \arrow[l, "p^I"'] \arrow[d, "(q_0,q_1)"] \\ B \arrow[r, "(id_B,id_B)"'] & B \times B & E \times E. \arrow[l, "p\times p"] \end{tikzcd}$$

By the dual of the gluing lemma [**1**, II.1.2], $p \times p$ is a fibration. By P4 and P2, p^I is a fibration. We can therefore form the pullbacks of the horizontal lines of the above diagram. Applying P4 to the final morphisms $E \twoheadrightarrow *$ and $B \twoheadrightarrow *$ we obtain that the vertical morphisms are fibrations. Since, again by P4, $(q_0, q_1, p^I) : E^I \to (E \times E) \times_{B\times B} B^I$ is a fibration, we may apply the dual of the gluing lemma to deduce that the morphism

$$id_B \times_{(q_0,q_1)} (q_0, q_1) : B \times_{B^I} E^I \to B \times_{B\times B} (E \times E) = E \times_B E$$

is a fibration. Consider now the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "id_B"] \arrow[d, "id_B"'] & B \arrow[d, "c"] & E \arrow[l, two heads, "p"'] \arrow[d, "c"] \\ B \arrow[r, "c"'] & B^I & E^I. \arrow[l, two heads, "p^I"] \end{tikzcd}$$

For any object X the natural morphism $c : X \to X^I$ is the weak equivalence of the mapping path factorization of id_X. Therefore the vertical morphisms in the diagram are weak equivalences and we may apply the dual of the gluing lemma to deduce that the morphism

$$id_B \times_c c : E = B \times_B E \to B \times_{B^I} E^I$$

is a weak equivalence. The composite

$$(id_B \times_{(q_0,q_1)} (q_0, q_1)) \circ (id_B \times_c c) : B \times_B E \to B \times_{B \times B} (E \times E)$$

is precisely the morphism $(id_E, id_E) : E \to E \times_B E$.

Let $f, g : X \to E$ be two morphisms such that $p \circ f = p \circ g$. By the dual of [**1**, II.2.2], we may replace the word "some" in Definition 5.7 by "any". It follows that f and g are homotopic over B if and only if there exists a morphism $H : X \to B \times_{B^I} E^I$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "(f,g)"'] & B \times_{B^I} E^I \arrow[d, "id_B \times_{(q_0,q_1)} (q_0,q_1)"] \\ E \times_B E \arrow[r, "id"'] & B \times_{B \times B} (E \times E). \end{tikzcd}$$

This is the case if and only if there exists a morphism $h : X \to E^I$ such that $q_0 \circ h = f$, $q_1 \circ h = g$ and $p^I \circ h = c \circ p \circ f = c \circ p \circ g$. The correspondance between H and h is given by $H = (p \circ f, h)$. □

Proposition 5.9. *Let $(C, \leqslant)$ be a pospace, $p : (E, \leqslant, \varepsilon) \to (B, \leqslant, \beta)$ be a $(C, \leqslant)$-difibration, and $f, g : (X, \leqslant, \xi) \to (E, \leqslant, \varepsilon)$ be two $(C, \leqslant)$-dimaps such that $p \circ f = p \circ g$. Then f and g are homotopic over $(B, \leqslant)$ in the fibration category $(C, \leqslant)$-***poTop** *if and only if there exists a dihomotopy relative to $(C, \leqslant)$ $H : (X, \leqslant) \times (I, \Delta) \to (E, \leqslant)$ from f to g such that $p(H(x, s)) = p(f(x)) = p(g(x))$ $(x \in X,\ s \in I)$. In particular, two $(C, \leqslant)$-dimaps are homotopic in the fibration category $(C, \leqslant)$-***poTop** *if and only if they are dihomotopic relative to $(C, \leqslant)$ and a $(C, \leqslant)$-dimap is a homotopy equivalence in the fibration category $(C, \leqslant)$-***poTop** *if and only if it is a dihomotopy equivalence relative to $(C, \leqslant)$.*

Proof. This follows from 5.8. □

6. Cofibrations in a fibration category

Throughout this section we work in a fibration category **F**. We suppose that all objects are cofibrant and $*$-fibrant and that a morphism is a weak equivalence if and only if it is a homotopy equivalence. By 5.5 and 5.9, the category of $(C, \leqslant)$-pospaces satisfies these hypotheses.

Definition 6.1. A *cofibration* is a morphism having the left lifting property with respect to the trivial fibrations. Cofibrations will be indicated by $\rightarrowtail$.

For general reasons we have

Proposition 6.2. *The class of cofibrations is closed under composition, retracts, and cobase change. Every isomorphism is a cofibration.*

Definition 6.3. A *trivial cofibration* is a cofibration which is also a weak equivalence.

Proposition 6.4. *A morphism is a trivial cofibration if and only if it has the left lifting property with respect to the fibrations.*

Proof. Let $i : A \to X$ be a morphism. Suppose first that i has the left lifting property with respect to the fibrations. Then i is a cofibration. Choose a factorization $i = p \circ h$ where $h : A \xrightarrow{\sim} E$ is a weak equivalence and $p : E \twoheadrightarrow X$ is a fibration. Thanks to our hypothesis there exists a morphism $\lambda : X \to E$ such that $p \circ \lambda = id_X$ and $\lambda \circ i = h$. The weak equivalence h is a homotopy equivalence. Let g be a homotopy inverse of h. We have $g \circ \lambda \circ i = g \circ h \simeq id_A$ and $i \circ g \circ \lambda = p \circ h \circ g \circ \lambda \simeq p \circ \lambda = id_X$. Thus, i is a homotopy equivalence. Thanks to our general hypothesis, i is a weak equivalence.

Now suppose that i is a trivial cofibration and consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"', "\sim"] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

where p is a fibration. Form the pullback

$$\begin{tikzcd} X \times_B E \arrow[r, "pr_E"] \arrow[d, two heads, "pr_X"'] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

and choose a factorization of the induced morphism $(i, f) : A \to X \times_B E$ into a weak equivalence $h : A \xrightarrow{\sim} Y$ and a fibration $q : Y \twoheadrightarrow X \times_B E$. Since fibrations are stable under base change and composition, $pr_X \circ q$ is a fibration. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "h", "\sim"'] \arrow[d, tail, "i"', "\sim"] & Y \arrow[d, "pr_X \circ q"] \\ X \arrow[r, "id_X"'] & X. \end{tikzcd}$$

By F1, $pr_X \circ q$ is a weak equivalence. Since i is a cofibration, there exists a morphism $\lambda : X \to Y$ such that $\lambda \circ i = h$ and $pr_X \circ q \circ \lambda = id_X$. We have $(pr_E \circ q \circ \lambda) \circ i = pr_E \circ q \circ h = f$ and $p \circ (pr_E \circ q \circ \lambda) = g \circ pr_X \circ q \circ \lambda = g$. This shows that i has the required lifting property. □

Corollary 6.5. *The class of trivial cofibrations is closed under cobase change, composition, and retracts. Every isomorphism is a trivial cofibration.*

Proposition 6.6. *Suppose that* **F** *has an initial object* $\emptyset$*. For each object* X *the initial morphism* $\emptyset \to X$ *is a cofibration.*

Proof. Let X be any object. Consider a commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & E \arrow[d, "\sim"', "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

where p is a trivial fibration. Form the pullback

$$\begin{tikzcd} X \times_B E \arrow[r, "pr_E"] \arrow[d, "pr_X"', "\sim"] & E \arrow[d, "\sim"', "p"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

By F2, pr_X is a trivial fibration. Since X is cofibrant, pr_X admits a section s. We have $p \circ pr_E \circ s = f$. This implies that the morphism $\emptyset \to X$ is a cofibration. □

Definition 6.7. [**1**, I.1] A category **C** equipped with two classes of morphisms, *weak equivalences* ($\xrightarrow{\sim}$) and *cofibrations* ($\rightarrowtail$), is a *cofibration category* if it has an initial object $\emptyset$ and if axioms C1, C2, C3, C4 dual to the axioms of a fibration category are satisfied. An object X is said to be $\emptyset$-*cofibrant* if the initial morphism $\emptyset \to X$ is a cofibration.

The concept of a cofibration category is formally dual to the one of a fibration category. For every concept or result concerning fibration categories there is a dual concept or result for cofibration categories and vice versa. In particular, there exists a notion of homotopy for cofibration categories which is dual to the notion of homotopy in fibration categories (cf. 5.7).

Theorem 6.8. *Suppose that* $\mathbf{F}$ *has an initial object* $\emptyset$*, that the pushout of two morphisms one of which is a cofibration exists, and that for every object* X *the morphism* $(id_X, id_X) : X \coprod X \to X$ *admits a factorization into a cofibration followed by a weak equivalence. Then* $\mathbf{F}$ *is a cofibration category. All objects are* $\emptyset$*-cofibrant and fibrant.*

Proof. C1 follows from 6.5, 6.2, and F1. By 6.6 and since all objects are $*$-fibrant, all objects are $\emptyset$-cofibrant. By 6.4, all objects are fibrant and C4 holds. We next prove C3. Let $f : X \to Y$ be a morphism. Choose a factorization of the morphism $(id_X, id_X) : X \coprod X \to X$ into a cofibration $j : X \coprod X \rightarrowtail IX$ and a weak equivalence $p : IX \xrightarrow{\sim} X$. Note that $X \coprod X$ exists by assumption since all objects are $\emptyset$-cofibrant. Denote the canonical morphisms $X \to X \coprod X$ by i_0 and i_1. Since all objects are $\emptyset$-cofibrant, by 6.2, i_0 and i_1 are cofibrations. By 6.2 and C1, both composites $j \circ i_0$ and $j \circ i_1$ are trivial cofibrations. Form the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "j\circ i_1"', "\sim"] & Y \arrow[d, tail, "\iota", "\sim"'] \\ IX \arrow[r, "\bar{f}"'] & Z. \end{tikzcd}$$

By 6.5, ι is a trivial cofibration. Let $r : Z \to Y$ be the morphism induced by the morphisms $f \circ p : IX \to Y$ and id_Y. Since $r \circ \iota = id_Y$ and ι and id_Y are weak equivalences, r is a weak equivalence. Let i be the composite of the morphisms $j \circ i_0 : X \to IX$ and $\bar{f} : IX \to Z$. Consider the following pushout diagram:

$$\begin{tikzcd} X \coprod X \arrow[r, "id_X \coprod f"] \arrow[d, tail, "j"'] & X \coprod Y \arrow[d, tail, "(i,\iota)"] \\ IX \arrow[r, "\bar{f}"'] & Z. \end{tikzcd}$$

Since j is a cofibration, (i, ι) is a cofibration. Since the initial morphism $\emptyset \to Y$ is a cofibration and cofibrations are closed under cobase change (cf. 6.2), the canonical morphism $\phi : X \to X \coprod Y$ is a cofibration. Since the composite of cofibrations is a cofibration (cf. 6.2) and $i = (i, \iota) \circ \phi$, i is a cofibration. We have $r \circ i = r \circ \bar{f} \circ j \circ i_0 = f \circ p \circ j \circ i_0 = f$. This shows that C3 holds. C2 follows from 6.2, 6.5, C1, C3, and [**1**, I.1.4]. □

Remark 6.9. The factorization $f = r \circ i$ constructed in the above proof is the *mapping cylinder factorization* of f which is dual to the mapping path factorization of 5.6.

Proposition 6.10. *Under the assumptions of Theorem 6.8, two morphisms of* $\mathbf{F}$ *are homotopic in the cofibration category* $\mathbf{F}$ *if and only if they are homotopic in the fibration category* $\mathbf{F}$*.*

Proof. Let $\simeq$ denote the homotopy relation of the fibration category $\mathbf{F}$ and $\sim$ denote the homotopy relation of the cofibration category $\mathbf{F}$. Both relations are natural equivalence relations (c.f. [**1**, II.3.2]) and we can form the quotient categories $\mathbf{F}/\simeq$ and $\mathbf{F}/\sim$. By [**1**, II.3.6], both quotient categories have the universal property of the localization of $\mathbf{F}$ with respect to the weak equivalences. This implies that there is an isomorphism of categories $\mathbf{F}/\simeq \to \mathbf{F}/\sim$ which is the identity on objects and which sends the $\simeq$-class of a morphism to its $\sim$-class. The result follows. □

7. $(C, \leqslant)$-dicofibrations

Definition 7.1. Let $(C, \leqslant)$ be a pospace. A $(C, \leqslant)$-*dicofibration* is a $(C, \leqslant)$-dimap having the left lifting property with respect to the trivial $(C, \leqslant)$-difibrations. If $C = \emptyset$ we simply talk of dicofibrations.

Theorem 7.2. *Let $(C, \leqslant)$ be a pospace. The category $(C, \leqslant)$-**poTop** is a cofibration category. The cofibrations are the $(C, \leqslant)$-dicofibrations and the weak equivalences are the dihomotopy equivalences relative to $(C, \leqslant)$. All objects are fibrant and $(C, \leqslant, id_C)$-cofibrant. Two $(C, \leqslant)$-dimaps are homotopic in the cofibration category $(C, \leqslant)$-**poTop** if and only if they are dihomotopic relative to $(C, \leqslant)$.*

Proof. Thanks to 2.4, 5.5, 5.9, 6.8, and 6.10 it is enough to show that for every $(C, \leqslant)$-pospace $(X, \leqslant, \xi)$ the $(C, \leqslant)$-dimap $(id_X, id_X)\colon (X, \leqslant, \xi) \coprod (X, \leqslant, \xi) \to (X, \leqslant, \xi)$ admits a factorization into a $(C, \leqslant)$-dicofibration and a dihomotopy equivalence relative to $(C, \leqslant)$. Let $(X, \leqslant, \xi)$ be a $(C, \leqslant)$-pospace. We have

$$(X, \leqslant, \xi) \coprod (X, \leqslant, \xi) = (X, \leqslant, \xi) \square_{(C, \leqslant)} \{0, 1\}$$

and (id_X, id_X) is the $(C, \leqslant)$-dimap $(X, \leqslant, \xi) \square_{(C, \leqslant)} \{0, 1\} \to (X, \leqslant, \xi)$, $[x, t] \mapsto x$. Let $\iota : \{0, 1\} \hookrightarrow I$ be the inclusion. We show that

$$(X, \leqslant, \xi) \square_{(C, \leqslant)} \iota : (X, \leqslant, \xi) \square_{(C, \leqslant)} \{0, 1\} \to (X, \leqslant, \xi) \square_{(C, \leqslant)} I$$

is a $(C, \leqslant)$-dicofibration and that the projection $r : (X, \leqslant, \xi) \square_{(C, \leqslant)} I \to (X, \leqslant, \xi)$, $r([x, t]) = x$ is a dihomotopy equivalence relative to $(C, \leqslant)$. The $(C, \leqslant)$-dimap $\sigma\colon (X, \leqslant, \xi) \to (X, \leqslant, \xi) \square_{(C, \leqslant)} I$ given by $\sigma(x) = [x, 0]$ is a dihomotopy inverse relative to $(C, \leqslant)$ of r. Indeed, $r \circ \sigma = id_X$ and a dihomotopy relative to $(C, \leqslant)$ from $\sigma \circ r$ to $id_{X \square_C I}$ is given by $F([x, t], s) = [x, st]$.

We now show that $(X, \leqslant, \xi) \square_{(C, \leqslant)} \iota$ is a $(C, \leqslant)$-dicofibration. Consider a commutative diagram of $(C, \leqslant)$-pospaces

$$\begin{tikzcd}
(X, \leqslant, \xi) \square_{(C, \leqslant)} \{0, 1\} \arrow[r, "f"] \arrow[d, "X \square_C \iota"'] & (E, \leqslant, \varepsilon) \arrow[d, "\sim"', "p"] \\
(X, \leqslant, \xi) \square_{(C, \leqslant)} I \arrow[r, "g"'] & (B, \leqslant, \beta)
\end{tikzcd}$$

where p is a trivial $(C, \leqslant)$-difibration. By the dual of the lifting lemma [**1**, II.1.11], there exists a section s of p such that $s \circ p$ is homotopic to $id_{(E, \leqslant, \varepsilon)}$ over $(B, \leqslant, \beta)$ in the fibration category $(C, \leqslant)$-**poTop**. By 5.9, there exists a dihomotopy relative to $(C, \leqslant)$ $H : (E, \leqslant) \times (I, \Delta) \to (E, \leqslant)$ from $s \circ p$ to $id_{(E, \leqslant, \varepsilon)}$ such that $p(H(x,\tau)) = p(x)$. Consider the following commutative diagram of spaces where j is the obvious inclusion and ϕ and Φ are given by $\phi(x,t,0) = s(g([x,t]))$, $\phi(x,0,\tau) = H(f([x,0]),\tau)$, $\phi(x,1,\tau) = H(f([x,1]),\tau)$, and $\Phi(x,t,\tau) = g([x,t])$:

$$\begin{CD} X \times (I \times \{0\} \cup \{0,1\} \times I) @>{\phi}>> E \\ @V{id_X \times j}VV @VV{p}V \\ X \times I \times I @>>{\Phi}> B. \end{CD}$$

Since g is a dimap, Φ is a dimap $(X,\leqslant) \times (I \times I, \Delta) \to (B,\leqslant)$. Moreover,

$$\Phi(\xi(c),t,\tau) = g(\bar{\xi}(c)) = \beta(c).$$

Let (x,t,τ), $(x',t',\tau') \in X \times (I \times \{0\} \cup \{0,1\} \times I)$ such that $(x,t,\tau) \leqslant (x',t',\tau')$ in $(X,\leqslant) \times (I \times \{0\} \cup \{0,1\} \times I, \Delta)$. Then $x \leqslant x'$, $t = t'$, and $\tau = \tau'$. It follows that $t = 0 \Rightarrow t' = 0$, $t = 1 \Rightarrow t' = 1$, and $\tau = 0 \Rightarrow \tau' = 0$. We obtain that $\phi(x,t,\tau) \leqslant \phi(x',t',\tau')$. Thus ϕ is a dimap $(X,\leqslant)\times(I\times\{0\}\cup\{0,1\}\times I,\Delta) \to (E,\leqslant)$. Moreover, $\phi(\xi(c),t,\tau) = \varepsilon(c)$. Since j is a trivial cofibration of spaces, there exists, by 4.7, a dimap $G : (X,\leqslant)\times(I\times I,\Delta) \to (E,\leqslant)$ such that $G\circ(id_X \times j) = \phi$, $p\circ G = \Phi$, and $G(\xi(c),t,\tau) = \varepsilon(c)$. Let $\lambda : (X,\leqslant,\xi)\Box_{(C,\leqslant)} I \to (E,\leqslant,\varepsilon)$ be the $(C,\leqslant)$-dimap given by $\lambda([x,t]) = G(x,t,1)$. We have $\lambda([x,0]) = G(x,0,1) = \phi(x,0,1) = H(f([x,0]),1) = f([x,0])$, $\lambda([x,1]) = G(x,1,1) = \phi(x,1,1) = H(f([x,1]),1) = f([x,1])$, and $(p\circ\lambda)([x,t]) = (p\circ G)(x,t,1) = \Phi(x,t,1) = g([x,t])$. This shows that $(X,\leqslant,\xi)\Box_{(C,\leqslant)}\iota$ is a $(C,\leqslant)$-dicofibration. □

Remark 7.3. Note that many interesting inclusions of pospaces are not dicofibrations. For example, the inclusion $i\colon (\{0,1\},\leqslant) \hookrightarrow (I,\leqslant)$ where $\leqslant$ is the natural order is not a dicofibration. Indeed, consider the subpospace $(Z,\leqslant)$ of $(I,\leqslant)\times(I,\Delta)$ given by $Z = I \times \{0\} \cup \{0,1\} \times I$. It is clear that the projection $r\colon (Z,\leqslant) \to (I,\leqslant)$ is a dihomotopy equivalence. Form the following commutative diagram where j is the inclusion given by $j(0) = (0,1)$ and $j(1) = (1,1)$:

$$\begin{CD} (\{0,1\},\leqslant) @>{j}>> (Z,\leqslant) \\ @V{i}VV @V{\sim}V{r}V \\ (I,\leqslant) @>{=}>> (I,\leqslant). \end{CD}$$

If i was a dicofibration we could apply the lifting lemma [**1**, II.1.11] to obtain a dimap $\lambda\colon (I,\leqslant) \to (Z,\leqslant)$ such that $\lambda \circ i = j$. It is clear that such a dimap cannot exist. Therefore i is not a dicofibration. Note, however, that the inclusion $i\colon (\{0,1\},\leqslant,id) \hookrightarrow (I,\leqslant,i)$ is a $(\{0,1\},\leqslant)$-dicofibration. Indeed, by 7.2, all $(\{0,1\},\leqslant)$-pospaces are $(\{0,1\},\leqslant,id)$-cofibrant.

8. The closed model category of pospaces

In this section we show that absolute pospaces form a closed model category. The axioms of closed model categories can for instance be found in the book [**7**] by P. Goerss and J.F. Jardine.

It is well-known that inclusions of DR-pairs are trivial cofibrations of spaces. We shall need the corresponding result for dimaps:

Lemma 8.1. *Let $i : (A, \leqslant) \to (X, \leqslant)$ be an inclusion of pospaces such that there exists a dihomotopy $H : (X, \leqslant) \times (I, \Delta) \to (X, \leqslant)$ and a dimap $\phi : (X, \leqslant) \to (I, \Delta)$ such that $A = \phi^{-1}(0)$, $H(x, 1) = x$ $(x \in X)$, $H(a, t) = a$ $(a \in A,\ t \in I)$, and $H(x, 0) \in A$ $(x \in X)$. Then i is a trivial dicofibration.*

Proof. The proof is similar to the one of 4.7. Consider a difibration $p : (E, \leqslant) \twoheadrightarrow (B, \leqslant)$, a dimap $f : (A, \leqslant) \to (E, \leqslant)$, and a dimap $g : (X, \leqslant) \to (B, \leqslant)$ such that $g(a) = p(f(a))$. Write $r(x) = H(x, 0)$. Consider the map $G : X \times I \to X$ defined by

$$G(x,t) = \begin{cases} H(x, \frac{t}{\phi(x)}) & t < \phi(x), \\ x & t \geqslant \phi(x). \end{cases}$$

As in [**13**, I.7.15] one can show that G is continuous. We have $G(x, 0) = (i \circ r)(x)$ for all $x \in X$. Let $(x, t), (x', t') \in X \times I$ such that $(x, t) \leqslant (x', t')$. Then $x \leqslant x'$ and $t = t'$. Since ϕ is a dimap, $\phi(x) \leqslant \phi(x')$ and hence $\phi(x) = \phi(x')$. It follows that $G(x, t) \leqslant G(x', t')$ and hence that G is a dimap. Consider the following commutative diagram of pospaces:

$$\begin{tikzcd} (X,\leqslant) \arrow[r, "f\circ r"] \arrow[d, "i_0"'] & (E,\leqslant) \arrow[d, two heads, "p"] \\ (X,\leqslant)\times(I,\Delta) \arrow[r, "g\circ G"'] & (B,\leqslant). \end{tikzcd}$$

Since p is a difibration, there exists a dimap $F : (X, \leqslant) \times (I, \Delta) \to (E, \leqslant)$ such that $F \circ i_0 = f \circ r$ and $p \circ F = g \circ G$. Consider the dimap $\lambda : (X, \leqslant) \to (E, \leqslant)$ defined by $\lambda(x) = F(x, \phi(x))$. We have $(p \circ \lambda)(x) = p(F(x, \phi(x))) = g(G(x, \phi(x))) = g(x)$ and $\lambda(a) = F(a, \phi(a)) = F(a, 0) = f(r(a)) = f(a)$. By 6.4, this shows that i is a trivial dicofibration. □

Theorem 8.2. *The category* **poTop** *of pospaces is a closed model category where weak equivalences are dihomotopy equivalences, fibrations are difibrations, and cofibrations are dicofibrations. All pospaces are fibrant and cofibrant. Two dimaps are homotopic in the closed model category* **poTop** *if and only if they are dihomotopic.*

Proof. By 2.2, **poTop** is complete and cocomplete. The “2=3” property is part of 3.4. The retract axiom follows from 3.4, 4.3, and 6.2. The lifting axiom follows from the definition of dicofibrations and 6.4. We now show the factorization axiom. Let $f : (X, \leqslant) \to (Y, \leqslant)$ be a dimap. We show first that f admits a factorization $f = p \circ i$ where p is a fibration and i is a trivial cofibration. We proceed as in [**12**]. Consider the mapping path factorization $f = q \circ j$ of 5.6; $j : (X, \leqslant) \to (X \times_Y Y^I, \leqslant)$ is the dihomotopy equivalence given by $j(x) = (x, \mathsf{c}_{f(x)})$ and q is the difibration $(X \times_Y Y^I, \leqslant) \to (Y, \leqslant)$, $(x, \omega) \mapsto \omega(1)$. Let $(E, \leqslant)$ be the subpospace of $(X \times_Y Y^I, \leqslant) \times (I, \Delta)$ defined by $E = j(X) \times I \cup (X \times_Y Y^I) \times]0, 1]$. Let $i : (X, \leqslant) \to (E, \leqslant)$ be the dimap defined by $i(x) = (j(x), 0)$. Let $p : (E, \leqslant) \to (Y, \leqslant)$ be the composite

$$(E, \leqslant) \hookrightarrow (X \times_Y Y^I, \leqslant) \times (I, \Delta) \overset{pr_{X \times_Y Y^I}}{\longrightarrow} (X \times_Y Y^I, \leqslant) \xrightarrow{q} (Y, \leqslant).$$

We have $p \circ i = f$. We show that i is a trivial dicofibration. Consider the dimap $H \colon (E, \leqslant) \times (I, \Delta) \to (E, \leqslant)$ defined by $H(x, \omega, s, t) = (x, \omega_t, st)$ where $\omega_t(\tau) = \omega(t\tau)$. We have $H(x, \omega, s, 1) = (x, \omega, s)$, $H(x, \mathsf{c}_{f(x)}, 0, t) = (x, \mathsf{c}_{f(x)}, 0)$, and $H(x, \omega, s, 0) = (x, \mathsf{c}_{f(x)}, 0) \in i(X)$. Moreover, $i(X) = \phi^{-1}(0)$ where $\phi : (E, \leqslant) \to (I, \Delta)$ is the dimap $(x, \omega, t) \mapsto t$. By 8.1, the inclusion $(i(X), \leqslant) \hookrightarrow (E, \leqslant)$ is a trivial dicofibration. The dimap $i : (X, \leqslant) \to (i(X), \leqslant)$ is an isomorphism of pospaces; the inverse is the composite

$$(i(X), \leqslant) \hookrightarrow (E, \leqslant) \hookrightarrow (X \times_Y Y^I, \leqslant) \times (I, \Delta) \overset{pr_{X \times_Y Y^I}}{\longrightarrow} (X \times_Y Y^I, \leqslant) \overset{pr_X}{\to} (X, \leqslant).$$

By 6.5, it follows that i is a trivial dicofibration.

We now show that p is a difibration. Since the composite of two difibrations is a difibration and q is a difibration, it suffices to show that the composite

$$\pi : (E, \leqslant) \hookrightarrow (X \times_Y Y^I, \leqslant) \times (I, \Delta) \overset{pr_{X \times_Y Y^I}}{\longrightarrow} (X \times_Y Y^I, \leqslant)$$

is a difibration. Consider a commutative diagram of pospaces

$$\begin{tikzcd}
(Z, \leqslant) \arrow[r, "g"] \arrow[d, "i_0"'] & (E, \leqslant) \arrow[d, "\pi"] \\
(Z, \leqslant) \times (I, \Delta) \arrow[r, "G"'] & (X \times_Y Y^I, \leqslant).
\end{tikzcd}$$

Consider the continuous map $F : Z \times I \to E$ defined by

$$F(z, t) = (G(z, t), t + (1 - t)\phi(g(z))).$$

If $t + (1 - t)\phi(g(z)) = 0$ then $t = 0$ and $\phi(g(z)) = 0$. Therefore $g(z) \in i(X)$ and

$$F(z, t) = (G(z, 0), 0) = (\pi(g(z)), 0) = g(z) \in i(X) \subset E.$$

This shows that F is well-defined. Let $(z, t) \leqslant (z', t') \in Z \times I$. Then $z \leqslant z'$ and $t = t'$. Since ϕ and g are dimaps, $\phi(g(z)) \leqslant \phi(g(z'))$, i.e., $\phi(g(z)) = \phi(g(z'))$. It follows that $F(z, t) \leqslant F(z', t')$ and hence that F is a dimap. We have

$$F(z, 0) = (G(z, 0), \phi(g(z))) = (\pi(g(z)), \phi(g(z))) = g(z)$$

and $\pi \circ F = G$. It follows that π is a difibration. This terminates the proof of the first part of the factorization axiom. For the second part of the factorization axiom we use 7.2 to obtain a factorization $f = r \circ j$ where j is a dicofibration and r is a dihomotopy equivalence. As we have seen, r admits a factorization $r = p \circ \iota$ where ι is a trivial dicofibration and p is a difibration. By "2=3", p is a dihomotopy equivalence.

Since the composite of two dicofibrations is a dicofibration, $i = \iota \circ j$ is a dicofibration. Thus, $f = p \circ i$ is a factorization as required. This terminates the proof of the factorization axiom. The remaining statements follow from Theorems 5.5 and 7.2. □

Remarks 8.3. (i) Model categories are sometimes required to admit functorial factorizations. The model category **poTop** satisfies this requirement.

(ii) Unfortunately, the proof of Theorem 8.2 does not extend to the relative setting. In the relative setting we would have to show that a $(C, \leqslant)$-dimap $f : (X, \leqslant, \xi) \to (Y, \leqslant, \theta)$ admits a factorization $f = p \circ i$ where p is a $(C, \leqslant)$-difibration and i is a trivial $(C, \leqslant)$-dicofibration. The proof of 8.2 relies on the construction of the subspace $E = j(X) \times I \cup (X \times_Y Y^I) \times]0, 1] \subset (X \times_Y Y^I) \times I$ and the projection $\phi : (E, \leqslant) \to (I, \Delta)$. (Note that in order to obtain such a projection, we have to construct E as a subspace of $X \times_Y Y^I \times I$ and we cannot use the relative cylinder $(X \times_Y Y^I)\Box_C I$ instead of the product.) If we had constructed ϕ as a $(C, \leqslant)$-dimap with respect to some $(C, \leqslant)$-dimap $\varepsilon : (C, \leqslant) \to (E, \leqslant)$ then we would have $\phi(\varepsilon(c)) = 0$ for all $c \in C$. This is because $i : X \to E$ would be a $(C, \leqslant)$-dimap so that $\varepsilon(c) = i(\xi(c)) \in i(X) = \phi^{-1}(0)$. Later in the proof we would have to show that π is a $(C, \leqslant)$-difibration. For this we would consider the dimap g that appears in the proof as a $(C, \leqslant)$-dimap with respect to some dimap $\zeta\colon (C, \leqslant) \to (Z, \leqslant)$. For the dihomotopy F one would obtain $F(\zeta(c), t) = (G(\zeta(c), t), t)$, because $\phi(g(\zeta(c))) = \phi(\varepsilon(c)) = 0$. But this would imply that F is not a relative dihomotopy, because otherwise $\varepsilon(c) = F(\zeta(c), t) = (G(\zeta(c), t), t)$ so that $\varepsilon(c)$ would depend on t. So it would not be possible to conclude that π is a $(C, \leqslant)$-difibration. For analogous reasons A. Strøm's proof of the fact that the category of topological spaces is a closed model category with respect to homotopy equivalences, (Hurewicz) fibrations, and (closed) cofibrations [**12**] does not extend to the relative setting.

(iii) It is well-known that every "undercategory" of a closed model category is a closed model category. Therefore the category of $(C, \leqslant)$-pospaces is a closed model category. The structure is, however, different from the one in 5.5 and 7.2. The homotopy category of the closed model category $(C, \leqslant)$-**poTop** is not the dihomotopy category relative to $(C, \leqslant)$ but equivalent to a subcategory of this category. Indeed, recall that the homotopy category of a closed model category, i.e., its localization with respect to the weak equivalences, is equivalent to the "true" homotopy category of fibrant and cofibrant objects. The fibrant and cofibrant objects of the closed model category $(C, \leqslant)$-**poTop** are the $(C, \leqslant)$-pospaces $(X, \leqslant, \xi)$ where ξ is a dicofibration, and two dimaps between fibrant and cofibrant $(C, \leqslant)$-pospaces are homotopic in the closed model category of $(C, \leqslant)$-pospaces if and only if they are dihomotopic relative to $(C, \leqslant)$. Therefore the homotopy category of the closed model category $(C, \leqslant)$-**poTop** is equivalent to the full subcategory of the dihomotopy category relative to $(C, \leqslant)$ consisting of the $(C, \leqslant)$-pospaces $(X, \leqslant, \xi)$ where ξ is a dicofibration. Unfortunately, for many interesting $(C, \leqslant)$-pospaces $(X, \leqslant, \xi)$, ξ is not a dicofibration (cf. 7.3), and therefore the closed model structure on $(C, \leqslant)$-**poTop** is not a satisfactory framework for relative dihomotopy theory.

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