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GLOBAL ACTIONS, GROUPOID ATLASES AND APPLICATIONS

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(*communicated by Hvedri Inassaridze*)

Abstract

Global actions were introduced by A. Bak to give a combinatorial approach to higher K-theory, in which control is kept of the elementary operations through paths and paths of paths. This paper is intended as an introduction to this circle of ideas, including the homotopy theory of global actions, which one obtains naturally from the notion of path of elementary operations. Emphasis is placed on developing examples taken from combinatorial group theory, as well as K-theory. The concept of groupoid atlas plays a clarifying role.

dedicated to the memory of Saunders MacLane (1909-2005).

Contents

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1. Introduction

The motivation for the introduction of global actions by A. Bak [2–4] was to provide an algebraic setting in which to bring higher algebraic K-theory nearer to the intuitions of the original work of J.H.C. Whitehead on $K_1(R)$. In this work, elementary matrices, and sequences of their actions on the general linear group, play a key rôle.

The geometric origin for K_1 came from Whitehead's plan to seek a generalisation to all dimensions of the Tietze equivalence theorem in combinatorial group theory: this theorem states that two finite presentations of isomorphic groups may be transformed from one to the other by a finite sequence of elementary moves, called Tietze transformations. These algebraic moves were translated by Whitehead, [20], into the geometric moves of *elementary collapses and expansions* of finite simplicial complexes: this gave the notion of *simple homotopy equivalence*. The astonishing conclusion of his work was that there was an obstruction to a homotopy equivalence $f : X \to Y$ of finite simplicial complexes being a simple homotopy equivalence, and that this lay in a quotient group of a group that he had defined, the latter being $K_1(R)$ where R is the integral group ring of $\pi_1(X)$.

Since then, there was a search for higher order groups $K_i(R)$, which were finally defined by Quillen as homotopy groups of a space $F(R)$, the homotopy fibre of the canonical map $B\,\mathrm{GL}(R) \to B\,\mathrm{GL}(R)^+$. This was a great result, but the excursion into topology has meant that the original combinatorial intuitions get somewhat lost.

Bak, in [2–4], for instance, found that to deal with stable and unstable higher K-groups it was useful to consider a family of subgroups of the elementary matrix group, and that the theory in general could be organised as a family of group actions, indexed by a set with a relation $\leqslant$, often a partial order, and with certain 'patching conditions'. This became his 'global action', which he viewed as a kind of 'algebraic manifold', analogous to the notion of topological manifold, but in an algebraic setting. A global action was, by analogy with the notion of atlas in the theory of manifolds, an *atlas of actions*. The individual group actions played a role analogous to that of charts in the definition of a manifold. Thus 'local' meant at one group action, and 'global' meant understanding the interaction of them all. The elaboration of the definition in Bak's work was intended to cope with paths, to define an analogue of K_1, and paths of paths, to deal with higher order questions. In an up-coming paper, Bak will use this algebraic approach to give a presentation of all stable and unstable higher K-theory groups.

In discussions at Bangor and Bielefeld, it was seen that: (a) there were interesting applications of global actions to identities among relations for groups with a specified family of subgroups, and (b) there were advantages in using the well known transition from group actions to groupoids defined by the action groupoid of an action, and to rephrase the definition of global action so that it became part of a wider concept, *atlases of groupoids*, or as we will nearly always say, *groupoid atlases*. This extension will be formulated in the current paper. The individual groupoids play a role analogous to *charts* in the definition of a manifold. (We could have used the rubric *global groupoid* for this concept, but chose instead to emphasise the atlas-like structure explicitly in the name. Similarly, emphasising the atlas-like structure in global actions would yield the name *action atlas*.) The generalisation to groupoid atlases allows a wider scope for the theory, since groupoids can generalise not only groups and group actions, but also equivalence relations.

The content and organization of the rest of the paper is as follows. Section 2 recalls the definition of global action and elucidates it with several remarks and examples. Section 3 recalls the principal notions of morphism, namely the strongest and the weakest. The latter is necessary for constructing a good homotopy theory. Section 4 makes the transition from global actions to groupoid atlases. It gives the important examples of the line action in the category global actions and the line atlas in the category of groupoid atlases. These objects are necessary for defining the notions of path, cylinder, and homotopy for global actions and groupoid atlases, respectively. Section 5 discusses curves, paths, and the functor π_0. The functor π_0 is computed for some interesting examples. It is shown that π_0 of the global action associated to the general linear group of a ring is the K-theory functor K_1 of the general linear group. Section 6 constructs products of objects and introduces the fundamental groupoid of an object and the cylinder and loop space constructions of the fundamental group of an object. The remainder of the paper revolves around developing the concept of fundamental group. Section 7 shows how to associate an abstract simplicial complex to an object and uses this association to develop tools for interpreting and computing fundamental groups. Section 8 computes the fundamental groups of some uncomplicated single domain global actions. A single domain global action has all groups acting on the same set. These are the kinds of ations which occur in constructing K-theory groups. Section 10 develops the notion of covering for global actions and proves a Galois-Poincaré correspondence between the isomorphism classes of connected coverings of a given connected action and the conjugacy classes of the fundamental group of the given action. Section 11 constructs up to isomorphism, all connected coverings of a given connected single domain action. Section 12 shows that the single domain action associated to the Steinberg group is a connected, simply connected covering of the single domain action associated to the elementary group defined by the Steinberg group. From this it follows that the K-theory functor K_2 of the general linear group is the fundamental group of the global action defined by that group.

The notes on which this paper is based were produced by Tim Porter, on the basis of lectures by Tony Bak, followed by discussions in Bangor and Bielefeld. The material in sections 10-12 is taken unadorned from the lectures. A version has been available since then as a Bangor Preprint [5]. [1]

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2. Global actions

The motivating idea is of a family of interacting and overlapping local G-sets for varying groups G. The prime example is the underlying set $\mathrm{GL}_n(R)$ operated on by the family of subgroups $\mathrm{GL}_n(R)_\alpha$ which are generated by elementary matrices of a certain form α. We will give the details of this example shortly, but first we will set up some basic terminology and notation concerning group actions (G-sets), before we give the definition of a global action.

A (left) group action consists of a group G and a set X on which G acts on the left; we will write $G \curvearrowright X$. This means that there is a function from the set $G \times X$ to X written as (g, x) goes to $g.x$, such that $g_1.(g_2.x) = (g_1g_2).x$ and $1_G.x = x$ for all $g_1, g_2 \in G$ and $x \in X$.

It is often convenient to omit the dot so we may write gx instead of $g.x$.

A morphism of group actions, $(\varphi, \psi) : G \curvearrowright X \to H \curvearrowright Y$, consists of a homomorphism of groups $\varphi : G \to H$ and a function $\psi : X \to Y$ such that $\psi(g.x) = \varphi(g).\psi(x)$.

The promised 'global version' of this is:

Definition 2.1. *A global action* A *consists of a set* X_A *together with:*

(i) *an indexing set* Φ_A*, called the coordinate system of* A*;*

(ii) *a reflexive relation, written* $\leqslant$*, on* Φ_A*;*

(iii) *a family* $\{(G_\mathsf{A})_\alpha \curvearrowright (X_\mathsf{A})_\alpha \mid \alpha \in \Phi_\mathsf{A}\}$ *of group actions on subsets* $(X_\mathsf{A})_\alpha \subseteq X_\mathsf{A}$*; the* $(G_\mathsf{A})_\alpha$ *are called the local groups of the global action;*

(iv) *for each pair* $\alpha \leqslant \beta$ *in* Φ_A*, a group morphism*

$$(G_\mathsf{A})_{\alpha\leqslant\beta} : (G_\mathsf{A})_\alpha \to (G_\mathsf{A})_\beta.$$

This data is required to satisfy:

(v) *if* $\alpha \leqslant \beta$ *in* Φ_A*, then* $(G_\mathsf{A})_{\alpha\leqslant\beta}$ *leaves* $(X_\mathsf{A})_\alpha \cap (X_\mathsf{A})_\beta$ *invariant;*

(vi) *if* $\sigma \in (G_\mathsf{A})_\alpha$ *and* $x \in (X_\mathsf{A})_\alpha \cap (X_\mathsf{A})_\beta$*, then*

$$\sigma x = ((G_\mathsf{A})_{\alpha\leqslant\beta}(\sigma))x. \qquad \square$$

The diagram $G_\mathsf{A} : \Phi_\mathsf{A} \to \mathsf{Groups}$ is called the *global group* of A. The set X_A is the *enveloping set* or *underlying set* of A. The notation $|X_\mathsf{A}|$ or $|\mathsf{A}|$ for X_A is sometimes used for emphasis or to avoid confusion since

$$X_\mathsf{A} : \Phi_\mathsf{A} \to \mathcal{P}(X_\mathsf{A})$$

is also a useful notation, where $\mathcal{P}(X_\mathsf{A})$ is the powerset of X_A.

Remarks 2.2.

a) *For technical reasons it is not assumed that the collection* $(X_\mathsf{A})_\alpha \subseteq X_\mathsf{A}$ *necessarily covers* X_A*. This holds in all the basic examples we will examine but is not a requirement.*

b) The relation $\leqslant$ is not assumed to be transitive on Φ_{A}, so really G_{A} is not a functor. However, the difference is minor as, if $F(\Phi_{\mathsf{A}})$ denotes the free category on the graph of $(\Phi_{\mathsf{A}}, \leqslant)$, then G_{A} extends to a functor $G_{\mathsf{A}} : F(\Phi_{\mathsf{A}}) \to \mathsf{Groups}$. We will usually refer, as here, to G_{A} as a diagram of groups. It will sometimes be useful to consider groups as single object groupoids, in which case the above yields a diagram of groupoids[2].

The simplest global actions come with just a single domain: a global action A is said to be *single domain* if for each $\alpha \in \Phi_{\mathsf{A}}, (X_{\mathsf{A}})_\alpha = |\mathsf{A}|$.

Example 2.3. *Let G be a group, $\mathcal{H} = \{H_i \mid i \in \Phi\}$ a family of subgroups of G. For the moment Φ is just a set (that is : $\alpha \leqslant \beta$ in Φ if and only if $\alpha = \beta$). Define $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$ to be the global action with*

$$\begin{aligned} X = |X_{\mathsf{A}}| &= |G|, \text{ the underlying set of } G \\ \Phi_{\mathsf{A}} &= \Phi \\ (X_{\mathsf{A}})_\alpha &= X_{\mathsf{A}} \text{ for all } \alpha \in \Phi \\ H_i &\curvearrowright X \text{ by left multiplication} \end{aligned}$$

(so the local orbits of the H_i-action are the left cosets of H_i).

Later on in section 7.4, we will need to refine this construction, taking Φ_{A} to be the family of finite non-empty subsets of Φ ordered by opposite inclusion and with $(G_{\mathsf{A}})_\alpha = \bigcap_{i\in\alpha} H_i$ if $\alpha \in \Phi_{\mathsf{A}}$. □

We will later look in some detail at certain specific such single domain global actions. The following prime motivating example is similar to these, but the indexing set/coordinate system is slightly more complex.

Example 2.4. The General Linear Global Action $\mathsf{GL}_n(R)$. *Let R be an associative ring with identity and n a positive integer.*

Let $\Delta = \{(i, j) \mid i \neq j, 1 \leqslant i, j \leqslant n\}$ be the set of non-diagonal positions in an $n \times n$ array. Call a subset $\alpha \subseteq \Delta$ closed *if*

$$(i, j) \in \alpha \text{ and } (j, k) \in \alpha \text{ implies } (i, k) \in \alpha$$

Note that if $(i, j) \in \alpha$ and α is closed then $(j, i) \notin \alpha$.

Let $\Phi = \{\alpha \subseteq \Delta \mid \alpha \text{ is closed}\}$. We put a reflexive relation $\leqslant$ on Φ by $\alpha \leqslant \beta$ if $\alpha \subseteq \beta$.

Now suppose $(i, j) \in \Delta$ and $r \in R$. The elementary matrix $\varepsilon_{ij}(r)$ is the matrix obtained from the identity $n \times n$ matrix by putting the element r in position (i, j),

$$\text{i.e.} \qquad \varepsilon_{ij}(r)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ r & \text{if } (k, l) = (i, j) \\ 0 & \text{otherwise .} \end{cases}$$

[2]By a groupoid we mean a small category in which every arrow is an isomorphism.

Let $\mathrm{GL}_n(R)_\alpha$*, for* $\alpha \in \Phi$*, denote the subgroup of* $\mathrm{GL}_n(R)$ *generated by*

$$\{\varepsilon_{ij}(r) \mid (i,j) \in \alpha, r \in R\}.$$

It is easy to see that $(a_{kl}) \in \mathrm{GL}_n(R)_\alpha$ *if and only if*

$$a_{k,l} = \begin{cases} 1 & \textit{if } k = l \\ \textit{arbitrary} & \textit{if } (k,l) \in \alpha \\ 0 & \textit{if } (k,l) \in \Delta \backslash \alpha. \end{cases}$$

For $\alpha \leqslant \beta$*, there is an inclusion of* $\mathrm{GL}_n(R)_\alpha$ *into* $\mathrm{GL}_n(R)_\beta$*. This will give the homomorphism*

$$\mathrm{GL}_n(R)_{\alpha \leqslant \beta} : \mathrm{GL}_n(R)_\alpha \to \mathrm{GL}_n(R)_\beta.$$

Let $\mathrm{GL}_n(R)_\alpha$ *act by left multiplication on* $\mathrm{GL}_n(R)$*.*

This completes the description of the single domain global action $\mathsf{GL}_n(R)$*. Later we will see how to define the fundamental group and more generally the higher homotopy groups of a global action. The* $(i-1)^{th}$*-homotopy group of* $\mathsf{GL}_n(R)$ *is the algebraic* K*-theory group* $K_i(n,R)$ *and the usual algebraic* K*-group,* $K_i(R)$ *is the direct limit of* $K_i(n,R)$*s by the obvious maps induced from the inclusions* $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$*.*
□

The way that a global action extends local information to become global information can be observed from the simplest cases of the $\mathsf{A}(G, \mathcal{H})$.

If $\mathcal{H}$ has just a single group H in it, then the global action is just the collection of orbits, i.e. right cosets. There is no interaction between them.

If $\mathcal{H}$ consists of distinct subgroups $\{H_1, H_2\}$, then any H_1-orbit intersects with some H_2-orbit, so now orbits do interact. How they interact can be very influential on the homotopy properties of the situation.

Example 2.5. *As a simple example consider the symmetric group* $S_3 \equiv \langle a, b \mid a^3 = b^2 = (ab)^2 = 1\rangle$*, with* a *denoting the 3-cycle (1 2 3) and* b *the transposition (1 2). Take* $H_1 = \langle a \rangle = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$ *yielding two orbits for its left action on* S_3, H_1 *and* $H_1 b$*. Similarly take* $H_2 = \langle b \rangle$ *giving local orbits* $H_2, H_2 a, H_2 a^2$*. Any* H_1*-orbit intersects with any* H_2*-orbit, but of course they do not overlap themselves. This gives an intersection diagram:*

$$\begin{tikzcd}[arrows=-]
 & H_1 \arrow[dl] \arrow[dr] \arrow[drrr] & & H_1 b \arrow[dlll] \arrow[dl] \arrow[dr] & \\
H_2 & & H_2 a & & H_2 a^2
\end{tikzcd}$$

The graph, defined by the intersection diagram, makes it clear that, even in such a simple case, it is possible to find loops and circuits within the global action, by moving an element at will through any local orbit and therfore into an intersection, crossing to the next orbit, etc. and eventually getting back to the starting position.

For example, the element $1 \in H_2$ multiplied on the left by $b \in H_2$ ends up in $H_2 \cap H_1 b$, multiplied on the left by $a \in H_1$ yields $ab \in H_1 b \cap H_2 a^2$ and so on as below. The circuit

$$
\begin{array}{ccccccccc}
H_2 & \longrightarrow & H_1 b & \longrightarrow & H_2 a^2 & \longrightarrow & H_1 & \longrightarrow & H_2 \\
\\
1 & \xrightarrow{b\times} & b & \xrightarrow{a\times} & ab & \xrightarrow{b\times} & bab & \xrightarrow{a\times} & abab = 1
\end{array}
$$

relates the structure of the single domain global action with the combinatorial information encoded in the presentation. This will be examined in more detail later.

3. Morphisms

Morphisms between global actions come in various strengths depending on what part of the data is preserved. Preservation of just the local orbit information corresponds to a "morphism", compatibility with the whole of the data then yields a "regular morphism".

First we introduce a subsidiary notion which will be important at several points in the later development.

Definition 3.1. *Let A be a global action. Let $x \in (X_{\mathsf{A}})_\alpha$ be some point in a local set of A.*

A local frame *at x in α or* α-frame *at x is a sequence $x = x_0, \cdots, x_p$ of points in the local orbit of the $(G_{\mathsf{A}})_\alpha$-action on $(X_{\mathsf{A}})_\alpha$ determined by x. Thus for each i, $1 \leqslant i \leqslant p$, there is some $g_i \in (G_{\mathsf{A}})_\alpha$ with $g_i x = x_i$.* □

Note that in extreme cases, such as a trivial action, all the x_i may be equal, but if the action is faithful, each α-frame at x consists essentially of x and a sequence $g_1, \cdots, g_p$ of elements of $(G_{\mathsf{A}})_\alpha$. For some of the homotopy theoretic side of the development this may be of use as $g_1, g_2 g_1^{-1}, \cdots$ yields a $(p-1)$-simplex in the nerve of the group $(G_{\mathsf{A}})_\alpha$.

Definition 3.2. *If A and B are global actions, a morphism $f : \mathsf{A} \to \mathsf{B}$ of global actions is a function $f : |\mathsf{A}| \to |\mathsf{B}|$ on their underlying sets, which preserves local frames. More precisely:*
if $x_0, \cdots, x_p$ is an α-frame at x_0 for some $\alpha \in \Phi_{\mathsf{A}}$ then $f(x_0), \cdots, f(x_p)$ is a β-frame at $f(x_0)$ for some $\beta \in \Phi_{\mathsf{B}}$. □

Note that not all α-frames may lead to the same β, so this notion is *not* saying that the whole of the local orbit of the $(G_{\mathsf{A}})_\alpha$-action corresponding to x_0 must end up within a single local orbit, merely that given $x_0, \cdots, x_p$, there is some β such that $f(x_0), \cdots, f(x_p)$ form a β-frame. This is of course only significant when there are infinitely many co-ordinates, as larger frames may lead to different "larger" βs.

Intuitively a *path* in a global action A is a sequence of points $a_0, \cdots, a_n$ in $|\mathsf{A}|$ so that each $a_i, a_{i+1}, i = 0, \cdots, n-1$ is a α-frame for some (varying) $\alpha \in \Phi_{\mathsf{A}}$. This idea can be captured using a morphism from a global action model of a line, and this is done in the initial papers on global actions, [2–4]. Here we postpone this until the third section as there is a certain technical advantage in considering the line with a groupoid atlas structure and that will be introduced there.

Definition 3.3. *A regular morphism $\eta : \mathsf{A} \to \mathsf{B}$ of global actions is a triple $(\eta_\Phi, \eta_G, \eta_X)$ satisfying the following*

– $\eta_\Phi : \Phi_{\mathsf{A}} \to \Phi_{\mathsf{B}}$ *is a relation preserving function:*
if $\alpha \leqslant \alpha'$, then $\eta_\Phi(\alpha) \leqslant \eta_\Phi(\alpha')$.

– $\eta_G : G_{\mathsf{A}} \to (G_{\mathsf{A}})_{\eta\Phi(\ )}$ *is a natural transformation of group diagrams over η_Φ,*

i.e. for each $\alpha \in \Phi_{\mathsf{A}}$,

$$\eta_G(\alpha) : (G_{\mathsf{A}})_\alpha \to (G_{\mathsf{B}})_{\eta_\Phi(\alpha)}$$

is a group homomorphism such that if $\alpha \leqslant \alpha'$ in Φ_{A}, the diagram

$$\begin{tikzcd}
(G_{\mathsf{A}})_\alpha \arrow[r, "\eta_G(\alpha)"] \arrow[d] & (G_{\mathsf{B}})_{\eta_\Phi(\alpha)} \arrow[d] \\
(G_{\mathsf{A}})_{\alpha'} \arrow[r, "\eta_G(\alpha')"'] & (G_{\mathsf{B}})_{\eta_\Phi(\alpha')}
\end{tikzcd}$$

where the vertical maps are the structure maps of the respective diagrams;
- $\eta_X : |\mathsf{A}| \to |\mathsf{B}|$ *is a function such that $\eta_X((X_{\mathsf{A}})_\alpha) \subseteq (X_{\mathsf{B}})_{\eta_\Phi(\alpha)}$ for all $\alpha \in \Phi_{\mathsf{A}}$;*
- *for each $\alpha \in \Phi_{\mathsf{A}}$, the pair*

$$(\eta_G, \eta_X) : (G_{\mathsf{A}})_\alpha \curvearrowright (X_{\mathsf{A}})_\alpha \to (G_{\mathsf{B}})_{\eta_\Phi(\alpha)} \curvearrowright (X_{\mathsf{B}})_{\eta_\Phi(\alpha)}$$

is a morphism of group actions. □

Remark 3.4. *If η is a regular morphism, it is clear that η_X preserves local frames and so is a morphism in the weaker sense.*

Composition of both types of morphism is defined in the obvious way and so one obtains categories of global actions and morphisms and of global actions and regular morphisms.

It is perhaps necessary to underline the meaning of a morphism of group actions: If $G \curvearrowright X$ and $H \curvearrowright Y$ are group actions of G on X and H on Y, respectively, a morphism from $G \curvearrowright X$ to $H \curvearrowright Y$ is a pair $(\varphi : G \to H, \psi : X \to Y)$ with φ a homomorphism and ψ a function such that for $g \in G, x \in X$,

$$\varphi(g).\psi(x) = \psi(g.x).$$

We need to note that, if x and x' are in the same orbit of $G \curvearrowright X$ then $\psi(x)$ and $\psi(x')$ are in the same orbit of $H \curvearrowright Y$. □

4. Actions as groupoids, and groupoid atlases.

4.1. Actions as groupoids

If $G \curvearrowright X$ is a group action then we can construct an *action groupoid* from it.

$\mathrm{Act}(G, X)$ or $G \ltimes X$ will denote the category with X as its set of objects and $G \times X$ as its set of arrows. Given an arrow (g, x), its source is x and its target $g.x$. We write $s(g, x) = x, t(g, x) = g.x$ and represent this diagrammatically by

$$x \xrightarrow{(g,x)} g.x\ .$$

The composite of (g, x) and (g', x') is defined only if the target of (g, x) is the source of (g', x') so $x' = g.x$, then

$$x \xrightarrow{(g,x)} g.x \xrightarrow{(g',gx)} g'gx$$

gives a composite $(g'g, x)$. The identity at x is $(1, x)$. The inverse of (g, x) is (g^{-1}, gx) so $G \ltimes X$ is in fact a groupoid.

Example 4.1. *Let $X = \{0, 1\}, G = C_2$ with the obvious action on X interchanging 0 and 1. If we write $C_2 = \{1, c\}$ we have $\mathrm{Ob}(G \ltimes X) = X = \{0, 1\}$,*

$$\mathrm{Arr}(G \ltimes X) = \{(1, 0) : 0 \to 0, (1, 1) : 1 \to 1,\ \ (c, 0) : 0 \to 1, (c, 1) : 1 \to 0\}$$

Thus diagrammatically the groupoid is just

$$G \ltimes X := \begin{tikzcd} \overset{\cdot}{0} \arrow[loop left] \arrow[r, bend left, "{(c,0)}"] & \overset{\cdot}{1} \arrow[l, bend left, "{(c,1)}"] \arrow[loop right] \end{tikzcd}$$

i.e. it is the groupoid often written as $\mathcal{I}$, the unit interval groupoid. □

Back to the general situation:

Suppose $(\varphi, \psi) : G \curvearrowright X \to H \curvearrowright Y$ is a morphism of group actions, then we can define a morphism of groupoids by

$$\begin{aligned} \varphi \ltimes \psi : G \ltimes X &\to H \ltimes Y \\ (\varphi \ltimes \psi)(x) &= \psi(x) && \text{on objects} \\ (\varphi \ltimes \psi)(g, x) &= (\varphi(g), \psi(x)) && \text{on arrows.} \end{aligned}$$

We check:

$$\begin{aligned} s(\varphi(g), \psi(x)) &= \psi(x) = \psi(s(g, x)), \\ t(\varphi(g), \psi(x)) &= \varphi(g).\psi(x) = \psi(g, x) \\ &= \psi t(g, x) \end{aligned}$$

so $\varphi \ltimes \psi$ preserves source and target. It also preserves identities and composition as is easily checked.

4.2. Groupoid atlases

The "language" of group actions thus translates well into the language of groupoids. The notion of an orbit of a group action becomes a connected component of a groupoid, so what is the analogue of a global action? The translation is not difficult, but the obvious term "global groupoid" does not seem to give the right intuition about the concept. We noted that a global action was similar to the notion of an atlas in the theory of manifolds, so is an *atlas of actions*, so instead we will use the term 'groupoid atlas' or, synonymously, 'atlas of groupoids'.

First a bit of notation: if G is a groupoid with object set X and $X' \subset X$ is a subset of X then $G \downharpoonleft_{X'}$ will denote the groupoid with object set X' having

$$G \downharpoonleft_{X'} (x, y) = G(x, y),$$

if $x, y \in X'$ and with the same composition and identities as G, when this makes sense. This groupoid $G \downharpoonleft_{X'}$ is the *full sub-groupoid of G determined by the objects in X'* or more simply, the *restriction* of G to X'.

Definition 4.2. *A groupoid atlas A consists of a set X_A together with:*

(i) *an indexing set Φ_A, called the coordinate system of A;*

(ii) *a reflexive relation, written $\leqslant$, on Φ_A;*

(iii) *a family $\mathcal{G}_\mathsf{A} = \{(G_\mathsf{A})_\alpha \mid \alpha \in \Phi_\mathsf{A}\}$ of groupoids with object sets $(X_\mathsf{A})_\alpha$; the $(G_\mathsf{A})_\alpha$ are called the local groupoids of the groupoid atlas;*

(iv) *if $\alpha \leqslant \beta$ in Φ_A, a groupoid morphism*

$$(G_\mathsf{A})_\alpha \downharpoonleft_{(X_\mathsf{A})_\alpha \cap (X_\mathsf{A})_\beta} \longrightarrow (G_\mathsf{A})_\beta \downharpoonleft_{(X_\mathsf{A})_\alpha \cap (X_\mathsf{A})_\beta}$$

which is the identity map on objects. The notation we will use for this morphism will usually be φ^α_β but the more detailed $(G_\mathsf{A})_{\alpha \leqslant \beta}$ may be used where more precision is needed. As before we write $|\mathsf{A}|$ for X_A, the underlying set of A.

This data is required to satisfy:

(v) *if $\alpha \leqslant \beta$ in Φ_A, then $(X_\mathsf{A})_\alpha \cap (X_\mathsf{A})_\beta$ is a union of components of $(G_\mathsf{A})_\alpha$, i.e. if $x \in (X_\mathsf{A})_\alpha \cap (X_\mathsf{A})_\beta$ and $g \in (G_\mathsf{A})_\alpha$ is such that $s(g) = x$ then $t(g) \in (X_\mathsf{A})_\alpha \cap (X_\mathsf{A})_\beta$.* □

A morphism of groupoid atlases comes in several strengths as with the special case of global actions.

A *local frame* in a groupoid atlas, A, is a sequence $(x_0, \cdots, x_p)$ of objcts in a single connected component of some $(\mathcal{G}_\mathsf{A})_\alpha$, i.e. there is some $\alpha \in \Phi_\mathsf{A}$, $x_0, \cdots, x_p \in (X_\mathsf{A})_\alpha$ and arrows $g_i : x_0 \to x_i, i = 1, \cdots, p$.

A function $f : |\mathsf{A}| \to |\mathsf{B}|$ supports a *weak morphism* structure if it preserves local frames. Similar comments apply to those made above about morphisms of global actions.

The stronger form of morphism of groupoid atlases will just be called a (strong) morphism.

A *strong morphism* $\eta : \mathsf{A} \to \mathsf{B}$ of groupoid atlases is a triple $(\eta_X, \eta_\Phi, \eta_g)$ satisfying the following

(i) $\eta_X : X_A \to X_B$ is a function between the underlying sets;

(ii) $\eta_\Phi : \Phi_\mathsf{A} \to \Phi_\mathsf{B}$ is a relation preserving function;

(iii) $\eta_G : \mathcal{G}_\mathsf{A} \to (\mathcal{G}_\mathsf{B})_{\eta_\Phi}$ is a (generalised) natural transformation of diagrams of groupoids *over* the function η_Φ on the objects.

To illustrate the difference between global actions and groupoid atlases, we consider some simple examples.

Example 4.3. *Let $X = \{0, 1, 2\}, G = C_3 = \{1, a, a^2\}$ (and, of course, $a^3 = 1$), the cyclic group of order 3, acting by $a.0 = 1, a.1 = 2$, on X. This gives us $C_3 \ltimes X$ with 9 arrows. We set $\mathsf{B} = C_3 \ltimes X$ as groupoid or $C_3 \curvearrowright X$ as C_3-set. We also have the example $\mathsf{A} = C_2 \ltimes \{0, 1\} = \mathcal{I}$ considered earlier.*

Both A and B will be considered initially as global actions having Φ_A and Φ_B a single element.

Any function $f : \{0, 1\} \to \{0, 1, 2\}$ supports the structure of a morphism of global actions since the only non-trivial frame in A is based on the set $\{x_0, x_1\}$, where $x_0 = 0$, and $x = 1$ and this must get mapped to a frame in B, since any non-empty subset of X is a frame in B. On the other hand, a regular morphism $\eta : \mathsf{A} \to \mathsf{B}$ must contain the information on a group homomorphism

$$\eta_G : C_2 \to C_3$$

which must, of course, be trivial. Hence the only regular morphism η must map all of A to a single point in B. There are thus 9 morphisms from A to B, but only 3 regular morphisms. The regular morphisms are very rigid. □

Remark 4.4. *It is not always the case that there are fewer regular morphisms than (general) morphisms. If A is a global action with one point and a group acting on that point and B is similar with group H, there is only one general morphism from A to B, but the set of regular morphisms is 'the same as' the set of group homomorphisms from G to H.* □

Example 4.5. *Now we continue the previous example by considering A and B as groupoid atlases. The element $(c, 0) : 0 \to 1$ in the single groupoid determining A, must be sent to some arrow in B. The inverse of $(c, 0)$ is $(c, 1)$, so as soon as a morphism, η_G is specified on $(c, 0)$, it is determined on $(c, 1)$ since $\eta_G(c, 1) = (\eta_G(c, 0))^{-1}$. Thus if we pick an arrow in B, say,*

$$(a^2, 0) : 0 \to 2,$$

we can define a morphism

$$\eta_G : \mathsf{A} \to \mathsf{B}$$

by specifying $\eta_G(c, 0) = (a^2, 0)$, so $\eta_G(0) = 0$, $\eta_G(1) = 2$ etc. In other words the fact that A uses an action by C_2 and B by C_3 does not inhibit the existence of morphisms from A to B. Any morphism of global actions from A to B in this case will support the structure of a morphism of the corresponding groupoid atlases, yet the extra structure of a "regularity condition" is supported in this latter setting. Of course the relationship between morphisms of global actions and morphisms of the corresponding groupoid atlases can be expected to be more subtle in general. □

Problem/Question 4.6. If A and B are global actions and $f : \mathsf{A} \to \mathsf{B}$ is a morphism, does f support the structure of a morphism of the corresponding groupoid atlases?

In general the answer is 'no' since if A is a global action with $\Phi_{\mathsf{A}} = \{a, b \mid a \leqslant b\}$ with both X_a and X_b single points, and B is similar but with Φ_{A} discrete, then the general morphism which corresponds to the identity does not support the structure of a (strong) morphism of the corresponding groupoid atlases because of the need for a relation function $\eta : \Phi_{\mathsf{A}} \to \Phi_{\mathsf{B}}$. Refining the question, suppose we have a general morphism of global actions together with a relation preserving function between the coordinate systems, which is compatible with the morphism. In that case the question is related to the following question about groupoids:
if we have two groupoids A and B and a function f from the objects of A to the objects of B which sends connected components of A to connected components of B, what obstructions are there for there to exist a functor F from A to B such that F restricted to the objects is the given f?

Clearly any global action determines a corresponding groupoid atlas as we have used above. As there are morphisms of action groupoids that do not come from regular morphisms of actions, the groupoid morphisms give a new notion of morphism of global actions, whose usefulness for the motivating examples will need investigating. Are there "useful" groupoid atlases other than those coming from global actions? The answer is most definitely: yes.

4.3. Equiv(A)

Equivalence relations are examples of groupoids.

Example 4.7. *Let X be a set. Any equivalence relation R on X determines a groupoid with object set X. We will denote this groupoid by R as well. It is specified by*

$$R(x, y) = \begin{cases} \{(x, y)\} & \textit{if } xRy \\ \emptyset & \textit{if } x \textit{ is not related to } y. \end{cases}$$

Now suppose $R_1, \cdots, R_n$ are a family of equivalence relations on X. Then define A to have coordinate system

$$\Phi_{\mathsf{A}} = \{1, \cdots, n\} \quad \textit{with discrete } \leqslant$$

and $(G_{\mathsf{A}})_i = R_i$. This gives a groupoid atlas that does not in general arise from a global action. □

Example 4.8. *Let G be a group, X a G-set and R an equivalence relation on X. Let $\Phi = \{1, 2\}$, with $\leqslant$ still to be specified. Take $G_1 = G \ltimes X$, $G_2 = R$ and $X_1 = X_2 = X$. Assume we have a groupoid atlas structure with this as partial data. If $\leqslant$ is discrete, there is no interaction between the two structures and no compatibility requirement. If $1 \leqslant 2$, each G-orbit is contained in an equivalence class with $\varphi_2^1(x, g) = (x, gx)$, i.e. the G-orbit structure is finer than the partition into equivalence classes. If $2 \leqslant 1$, the partition is finer than the orbit structure (the connected components of the groupoid G_1) and if xRy then there is some $g_{x,y} \in G$ such that $g_{x,y}x = y$.* □

This last case is closely related to a useful construction on global actions.

Example 4.9. *Let $\mathsf{A} = (X_\mathsf{A}, G_\mathsf{A}, \Phi_\mathsf{A})$ be a global action. Let $\alpha \in \Phi_\mathsf{A}$ and $(G_\mathsf{A})_\alpha \curvearrowright (X_\mathsf{A})_\alpha$ be the corresponding action. Set R_α to be the equivalence relation determined by the $(G_\mathsf{A})_\alpha$-action. Thus $xR_\alpha x'$ if and only if there is some $g \in (G_\mathsf{A})_\alpha$ with $gx = x'$. Of course the partition of $(X_\mathsf{A})_\alpha$ into R_α-equivalence classes is exactly that given by the $(G_\mathsf{A})_\alpha$-orbits (or the $(\mathcal{G}_\mathsf{A})_\alpha$-components where $(\mathcal{G}_\mathsf{A})_\alpha$ is the corresponding groupoid).*

If $\alpha \leqslant \beta$ then the compatibility conditions are satisfied between R_α and R_β making $(X_\mathsf{A}, R_\mathsf{A}, \Phi_\mathsf{A})$ with $R_\mathsf{A} = \{R_\alpha \mid \alpha \in \Phi_\mathsf{A}\}$ into a groupoid atlas which will be denoted $\mathrm{Equiv}(\mathsf{A})$.

The functions $(\mathcal{G}_\mathsf{A})_\alpha \to R_\alpha$ mapping the groupoid of the $(G_\mathsf{A})_\alpha$-action to the corresponding equivalence relation yield a natural transformation of groupoid diagrams and hence a strong morphism

$$\mathsf{A} \to \mathrm{Equiv}(\mathsf{A})$$

with obvious universal properties. Of course the same construction works if A is an arbitrary groupoid atlas, that is, one not necessarily arising from a global action. The result gives a left adjoint to the inclusion of the full subcategory of atlases of equivalence relations into that of groupoid atlases. The usefulness of this construction is another reason for extending our view beyond global actions to include groupoid atlases. The notion of morphism of global actions, $f : \mathsf{A} \to \mathsf{B}$, translates to the notion of strong morphism, $f : \mathrm{Equiv}(\mathsf{A}) \to \mathrm{Equiv}(\mathsf{B})$ of the corresponding groupoid atlases, at least for examples with finite orbits. □

4.4. The Line

We have seen that the simple action with $G = C_2$, $X = \{0, 1\}$ gives the groupoid $\mathcal{I}$ (also sometimes written [1] as it is the groupoid version of the 1-simplex). We want an analogue of a line so as to describe paths and loops. The line, L, is obtained by placing infinitely many copies of $\mathcal{I}$ end to end. It is a global action, but, as the morphisms that give paths in a global action A will need to be non-regular morphisms in general, it is often expedient to think of it as a groupoid atlas.

The set, $|\mathsf{L}|$, of points of L is $\mathbb{Z}$, the set of integers; $\Phi_\mathsf{L} = \mathbb{Z} \cup \{\square\}$, where $\square$ is an element satisfying $\square < n$ for all $n \in \mathbb{Z}$, and otherwise the relation $\leqslant$ is equality.

(Thus $\square \leqslant \square$, for all $n \in \mathbb{Z}$, $\square < n$ and $n \leqslant n$, but that gives all related pairs.) If $n \in \Phi_{\mathsf{L}}$, $(X_{\mathsf{L}})_n = \{n, n+1\}$, whilst $(X_{\mathsf{L}})_\square = |\mathsf{L}|$ itself.

The groupoid $(\mathcal{G}_{\mathsf{L}})_n$ is a copy of $\mathcal{I}$, whilst $(\mathcal{G}_{\mathsf{L}})_\square$ is discrete with trivial vertex groups.

The underlying structure of L rests firmly on the locally finite simplicial complex structure of the ordinary real line. There the (abstract) simplicial complex structure is given by:

$$\begin{aligned} \text{Vertices } &= \mathbb{Z}, \text{ the set of integers;} \\ \text{Set of 1-simplices } &= \{\{n, n+1\} \mid n \in \mathbb{Z}\}, \text{ the set of adjacent pairs in } \mathbb{Z}. \end{aligned}$$

We will see shortly that there is a close link between simplicial complexes and this context of global actions/ groupoid atlases.

5. Curves, paths and connected components

Suppose A is a global action or more generally a groupoid atlas. A curve in A is simply a (weak) morphism

$$f : \mathsf{L} \to \mathsf{A}$$

where L is the line groupoid atlas introduced above.

This implies that $f : |\mathsf{L}| \to |\mathsf{A}|$ is a function for which local frames are preserved. In L the local frames are simply the adjacent pairs $\{n, n+1\}$ and the singleton sets $\{n\}$. Thus the condition that $f : \mathsf{L} \to \mathsf{A}$ be a path is that the sequence of points

$$\cdots, f(n), f(n+1), \cdots$$

is such that for each n, there is a $\beta \in \Phi_{\mathsf{A}}$ and $g_\beta : f(n) \to f(n+1)$ in $(\mathcal{G}_{\mathsf{A}})_\beta$. (If you prefer global action notation $g_\beta \in (G_{\mathsf{A}})_\beta$ and $g_\beta f(n) = f(n+1)$.)

Note that f does not specify β and g_β, merely requiring their existence. This observation leads to a notion of a *strong curve* in A which is a morphism of groupoid atlases

$$f : \mathsf{L} \to \mathsf{A}$$

so for each n one gets a $\beta = \eta_\Phi(n) \in \Phi_{\mathsf{A}}$ and $\eta_G : \mathcal{G}_{\mathsf{L}} \to (\mathcal{G}_{\mathsf{A}})_{\eta_\Phi}$ is a natural transformation of groupoid diagrams. This condition only amounts to specifying $\eta_G(n, \mathsf{L}) = g : f(n) \to f(n+1)$, but this time the data is part of the specification of the curve. We can thus write a strong curve as $(\cdots, f(n), g_n, f(n+1), \cdots)$, that is a sequence of points of $|\mathsf{A}|$ together with locally defined arrows

$$g_n : f(n) \to f(n+1)$$

in the chosen local groupoid $(\mathcal{G}_{\mathsf{A}})_\beta$. Changing the β or the g_n changes the morphism. We will later see the rôle of strong curves, strong paths, etc.

A (*free*) *path* in A will be a curve that stabilises to a constant value on both its left and right ends. More precisely it is a curve $f : \mathsf{L} \to \mathsf{A}$ such that there are integers

$N^- \leqslant N^+$ with the property that

$$\text{for all } n \leqslant N^-, \ f(n) = f(N^-);$$

$$\text{for all } n \geqslant N^+, \ f(n) = f(N^+).$$

We will call (N^-, N^+) a *stabilisation pair* for f.

A "based path" can be defined if A has a distinguished base point. This occurs naturally in such cases as $\mathsf{A} = \mathsf{GL}_n(R)$ or $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$ for $\mathcal{H}$ a family of subgroups of a group G, but is also defined abstractly by adding the specification of the chosen base point explicitly to the data. This situation is well known from topology where a notation such as (A, a_0) would be used. We will adopt similar conventions.

If (A, a_0) is based groupoid atlas, a *based path* in (A, a_0) is a free path that stabilises to a_0 on the left, i.e., in the notation above, $f(N^-) = a_0$.

A *loop* in (A, a_0) is a based path that stabilises to a_0 on both the left and the right so $f(N^-) = f(N^+) = a_0$.

The analogue in this setting of concepts such as "connected component" should now be clear. We say that points p and q of A, a global action or groupoid atlas, are *free path equivalent* if there is a free path in A which stabilises to p on the left and to q on the right.

Clearly free path equivalence is reflexive. It is also symmetric since if $g_n : f(n) \to f(n+1)$ in a local patch then $g_n^{-1} : f(n+1) \to f(n)$. Once a free path from p to q has reached q (i.e. has stabilised at q) then it can be concatenated with a path from q to r, say, hence free path equivalence is also transitive. The equivalence classes for free path equivalence will be called *connected components*, with $\pi_0(\mathsf{A})$ denoting the *set of connected components* of A. If A has just one connected component then it is said to be *connected*.

Examples 5.1.

1. *The prime and motivating example is the set of connected components $\pi_0\mathsf{GL}_n(R)$ of the general linear global action.*
   Suppose $x, y \in \mathsf{GL}_n(R)$. Suppose $f : \mathsf{L} \to \mathsf{GL}_n(R)$ is a free path from x to y, so there are $N^- \leqslant N^+$ as above with

   $$\text{if } n \leqslant N^-, \ f(n) = f(N^-) = x,$$
   $$\text{if } n \geqslant N^+, \ f(n) = f(N^+) = y.$$

   For each $i \in [N^-, N^+]$, there is some local arrow

   $$g_i : f(n) \to f(i+1)$$

   and since $\mathsf{GL}_n(R)$ is a global action, this means there is some $\alpha_i \in \Phi$ and $\varepsilon_i \in \mathsf{GL}_n(R)_{\alpha_i}$ such that $\varepsilon_i f(i) = f(i+1)$. (The specification of f gives the existence of such an ε_i but does not actually specify which of possibly many ε_is *to take, so we choose one. In fact of course, $f(i)$, and $f(i+1)$ are invertible matrices, so there is only one ε_i possible.) We thus have*

$$\varepsilon_{N^+}\varepsilon_{N^+-1}\cdots\varepsilon_{N^-}x = y.$$

If $\mathrm{E}_n(R)$ *is the subgroup of elementary matrices of* $\mathrm{GL}_n(R)$*, this is the subgroup generated by all the* $\mathrm{GL}_n(R)_\alpha$ *for* $\alpha \in \Phi$ *and so if x and y are free path equivalent*

$$y \in \mathrm{E}_n(R)x,$$

i.e., x and y are in the same right coset of $\mathrm{E}_n(R)$.
Conversely if $y \in \mathrm{E}_n(R)x$*, there is an element* $\varepsilon \in \mathrm{E}_N(R)$ *such that* $y = \varepsilon x$*, but ε can be written (in possibly many ways) as a product of elementary matrices*

$$\varepsilon = \varepsilon_N \cdots \varepsilon_1$$

with $\varepsilon_i = \mathrm{GL}_n(R)_{\alpha_i}$*, say. Then defining*

$$f : \mathsf{L} \to \mathsf{GL}_n(R)$$

by

$$f(n) = \begin{cases} x & n \leqslant 0 \\ \varepsilon_n \cdots \varepsilon_1 x & 1 \leqslant n \leqslant N \\ y & n \geqslant N \end{cases}$$

gives a free path from x to y in $\mathsf{GL}_n(R)$.
Thus $\pi_0(\mathsf{GL}_n(R)) = \mathrm{GL}_n(R)/\,\mathrm{E}_n(R)$*, the set of right cosets of* $\mathrm{GL}_n(R)$ *modulo elementary matrices. This is, of course, the algebraic K-group $K_1(n,R)$ if R is a commutative ring.*
We can naturally ask the question: 'is $K_2(n,R) \cong \pi_1(\mathsf{GL}_n(R))$*?' even if we have not yet defined the righthand side of this.*
We note the use of the strong rather than the weak version of paths would not change the resulting π_0.

2. *Suppose* $\mathsf{A} = \mathsf{A}(G,\mathcal{H})$*. Can one calculate* $\pi_0(\mathsf{A})$*? A similar argument to that in 1 above shows that if* $x, y \in |\mathsf{A}| = |G|$*, then they are free path equivalent if and only if there are indices* $\alpha_i \in \Phi$ *and elements* $h_{\alpha_i} \in H_{\alpha_i}$*, such that*

$$h_{\alpha_n}\cdots h_{\alpha_0}x = y$$

for some n. Thus writing $\langle\mathcal{H}\rangle = \langle H_i \mid i \in \Phi\rangle$ *for the subgroup of G generated by the family* $\mathcal{H} = \{H_i \mid i \in \Phi\}$*, we clearly have*

$$\pi_0(\mathsf{A}(G,\mathcal{H})) = G/\langle\mathcal{H}\rangle.$$

Again the question arises as to $\pi_1(\mathsf{A}(G,\mathcal{H}))$*: what is it and what does it tell us?*

6. Fundamental groups and fundamental groupoids.

Ideas for the construction of $\pi_1(\mathsf{A}, a_0)$ for a pointed global action or groupoid atlas, A, seem clear enough. There are three possible approaches:

(i) take some notion of homotopy of paths and define $\pi_1(\mathsf{A}, a_0)$ to be the set of homotopy classes of loops at a_0; hopefully this would be a group for the natural notion of composition via concatenation of paths. Alternatively define a fundamental groupoid $\Pi_1\mathsf{A}$ using a similar plan and then take $\pi_1(\mathsf{A}, a_0)$ to be the vertex group of $\Pi_1\mathsf{A}$ at a_0.

(ii) define a global action or groupoid atlas structure on the set $\Omega(\mathsf{A}, a_0)$ of loops at a_0, then take $\pi_1(\mathsf{A}, a_0) = \pi_0(\Omega(\mathsf{A}, a_0))$.

(iii) define covering morphisms of global actions or groupoid atlases, then use a universal or simply connected covering to find a classifying group for connected coverings. This should be $\pi_1(\mathsf{A}, a_0)$.

We will look at the first two of these in this section, handling the third one later.

6.1. Products

We start by checking that products exist in the various categories we are looking at. In this section we will only need them in very special cases, but they will be needed later on in full strength.

Let A and B be groupoid atlases, $\mathsf{A} = (X_\mathsf{A}, \mathcal{G}_\mathsf{A}, \Phi_\mathsf{A}), \mathsf{B} = (X_\mathsf{B}, G_\mathsf{B}, \Phi_\mathsf{B})$, and consider the structure that we will denote by $\mathsf{A}\times\mathsf{B}$ and which is given by

$$|X_{\mathsf{A}\times\mathsf{B}}| = |X_\mathsf{A}| \times |X_\mathsf{B}|;$$

$$\Phi_{\mathsf{A}\times\mathsf{B}} = \Phi_\mathsf{A} \times \Phi_\mathsf{B}$$

with $(\alpha, \beta) \leqslant (\alpha', \beta')$ if and only if $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$;

$$(\mathcal{G}_{\mathsf{A}\times\mathsf{B}})_{(\alpha,\beta)} = (\mathcal{G}_\mathsf{A})_\alpha \times (\mathcal{G}_\mathsf{B})_\beta,$$

the product groupoid, for $(\alpha, \beta) \in \Phi_{\mathsf{A}\times\mathsf{B}}$ with the obvious product homomorphisms as coordinate changes. (We thus have $\mathcal{G}_{\mathsf{A}\times\mathsf{B}}$ is the product groupoid diagram

$$\Phi_{\mathsf{A}\times\mathsf{B}} = \Phi_\mathsf{A} \times \Phi_\mathsf{B} \xrightarrow{\mathcal{G}_\mathsf{A}\times\mathcal{G}_\mathsf{B}} \mathsf{Groupoids} \times \mathsf{Groupoids} \longrightarrow \mathsf{Groupoids}.\ )$$

Lemma 6.1. $\mathsf{A}\times\mathsf{B}$ *with this structure is a groupoid atlas.*

Proof. The proof is by routine checking so is omitted. □

There are obvious 'projections' $p_\mathsf{A} : \mathsf{A}\times\mathsf{B} \to \mathsf{A}$, $p_\mathsf{B} : \mathsf{A}\times\mathsf{B} \to \mathsf{B}$, at least at the level of underlying sets, so we need to check that they enrich nicely to give the various strengths of morphism. We start with weak morphisms.

Lemma 6.2. *p_{A} and p_{B} support the structure of weak morphisms.*

Proof. A local frame in $\mathsf{A}\times\mathsf{B}$ will be a sequence $((x_0, y_0), \dots, (x_p, y_p))$ of objects in a connected component of some $(\mathcal{G}_{\mathsf{A}\times\mathsf{B}})_{(\alpha,\beta)}$, so there are arrows $(g_i, g_i') : (x_0, y_0) \to (x_i, y_i)$ in $(\mathcal{G}_{\mathsf{A}})_\alpha \times (\mathcal{G}_{\mathsf{B}})_\beta$. Clearly this means that $(x_0, \dots, x_p)$ is an α-frame and $(y_0, \dots, y_p)$ is a β-frame, so p_{A} and p_{B} are weak morphisms. ☐

This lemma is an immediate corollary of the next one, but has the advantage that its very simple proof shows directly the structure of local frames in $\mathsf{A}\times\mathsf{B}$ and how they relate to those in A and B. This will aid our intuition when looking at the links with simplicial complexes later on.

Lemma 6.3. *a) p_{A} and p_{B} are strong morphisms of groupoid atlases.*
b) If A and B are global actions, considered as groupoid atlases, then so is $\mathsf{A}\times\mathsf{B}$

Proof. (b) will be left as an 'exercise', as it is easy to check.)

a) We have already specified p_{A} at set level. On the coordinate system $p_{\mathsf{A},\Phi} : \Phi_{\mathsf{A}\times\mathsf{B}} \to \Phi_{\mathsf{A}}$ is again just the projection and this is clearly order preserving. Finally $p_{\mathsf{A},\mathcal{G}} : (\mathcal{G}_{\mathsf{A}\times\mathsf{B}})_{(\alpha,\beta)} \to (\mathcal{G}_{\mathsf{A}})_{p_{\mathsf{A},\Phi}(\alpha,\beta)}$ is the projection from $(\mathcal{G}_{\mathsf{A}})_\alpha \times (\mathcal{G}_{\mathsf{B}})_\beta$ to its first factor. This coincides with (p_{A}) on objects and satisfies all the naturality conditions. ☐

We next return to weak morphisms to see if $(\mathsf{A}\times\mathsf{B}, p_{\mathsf{A}}, p_{\mathsf{B}})$ has the universal property for products (in the relevant category of groupoid atlases and weak morphisms).

Suppose that $f : \mathsf{C} \to \mathsf{A}$ and $g : C \to \mathsf{B}$ are weak morphisms, then we clearly get a mapping $(f, g) : \mathsf{C} \to \mathsf{A}\times\mathsf{B}$ given by $(f, g)(c) = (f(c), g(c))$. This preserves local frames as is easily seen and is unique with the property that $p_{\mathsf{A}}(f, g) = f$ and $p_{\mathsf{B}}(f, g) = g$. We thus nearly have:

Proposition 6.4. *The categories of groupoid atlases and of global actions, both with weak morphisms, have all finite products.*

Proof. The only thing left to note is that these categories have a terminal object, namely the singleton trivial global action. ☐

As we would expect we have a similar result for strong morphisms.

Proposition 6.5. *The categories of groupoid atlases and of global actions, with strong morphisms, has all finite products.*

Proof. The only thing left to prove is the universal property of the 'product' $\mathsf{A}\times\mathsf{B}$ in ths strong case, so suppose $f : \mathsf{C} \to \mathsf{A}$ and $g : C \to \mathsf{B}$ are now morphisms, then we have $f = (f_X, f_\Phi, f_{\mathcal{G}})$, etc. As the construction of $\mathsf{A}\times\mathsf{B}$ uses products of sets, reflexive relations and groupoids, we get the corresponding existence and uniqueness of $(f, g) = ((f_X, g_X), (f_\Phi, g_\Phi), (\mathrm{f}_{\mathcal{G}}, g_{\mathcal{G}}))$ more or less for free. The terminal object is as before the singleton trivial global action. ☐

Remarks 6.6.

(i) *It is now easy to generalise from finite products to arbitrary products* $\prod \mathsf{A}_i = (\prod X_{\mathsf{A}_i}, \prod \Phi_{\mathsf{A}_i}, \prod \mathcal{G}_{\mathsf{A}_i})$. *We leave the detailed check to the reader as we will not need this result for the moment.*

(ii) *If we are looking at products of global actions, the corresponding projection morphisms are regular.*

6.2. Homotopies of paths

We only need one example of a product for the moment, namely $\mathsf{L} \times \mathsf{L}$, which is a groupoid atlas model of $\mathbb{R}^2$. Just as a path has to start and end somewhere so does a homotopy of paths.

Given a global action A, points $a, b \in \mathsf{A}$ and paths $f_0, f_1 : \mathsf{L} \to \mathsf{A}$ joining a and b (and hence stabilising to these values to the left and right respectively), a (fixed end point) *homotopy* between f_0 and f_1 is a morphism

$$h : \mathsf{L} \times \mathsf{L} \to \mathsf{A}$$

such that:
there exist $N^-, N^+ \in \mathbb{Z}$, $N^- \leqslant N^+$ such that

- for all $n \leqslant N^-$, and all $m \in |\mathsf{L}|$, $h(m, n) = f_0(m)$;
- for all $n \geqslant N^+$, and all $m \in |\mathsf{L}|$, $h(m, n) = f_1(m)$;
- for all $m \leqslant N^-$, and all $n \in \mathbb{Z}$, $h(m, n) = a$;
- for all $m \geqslant N^+$, and all $n \in \mathbb{Z}$, $h(m, n) = b$.

Remarks 6.7.

(i) *The idea is that if we consider* $\mathsf{L} \times \mathsf{L}$ *as being based on the integer lattice of the plane, the morphism* h *must stabilise along all horizontal and vertical lines outside the square with corners* (N^+, N^+), (N^-, N^+), (N^-, N^-) *and* (N^+, N^-). *Although the paths* f_0, f_1 *coming with given "lengths", i.e. a given number of steps from* a *to* b, *we allow a homotopy to increase, or decrease, the number of those steps an arbitrary (finite) amount.*

(ii) *Given any* $f : \mathsf{L} \to \mathsf{A}$, *a path from* a *to* b, *we can re-index* f *to get*

$$f' : \mathsf{L} \to \mathsf{A}$$

with $f'(n) = a$ *if* $n \leqslant 0$, *and a new* $N^{+\prime}$ *so that* $f'(n) = b$ *if* $n \geqslant N^{+\prime}$, *simply by taking the old stabilisation pair* (N^-, N^+) *for* f *and defining*

$$f'(n) = f(n - N^-), \quad n \in \mathbb{Z} = |\mathsf{L}|.$$

The resulting f' *is homotopic to* f. *Although this is fairly clear intuitively, it is useful as an exercise as it brings home the complexity of the processes involved, but also their inherent simplicity.*

Example 6.8. *Let $f_0 : \mathsf{L} \to \mathsf{A}$ be a path from a to b with (N_0^-, N_0^+) being a stabilisation pair for f. Define $f_1 : \mathsf{L} \to \mathsf{A}$ by*

$$f_1(n) = f_0(n-1), \qquad n \in |\mathsf{L}|,$$

i.e., f_1 is f_0 shifted one "notch" to the right on L. Then
(i) a suitable stabilisation pair for f_1 is $(N_0^- + 1, N_0^+ + 1)$
(ii) f_1 is a path from a to b
and (iii) f_1 is homotopic to f_0.

The only claim that is not obvious is (iii). To construct a suitable homotopy h, we construct many intermediate steps. For simplicity we will start defining h on the upper half-plane (we can always extend it to a suitable square afterwards by a vertically constant extension):

$$h(n, 0) = f_0(n), \qquad n \in |\mathsf{L}|$$

and we make a choice of a local arrow $g_n : f_0(n) \to f_0(n+1)$, for each n (of course, for $n \leqslant N_0^-$ or $n \geqslant N_0^+$, g_n will be an identity of the local groupoid patch),

$$\begin{aligned} h(n,1) &= \quad f_0(n) \qquad \text{for } n \leqslant N_0^+ - 1 \\ h(N_0^+, 1) &= f_0(N_0^+ - 1) \end{aligned}$$

with the identity on $f_0(N_0^+ - 1)$ as corresponding local arrow from $h(N_0^+ - 1, 1)$ to $h(N_0^+, 1)$.

$$h(N_0^+ + 1, 1) = f_0(N_0^+) \qquad \text{and stabilise horizontally.}$$

Thus so far we have inserted an identity one place from the end and shifted the end stage one to the right. We give next a local arrow from $h(n,0)$ to $h(n,1)$ for each n. For most this will be the identity arrow but for the local arrow from $h(N_0^+, 0)$ to $h(N_0^+, 1)$ we take $g^{-1}_{N_0^+}$. The same idea is used for $h(n,2)$ but with the identity inserted one step back to the left. At each successive stage of the homotopy, the "ripple" that is the identity moves an extra step to the left. (In the diagram we write N for N_0^+.)

$$\begin{tikzcd}
h(-,2) : \cdots \arrow[r] & f_0(N-2) \arrow[r, "id"] & f_0(N-2) \arrow[r, "g_{N-2}"] & f_0(N-1) \arrow[r, "g_{N-1}"] & f_0(N) \arrow[r] & S \\
h(-,1) : \cdots \arrow[r] & f_0(N-2) \arrow[r, "g_{N-2}"] \arrow[u, "id"'] & f_0(N-1) \arrow[r, "id"] \arrow[u, "g^{-1}_{N-2}"'] & f_0(N-1) \arrow[r, "g_{N-1}"] \arrow[u, "id"'] & f_0(N) \arrow[r] \arrow[u, "id"'] & S \\
h(-,0) : \cdots \arrow[r] & f_0(N-2) \arrow[r, "g_{N-2}"] \arrow[u, "id"'] & f_0(N-1) \arrow[r, "g_{N-1}"] \arrow[u, "id"'] & f_0(N) \arrow[r, "id"] \arrow[u, "g^{-1}_{N-2}"'] & f_0(N) \arrow[r] \arrow[u] & S
\end{tikzcd}$$

(the symbol S indicates the sequence stabilises to the last specified value.)

Thus within a homotopy class we can "ripple homotopy" a path to have specified N^- (or for that matter N^+).

Now suppose $f : \mathsf{L} \to \mathsf{A}$ is a path from a to b and $g : \mathsf{L} \to \mathsf{A}$ one from b to c. We can assume that the stabilisation pair for g is to the right of that for f, i.e., if $(N^-(f), N^+(f))$ and $(N^-(g), N^+(g))$ are suitable stabilisation pairs, $N^+(f) \leqslant N^-(g)$. Then we can form a concatenated path: $f * g$ by first going along f until it stabilises at b then along g. Of course $f * g$ will depend on the choice of stabilisation pairs, but using "ripple homotopies" we can change positions of f and g at will and these homotopies will be reflected by homotopies of the corresponding $f * g$. We may, for instance, start g immediately after f stabilises to b. This means that the composition is well defined on homotopy classes of paths.

Likewise using vertical composition, i.e., exchanging the roles of horizontal and vertical on homotopies it is elementary to prove that (fixed end point) homotopy is an equivalence relation on paths. Reflexivity of the homotopy relation is proved by taking the inverses of all vertical local arrows in a homotopy. To reverse paths, $f \rightsquigarrow f^{(r)}$, and to prove the "reverse" is an inverse modulo homotopy is also simple using the move:

$$\begin{tikzcd} \cdot \arrow[r, "g_k"] & \cdot \arrow[r, "g_k^{-1}"] & \cdot \\ \cdot \arrow[u, "id"] \arrow[r, "id"'] & \cdot \arrow[u, "g_k"'] \arrow[r, "id"'] & \cdot \arrow[u, "id"'] \end{tikzcd}$$

followed by a ripple homotopy to move the identities to the end. As concatenation does not require reindexation (unlike paths in spaces where $f : [0, 1] \to X$ uses a *unit length* interval) proof of associativity is easy: one concatenates immediately on stabilisation to get a unique chosen composite and then associativity is assured.

This set of properties allows one to define the *fundamental groupoid* $\Pi_1\mathsf{A}$ of a global action or groupoid atlas A in the obvious way. The objects of $\Pi_1\mathsf{A}$ are the points of $|\mathsf{A}|$ whilst if a, b are points of $|\mathsf{A}|$, $\Pi_1\mathsf{A}(a, b)$ will be the set of (fixed end point) homotopy classes of paths from a to b within A. Composition is by concatenation as above and "inversion is by reversion": if $w = [f]$, $w^{-1} = [f^{(r)}]$, where $f^{(r)}$ is the "reverse" of f.

There is a strong variant of this construction. All the homotopies etc. used above manipulate a strong path that represents the chosen path, i.e., we chose the local arrows $g_n : f(n) \to f(n+1)$ and worked with them. There is a clear notion of (fixed end point) strong homotopy of paths and strong homotopy classes of strong paths compose in the same way giving a *strong fundamental groupoid* $\Pi_1^{\mathrm{Str}}\mathsf{A}$.

6.3. Objects of curves, paths and loops

We aim to define for at least a large class of groupoid atlases (including most if not all global actions of significance for K-theory), a "loop space" analogous to that defined for topological spaces. We expect to be able to concatenate loops within that structure giving some embryonic analogue of the H-space structure on a loop space. More precisely given a nice enough groupoid atlas $\mathsf{A} = (X_\mathsf{A}, \mathcal{G}_\mathsf{A}, \Phi_\mathsf{A})$, we want a new groupoid atlas $\Omega\mathsf{A}$ and a concatenation operator

$$\Omega\mathsf{A} \times \Omega\mathsf{A} \to \Omega\mathsf{A},$$

which will induce at least a monoid structure on $\pi_0(\Omega\mathsf{A})$ and hopefully for a large class of examples will give a group isomorphic to $\pi_1(\mathsf{A})$.

Loops are best thought of via paths and thus via curves. We thus start by searching for a suitable structure on the set of all curves $\mathrm{Mor}(\mathsf{L},\mathsf{A})$ in A. (Remember a curve is merely a (weak) morphism from L to A.) If $f:\mathsf{L}\to\mathsf{A}$ is a curve of A, f will pass through a sequence of local sets. The intuition is that a local set containing f in $\mathrm{Mor}(\mathsf{L},\mathsf{A})$ will consist of curves passing through the same local sets $(X_\mathsf{A})_\alpha$ in the same sequence.

To simplify notation we will write (L,A) for the set of morphisms from L to A.

Definition 6.9. *Given a curve $f:\mathsf{L}\to\mathsf{A}$ in A, a function $\beta:|\mathsf{L}|\to\Phi_\mathsf{A}$ frames f if β is a function such that*
(i) for $m\in|\mathsf{L}|=\mathbb{Z}$, $f(m)\in(X_\mathsf{A})_{\beta(m)}$;
(ii) for $m\in|\mathsf{L}|$, there is a b in Φ_A with $b\geqslant\beta(m)$, $b\geqslant\beta(m+1)$ and a $g:f(m)\to f(m+1)$ in $(\mathcal{G}_\mathsf{A})_b$.

Thus β picks out the local sets $(X_\mathsf{A})_{\beta(m)}$ which are to receive $f(m)$. The condition (ii) ensures that these choices are compatible with the requirement that f be a curve. Note that there may be curves that have no framing, especially if the pseudo-order on Φ_A has few related pairs, e.g. is discrete. For instance in a single domain global action of the form $\mathsf{A}(G,\mathcal{H})$ we have used a discrete order on Φ_A. Hence the condition $b\geqslant\beta(m)$, $b\geqslant\beta(m+1)$ must imply that $\beta(m)=\beta(m+1)$, yet in our examples we have seen non-trivial paths going through several local orbits. Thus in such a case the coordinate system we will define shortly does not cover the set of paths in A and for this reason we will consider in detail, later on, the second global action structure on such A that was mentioned at the start of example 2.3.

Lemma 6.10. *Let $f:\mathsf{L}\to\mathsf{A}$ be a curve and $\beta:|\mathsf{L}|\to\Phi_\mathsf{A}$ a framing for f.*

Suppose $\sigma:|\mathsf{L}|\to|\prod(\mathcal{G}_\mathsf{A})_{\beta(\cdot)}|$ is a function defining a sequence (σ_m) of arrows in the local groupoids of A such that $\sigma_m\in(\mathcal{G}_\mathsf{A})_{\beta(m)}$ and, in fact, the source sequence of σ is the underlying function of f, i.e.,

$$s(\sigma_m)=f(m).$$

Then the sequence $(f'(m))$, where $f'(m)=t(\sigma_m)$, supports a weak morphism structure, $f':\mathsf{L}\to\mathsf{A}$, and β frames f' as well.

Proof. We have $b\geqslant\beta(m)$ and $b\geqslant\beta(m+1)$ with $g:f(m)\to f(m+1)$ in $(\mathcal{G}_\mathsf{A})_b$. We have $\sigma_m:f(m)\to f'(m)$ and hence its inverse is in $(\mathcal{G}_\mathsf{A})_{\beta(m)}$, so using the structural morphism

$$\varphi_b^{\beta(m)}:(\mathcal{G}_\mathsf{A})_{\beta(m)}\to(\mathcal{G}_\mathsf{A})_b,$$

we get $\varphi_b^{\beta(m)}(\sigma_m):f(m)\to f'(m)$ in $(\mathcal{G}_\mathsf{A})_b$. Similarly $\varphi_b^{\beta(m)}(\sigma_{m+1}):f(m+1)\to f'(m+1)$.

It is now clear that using the composite

$$f'(m)\to f(m)\to f(m+1)\to f'(m+1)$$

of these three arrows in $(\mathcal{G}_{\mathsf{A}})_b$ shows that β frames f'. As the only frames in L have size two, i.e., are adjacent pairs $\{m, m+1\}$, this also shows that f' is a curve in A. $\square$

It should be clear what our next step will be.

Take $\Phi_{(\mathsf{L},\mathsf{A})} = \{\beta : |\mathsf{L}| \to \Phi_{\mathsf{A}} \mid \beta \text{ frames some curve } f\}$.

The set of objects $(X_{(\mathsf{L},\mathsf{A})})_\beta$ of $(\mathcal{G}_{(\mathsf{L},\mathsf{A})})_\beta$ will be

$$(X_{(\mathsf{L},\mathsf{A})})_\beta = \{f : \mathsf{L} \to \mathsf{A} \mid f \in (\mathsf{L},\mathsf{A}), \beta \text{ frames } f\}$$

then take $(\mathcal{G}_{(\mathsf{L},\mathsf{A})})_\beta$ to be the set of sequences (σ_m) with σ_m an arrow in $(\mathcal{G}_{\mathsf{A}})_{\beta(m)}$, with the property that $(s(\sigma_m))$ supports the structure of a curve in $(X_{(\mathsf{L},\mathsf{A})})_\beta$, i.e., framed by β. The lemma above ensures that in this case $(t(\sigma_m))$ is also in $(X_{(\mathsf{L},\mathsf{A})})_\beta$, and that $(\mathcal{G}_{(\mathsf{L},\mathsf{A})})_\beta$ is a groupoid.

We have yet to specify the relation on $\Phi_{(\mathsf{L},\mathsf{A})}$, but a component-wise definition is the obvious one to try:

$$\beta \leqslant \beta' \text{ if and only if } \beta(m) \leqslant \beta'(m) \text{ for all } m \in |\mathsf{L}|.$$

As each $\beta(m) \leqslant \beta'(m)$ results in an induced morphism of groupoids

$$(\mathcal{G}_{\mathsf{A}})_{\beta(x)} \downharpoonleft \to (\mathcal{G}_{\mathsf{A}})_{\beta'(x)} \downharpoonleft$$

over the intersection $(X_{\mathsf{A}})_{\beta(m)} \cap (X_{\mathsf{A}})_{\beta'(x)}$, there is an induced morphism of groupoids

$$(\mathcal{G}_{(\mathsf{L},\mathsf{A})})_\beta \downharpoonleft \to (\mathcal{G}_{(\mathsf{L},\mathsf{A})})_{\beta'} \downharpoonleft$$

over $(X_{(\mathsf{L},\mathsf{A})})_\beta \cap (X_{(\mathsf{L},\mathsf{A})})_{\beta'}$.

If $f \in (\mathsf{L},\mathsf{A})$ is framed by both β and β', then for any $\sigma : f \to f'$ in $(\mathcal{G}_{(\mathsf{L},\mathsf{A})})_\beta$, we have seen in the above lemma that f' is framed by both β and β'. We thus have all the elements of the verification of the following:

Proposition 6.11. *With the above notation,* $\mathsf{A}^{\mathsf{L}} = ((\mathsf{L},\mathsf{A}), \mathcal{G}_{(\mathsf{L},\mathsf{A})}, \Phi_{(\mathsf{L},\mathsf{A})})$ *is a groupoid atlas. If* A *is a global action, then so is* A^{L}*.* $\square$

The only part not covered by the previous discussion is that relating to global actions, however taking $(G_{(\mathsf{L},\mathsf{A})})_\beta$ to be the product of the $(G_{\mathsf{A}})_{\beta(m)}$, one gets an action of this on $(X_{\mathsf{A}})_\beta$ giving exactly the groupoid $(\mathcal{G}_{(\mathsf{L},\mathsf{A})})_\beta$ of the proposition.

We thus have a groupoid atlas A^{L} of curves in A, and if A is a global action, A^{L} is one as well. Note however that A^{L} will not usually be a single domain global action even when A is one. This is the reason why Bak introduced the general notion of global action, where each group is given its own, possibly distinct, domain of action.

Our earlier comments also show that to assume that the $(X_{\mathsf{A}})_\alpha$ cover X_{A} in our original definition would have been unduly retrictive here. Of course, our ability to study properties of elements of X_{A} which lie outside the coordinate patches is restricted.

To obtain a path space $P(\mathsf{A})$, we merely restrict to those curves that are paths with an adjustment made to the local groupoids to allow for the fact that a path can be linked by an arrow to a general curve.

More explicitly we have

$$\begin{array}{ll} |P(\mathsf{A})| & = \text{ the set of paths } f : \mathsf{L} \to \mathsf{A} \\ \Phi_{P(\mathsf{A})} & = \{\beta : |\mathsf{L}| \to \Phi_{\mathsf{A}} \mid \beta \text{ stably frames some path } f\} \\ (X_{P(\mathsf{A})})_\beta & = \{f \in |P(\mathsf{A})| \mid \beta \text{ stably frames } f\} \\ (\mathcal{G}_{P(\mathsf{A})})_\beta & = \text{ the groupoid of stable sequences } (\sigma_m) \text{ with } \sigma_m \text{ an arrow in} \\ & \quad (\mathcal{G}_{P(\mathsf{A})})_{\beta(m)} \text{ and such that } (s(\sigma_m)) \text{ supports the structure of} \\ & \quad \text{a path in } (X_{P(\mathsf{A})})_\beta, \text{ i.e. is stably framed by } \beta. \end{array}$$

The references to "stable" in this are the needed restriction to ensure "paths" not "curves" are involved. Recall $f : \mathsf{L} \to \mathsf{A}$ is a path if it is a curve and there are integers $N_f^- \leqslant N_f^+$ with $f(n) = f(N_f^-)$ for $n \leqslant N_f^-$ and $f(n) = f(N_f^+)$ for $n \geqslant N_f^+$. (The pair (N_f^-, N_f^+) was earlier called a stabilisation pair.) When $\beta : |\mathsf{L}| \to \Phi_{\mathsf{A}}$ frames a path, f, it would clearly be possible to have β varying beyond the end of the "active interval", $N_f^- \leqslant n \leqslant N_f^+$, but is this necessary? We will use the term "stable frame" of f if for some $(N_\beta^-, N_\beta^+), \beta(n)$ is constant for smaller and for larger n. We do not specify how the stabilisation pair for f is related to one for β if at all. Similar comments apply to stable sequences, (σ_m). There is a stable version of the above lemma. The proof should be clear.

Lemma 6.12. *If f is a path, $\beta : \mathsf{L} \to \Phi_{\mathsf{A}}$ a stable framing of f and $\sigma = (\sigma_m)$ a stable sequence of arrows with $s(\sigma_n)$ the underlying function of f then $f' = t(\sigma)$ supports the structure of a path and β stably frames f' as well.* □

Finally we will want a based path space $\Gamma(\mathsf{A}, a_0)$ and a "loop space" $\Omega\mathsf{A}$. We note first that if $f : \mathsf{L} \to \mathsf{A}$ is in $P(\mathsf{A})$ then there are integers $N_f^- \leqslant N_f^+$ with $f(n) = f(N_f^-)$ for $n \leqslant N_f^-$ and $f(n) = f(N_f^+)$ for $n \geqslant N_f^+$. Define two functions

$$e^0, e^1 : P(\mathsf{A}) \to \mathsf{A}$$

by $e^0(f) = f(N_f^-), e^1(f) = f(N_f^+)$. These are clearly independent of the choice of stabilisation pair for f used in their definition. Clearly we expect these functions to support (weak) morphisms on the corresponding groupoid atlases or global actions.

Suppose we have a local frame in the groupoid atlas $P\mathsf{A}$. Then we have some $\beta \in \Phi_{P\mathsf{A}}$, and paths $f_0, f_1, \cdots, f_p$ in some connected component of $(\mathcal{G}_{P(\mathsf{A})})_\beta$. Thus β is a stable framing of f_0 and we have that there exist $\sigma^{(i)} : f_0 \to f_i$, $\sigma^{(i)} = \left(\sigma_m^{(i)}\right)$. By the two lemmas, β is a stable framing of each f_i as well - with the same "b" for all of them.

Each f_i, each $\sigma^{(i)}$ and β itself have stabilisation pairs so we can find N^- smaller than all the left hand ends of these and N^+ larger than the right hand ends. Since $\sigma_m^{(i)}$ will be constant for $n \leqslant N^-$ and also for $n \geqslant N^+$, we have

$$f_0(n), f_1(n), \cdots, f_p(n)$$

is a frame in $\beta(n)$, i.e., $e^0(f_0), \cdots, e^0(f_p)$ is a frame in A and we have shown e^0 is a morphism. Of course a completely similar argument shows e^1 is a morphism.

We now assume that a base point a_0 is given in A. We can take $\Gamma(\mathsf{A}, a_0)$ to be the global action or groupoid atlas defined by $e_0^{-1}(a_0)$. As we have not yet described how to do such a construction, we consider a more general situation.

Suppose $f : \mathsf{A} \to \mathsf{B}$ is a weak morphism of global actions or, more generally, groupoid atlases and let $b \in \mathsf{B}$, then the set $f^{-1}(b) \subset \mathsf{A}$ supports the following structure

$$\begin{array}{ll} |f^{-1}(b)| & = \{\alpha \in \mathsf{A} : f(a) = b\} \\ \Phi_{f^{-1}(b)} & = \{a \in \Phi_{\mathsf{A}} \mid (X_{\mathsf{A}})_\alpha \cap f^{-1}(b) \neq \emptyset\} \\ (X_{f^{-1}(b)})_\alpha & = (X_{\mathsf{A}})_a \cap f^{-1}(b), \end{array}$$

and the local action / local groupoid is the restriction of that in A. That this last specification works needs a bit of care. We first look at the global action case.

If $f(a) = b$, and $g \in (G_{\mathsf{A}})_\alpha$ one usually would not expect $g.a$ to be still "over b" so one has to take

$$(G_{f^{-1}(b)})_\alpha = \{g \in (G_{\mathsf{A}})_\alpha : g.f^{-1}(b) = gf^{-1}(b)\},$$

then no problem arises. For the groupoid case the corresponding $(\mathcal{G}_{f^{-1}(b)})_\alpha$ is just the full sub-groupoid of $(\mathcal{G}_{\mathsf{A}})_\alpha$ determined by the objects $(X_{f^{-1}(b)})_\alpha$. The induced morphisms when $\alpha \leqslant \alpha'$ in $\Phi_{f^{-1}(b)}$ are now easy to handle.

We thus have a global action/groupoid atlas $\Gamma(\mathsf{A}, a_0)$ of based paths in (A, a_0) and a restricted "end point morphism"

$$e_1 : \Gamma(\mathsf{A}, a_0) \to \mathsf{A}.$$

Of course, $\Omega(\mathsf{A}, a_0)$, the *object of loops at* a_0, will be $e_1^{-1}(a_0)$ considered as a sub-object of $\Gamma(\mathsf{A}, a_0)$ or $e_0^{-1}(a_0) \cap e_1^{-1}(a_0)$ when thought of as being within $P(\mathsf{A})$.

The discussion of concatenation of paths in the lead up to the fundamental groupoid indicates that there is a concatenation of loops, but that such an operation depends strongly on the choice of stabilisation pairs for the individual loops. There is thus no single composition map

$$\Omega(\mathsf{A}, a_0) \times \Omega(\mathsf{A}, a_0) \to \Omega(\mathsf{A}, a_0)$$

that is "best possible" or "most natural". Composition can be defined if stabilisation pairs are chosen:

Given $f, g \in |\Omega(\mathsf{A}, a_0)|$ with chosen stabilisation pairs $(N_f^-, N_f^+), (N_g^-, N_g^+)$ then define

$$f * g : \mathsf{L} \to \Omega(\mathsf{A}, a_0)$$

$$\text{by} \qquad (f * g)(n) = \begin{cases} f(n) & \text{if } n \leqslant N_f^+ \\ g(n - N_f^+ + N_g^-) & \text{if } n \geqslant N_f^+ \end{cases}$$

The obvious stabilisation pair is $(N_f^-, N_f^+ + N_g^+)$ and with this choice we get an associative composition, <u>but</u> on loops with chosen stabilisation pairs. Of course the composition is not really well defined on the loops alone. As before it is well defined up to "ripple homotopies". This structure is analogous to the 'Moore loop space' construction in a topological setting.

Our next task is to calculate $\pi_0\Omega(\mathsf{A}, a_0)$ comparing it with $\pi_1(\mathsf{A}, a_0)$ as defined as the vertex group at a_0 of $\Pi_1\mathsf{A}$, the fundamental groupoid of A.

Suppose f_0 and f_1 are loops at a_0. When are they free path equivalent as points of $\Omega(\mathsf{A}, a_0)$? Suppose $h : \mathsf{L} \to \Omega(\mathsf{A}, a_0)$ is a free path in $\Omega(\mathsf{A}, a_0)$ that stabilises to f_0 on the left and to f_1 on the right. The "obvious" way to proceed is to try to use h to construct a homotopy between f_0 and f_1 as paths in A. Picking a stabilisation pair (N_h^-, N_h^+) for h, we, of course, have

$$\begin{aligned} h(n) &= f_0 && \text{for } n \leqslant N_h^-, \\ h(n) &= f_1 && \text{for } n \geqslant N_h^+. \end{aligned}$$

Define

$$\begin{aligned} H : |\mathsf{L}| \times |\mathsf{L}| &\to \mathsf{A} \text{ by} \\ H(m,n) &= \ h(n)(m), \end{aligned}$$

and, noting that there are finitely many <u>different</u> $h(n)$ involved, pick for each of these $h(n)$ a suitable stabilisation pair $(N_{h(n)}^-), (N_{h(n)}^+)$ and set

$$\begin{aligned} N^- &= \min\Big(\{N_{h(n)}^- : n \in [N_h^-, N_h^+]\}, N_h^-\Big) \\ N^+ &= \max\Big(\{N_{h(n)}^+ : n \in [N_h^-, N_h^+]\}, N_h^+\Big) \end{aligned}$$

We claim that outside the square $[N^-, N^+] \times [N^-, N^+]$, $H(m,n) = a_0$, since, for instance, if $m < N^-$ and $n > N^+$, then $n > N_h^+$ so $h(n) = f_1$ and as $m < N_{h(n)}^-$, $h(n)(m) = f_1(m) = a_0$, as required. The other cases are similar.

Now assume that f_0 and f_1 are loops at a_0 in A which yield the same element of the fundamental group $\pi_1(\mathsf{A}, a_0)$. Then there is a fixed end point homotopy

$$H : f_0 \simeq f_1$$

between them and reversing the above process, we obtain a free path in $\Omega(\mathsf{A}, a_0)$. We have only to note that the definitions of the concatenation operations in $\pi_1(\mathsf{A}, a_0)$ and $\pi_0(\Omega(\mathsf{A}, a_0))$ correspond exactly, to conclude that

$$\pi_1(\mathsf{A}, a_0) \cong \pi_0(\Omega(\mathsf{A}, a_0)).$$

7. Simplicial complexes from global actions.

If A is a global action, a path f in A is a sequence of points $\cdots, f(n), \cdots$ so that pairs of successive points are in the orbit of some local action. A path can thus wander from one local "patch" to the next by going via a point in their intersection. It is only that each $f(n)$ is in a local orbit with $f(n-1)$ and in a local orbit with

$f(n+1)$, not the group elements used that matter. This suggests that f yields a path in a combinatorially defined simplicial complex constructed by considering finite families of points within local orbits. This is the case. We will describe this for a general groupoid atlas, A.

The simplicial complex $V(\mathsf{A})$ of A has $|\mathsf{A}|$ as its set of vertices. A subset

$$\sigma = \{x_0, \cdots, x_p\}$$

is a p-simplex of $V(\mathsf{A})$ if there is some $\alpha \in \Phi_{\mathsf{A}}$ for which σ is an α-frame, i.e., $\sigma \subseteq (X_{\mathsf{A}})_\alpha$ and there are $g_i : x_0 \to x_i \in (\mathcal{G}_{\mathsf{A}})_\alpha$, so σ is contained in a single connected component of $(\mathcal{G}_{\mathsf{A}})_\alpha$.

It is clear that $\Pi_1 V(\mathsf{A}) \cong \Pi_1\mathsf{A}$ and if $a_0 \in \mathsf{A}$ is a base point then

$$\pi_1(V(\mathsf{A}), \{a_0\}) \cong \pi_1(\mathsf{A}, a_0).$$

The one disadvantage of $V(\mathsf{A})$ is that it has as many vertices as A has elements and this can obscure the essential combinatorial structure involved. This construction of $V(\mathsf{A})$ is an analogue of the Vietoris construction used in Alexander-Čech cohomology theory in algebraic topology. It is an instance of a general construction that associates two simplicial complexes to a relation from one set to another. This construction was studied by C. H. Dowker, [12]. We outline his results.

7.1. The two nerves of a relation: Dowker's construction

Let X, Y be sets and R a relation between X and Y, so $R \subseteqq X \times Y$. We write xRy for $(x, y) \in R$.

For our case of interest X is the set of points of A and Y is the set of local components of the local groupoids if A is groupoid atlas and is thus the set of local orbits in the global action case. The relation is 'xRy if and only if $x \in y$'. Two other exemplary cases should be mentioned.

Example 7.1. *Let X be a set (usually a topological space) and Y be a collection of subsets of X covering X, i.e. $\bigcup Y = X$. The classical case is when Y is an index set for an open cover of X. The relation is the same as above i.e. xRy if and only if $x \in y$ or more exactly x is in the subset indexed by y.* □

Example 7.2. *If K is a simplicial complex, its structure is specified by a collection of non-empty finite subsets of its set of vertices namely those sets of vertices declared to be simplices. This collection of simplices is supposed to be downward closed, i.e., if σ is a simplex and $\tau \subseteqq \sigma$ with $\tau \neq \emptyset$, then τ is a simplex. For our purposes here, set $X = V_K$ to be the set of vertices of K and $Y = S_K$, the set of simplices of K with xRy if x is a vertex of the simplex y.* □

Returning to the general situation we define two simplicial complexes associated to R, as follows:

(i) $K = K_R$:
   - the set of vertices is the set X;
   - a p-simplex of K is a set $\{x_0, \cdots, x_p\} \subseteq X$ such that there is some $y \in Y$ with $x_i R y$ for $i = 0, 1, \cdots, p$.

(ii) $L = L_R$:
   - the set of vertices is the set, Y;
   - p-simplex of K is a set $\{y_0, \cdots, y_p\} \subseteq Y$ such that there is some $x \in X$ with $x R y_j$ for $j = 0, 1, \cdots, p$.

Clearly the two constructions are in some sense dual to each other. For our situation of global actions/groupoid atlases, K_R is $V(\mathsf{A})$. The corresponding L_R does not yet seem to have been considered in exactly this context. We will denote it $N(\mathsf{A})$ so if $\sigma \in N(\mathsf{A})_p$

$$\sigma = \left\{U_{\alpha_0}, \cdots, U_{\alpha_p}\right\}$$

with U_{α_i} a local orbit for $G_{\alpha_i} \curvearrowright X_{\alpha_i}$ or a connected component of $\mathcal{G}_{\alpha_i}$ with the requirement that $\bigcap \sigma = \bigcap_{i=0}^{p} U_{\alpha_i} \neq \emptyset$.

In the case of X a space with Y an open cover, K_R is the *Vietoris complex* of X relative to Y whilst L_R is the nerve of the open cover (often called the *Čech complex* of X relative to the cover). We will consider the other example in detail later on.

7.2. Barycentric subdivisions

Combinatorially, if K is a simplicial complex with vertex set V_K, then one associates to K the partially ordered set of its simplices. Explicitly we write S_K for the set of simplices of K and $(S_K, \subseteq)$ for the partially ordered set with $\subseteq$ being the obvious inclusion. The barycentric subdivision, K', of K has S_K as its set of vertices and a finite set of vertices of K' (i.e. simplices of K) is a simplex of K' if it can be totally ordered by inclusion. (Thus K' is the simplicial complex given by taking the nerve of the poset, $(S_K, \subseteq)$. We may sometimes write $Sd(K)$ instead of K'.)

Remark 7.3. *It is important to note that there is in general no natural simplicial map from K' to K. If however V_K is ordered in such a way that the vertices of any simplex in K are totally ordered (for instance by picking a total order on V_K), then one can easily specify a map*

$$\varphi : K' \to K$$

by:
if $\sigma' = \{x_0, \cdots, x_p\}$ is a vertex of K' (so $\sigma' \in S_K$), let $\varphi\sigma'$ be the least vertex of σ' in the given fixed order.

This preserves simplices, but reverses order so if $\sigma'_1 \subset \sigma'_2$ then $\varphi(\sigma'_1) \geqslant \varphi(\sigma'_2)$.

If one changes the order, then the resulting map is *contiguous*:
Let $\varphi,\ \psi : K \to L$ be two simplicial maps between simplicial complexes. They are *contiguous* if for any simplex σ of K, $\varphi(\sigma) \cup \psi(\sigma)$ forms a simplex in L.
Contiguity gives a constructive form of homotopy applicable to simplicial maps.

If $\psi : K \to L$ is a simplicial map, then it induces $\psi' : K' \to L'$ after subdivision. As there is no way of knowing/picking compatible orders on V_K and V_L in advance, we get that on constructing

$$\varphi_K : K' \to K$$

and

$$\varphi_L : L' \to L$$

that $\varphi_L \psi'$ and $\psi\varphi$ will be contiguous to each other but rarely equal.

Returning to K_R and L_R, we order the elements of X and Y. Then suppose y' is a vertex of L'_R, so $y' = \{y_0, \cdots, y_p\}$, a simplex of L_R and there is an element $x \in X$ with $xRy_i, i = 0, 1, \cdots, p$. Set $\psi y' = x$ for one such x.

If $\sigma = \{y'_0, \cdots, y'_q\}$ is a q-simplex of L'_R, assume y'_0 is its least vertex (in the inclusion ordering)

$$\varphi_L(y'_0) \in y'_0 \subset y' \text{ for each } y_i \in \sigma$$

hence $\psi y'_i R \varphi_L(y'_0)$ and the elements $\psi y'_0, \cdots, \psi y'_q$ form a simplex in K_R, so $\psi : L'_R \to K_R$ is a simplicial map. It, of course, depends on the ordering used and on the choice of x, but any other choice $\bar{x}$ for $\psi y'$ gives a contiguous map.

Reversing the rôles of X and Y in the above we get a simplicial map

$$\bar{\psi} : K'_R \to L_R.$$

Applying barycentric subdivisions again gives

$$\bar{\psi}' : K''_R \to L'_R$$

and composing with $\psi : L'_R \to K_R$ gives a map

$$\psi\bar{\psi}' : K''_R \to K_R.$$

Of course, there is also a map

$$\varphi_K \varphi'_K : K''_R \to K_R.$$

Proposition 7.4. *(Dowker, [12] p.88). The two maps $\varphi_K \varphi'_K$ and $\psi\bar{\psi}'$ are contiguous.*
□

Before proving this, note that contiguity implies homotopy and that $\varphi\varphi'$ is homotopic to the identity map on K_R after realisation, i.e., this shows that

Corollary 7.5.

$$|K_R| \simeq |L_R|.$$

□

The homotopy depends on the ordering of the vertices and so is not natural.

Proof. of Proposition.

Let $\sigma''' = \{x''_0, x''_1, \cdots, x''_q\}$ be a simplex of K''_R and as usual assume x''_0 is its least vertex, then for all $i > 0$

$$x''_0 \subset x''_i.$$

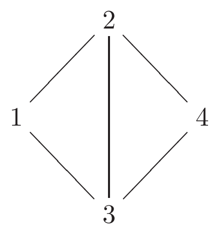
We have that φ'_K is clearly order reversing so $\varphi'_K x''_i \subseteq \varphi'_K x''_0$. Let $y = \bar{\varphi}\varphi'_K x''_0$, then for each $x \in \varphi'_K x''_0$, xRy. Since $\varphi_K \varphi'_K x''_i \in \varphi'_K x''_i \subseteq \varphi'_K x''_0$, we have $\varphi_K \varphi'_K x''_i Ry$.

For each vertex x' of $x''_i, \bar{\psi} x' \in \bar{\psi}' x''_i$, hence as $\varphi'_K x''_0 \in x''_0 \subset x''_i, y = \bar{\psi}\varphi'_K x x''_0 \in \bar{\psi}' x''_i$ for each x''_i, so for each $x''_i, \psi\bar{\psi}' x''_i Ry$, however we therefore have

$$\varphi_k \varphi'_K(\sigma'') \cup \psi\bar{\psi}(\sigma''') = \bigcup \varphi_k \varphi'_K(x''_i) \cup \psi\bar{\psi}; x''_i$$

forms a simplex in K_R, i.e. $\varphi_K \varphi'_K$ and $\psi\bar{\psi}'$ are contiguous. □

Example 7.6. *To illustrate both the Proposition and the remaining example of K_R and L_R, consider the simplicial complex, K:*



consisting of two 2-simplices joined along a common edge. More precisely, take $X = V_K = \{1, 2, 3, 4\}$ with this as given order and Y to be the set S_K of simplices of K, so S_K consists of all non-empty subsets of V_K that do not contain $\{1, 4\}$.

There are 11 elements in Y.

The relation R from X to Y in xRy if and only if x is a vertex of simplex y.

In $K_R, \{x_0, \cdots, x_p\}$ is a simplex if there is a $y \in Y$ such that each $x_i \in y$, so with K_R we retrieve exactly K itself.

Before looking at L_R consider a simpler example.

If we consider $\Delta[n]$, the n-simplex, with vertices $X = \{0, 1, \cdots, n\}$ and the non-empty subsets of X as simplices then K_R will be $\Delta[n]$, but the vertices of L_R will be the set of simplices of $\Delta[n]$, the 1-simplices of L_R will be pairs of simplices with non-empty intersection. In particular for each vertex, i, of $\Delta[n]$, there will be a $(2^n - 1)$-simplex in L_R namely that obtained by considering the power set of $X \backslash \{i\}$ (yielding 2^n elements) and adding in the singleton $\{i\}$ to each of these sets. For instance for $n = 2$, $X = \{0, 1, 2\}$ and there is a 3-simplex $\{\{0\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}\}$ in L_R. Thus L_R has much higher dimension than the original K.

Among the simplices of L_R, however, we have all of those that are totally ordered in the inclusion ordering and these give a sub-complex of L_R that is isomorphic to

K', the barycentric subdivision of K. This is true in general and in our example of the two 2-simplices with a shared edge, the complex L_R contains the barycentric subdivision of K_R, but also has some higher dimensional simplices such as

$$\sigma_2^{\max} = \{\{2\}, \{1,2\}, \{2,3\}, \{2,4\}, \{1,2,3\}, \{2,3,4\}\}.$$

Of course the inclusion map $K'_R \to L_R$ is part of that structure used in the lemma. The map from L'_R to K_R is now relatively easy to describe. The above 5-simplex $\sigma_2^{\max}$ is a simplex because the element 2 is in all of the parts so $\psi\sigma_2^{\max} = 2$. In general of course there will be a choice of element, for instance,

$$\sigma_{\{2,3\}}^{\max} = \{\{2,3\}, \{1,2,3\}, \{2,3,4\}\},$$

and is a simplex of L_R because its intersection is non-empty as it contains both 2 and 3, thus there are two different maps one using $\psi\sigma_{\{2,3\}}^{\max} = 2$, the other using 3 as image point. Of course they are contiguous. The complex L_R seems to include aspects of both the barycentric subdivision and the dual complex. The explosion in dimension is, of course, typical here as, for instance, in the case of an open cover of a topological space the nerve of the cover yields a simplicial complex whose dimension indicates the multiple overlaps in the cover and as the cover is varied reflects the covering dimension of the space but typically the Vietoris complex is of unbounded dimension.

Returning to global actions and groupoid atlases, combining earlier results, we have that:

Proposition 7.7. *If A is a global action or groupoid atlas, pointed at a_0, then*

$$\pi_1(\mathsf{A}, a_0) \cong \pi_1(N(\mathsf{A}), [a_0])$$

where $[a_0]$ is a connected component of some local groupoid $(\mathcal{G}_\mathsf{A})_\alpha$ with $a_0 \in (X_\mathsf{A})_\alpha$. □

7.3. V and N on morphisms

We would clearly expect these constructions, V and N to extend to give us functors from global actions /groupoid atlases to simplicial complexes. Life is not quite that simple, but almost. We have to check what they do to the various strengths of morphism.

Weak morphisms

As weak morphisms are defined in terms of local frames, and simplices in $V(\mathsf{A})$ are essentially just local frames, it is clear that

Lemma 7.8. *If $f : \mathsf{A} \to \mathsf{B}$ is a weak morphism, then f induces a simplicial map $V(f) : V(\mathsf{A}) \to V(\mathsf{B})$, by $V(f)\langle x_0, \ldots, x_n\rangle = \langle f(x_0), \ldots, f(x_n)\rangle$, followed by elimination of repeats.*

The only point for comment is the last phrase: $\langle f(x_0), \ldots, f(x_n)\rangle$ may not actually be a simplex as it may involve repeated elements, but on eliminating these repeats we will get a simplex. This minor technicality can be avoided using simplicial sets where degenerate simplices are part of the structure, but their use would entail other complications so we merely note that 'technicality' here. It causes no real problem.

The corresponding result for N is much more complicated. For $\{U_\alpha\}$ in $N(\mathsf{A})$, we know there is some $x \in U_\alpha$ and hence $\{f(x)\}$ is a local frame in some $(X_{\mathsf{B}})_\beta$, so there is some $\beta \in \Phi_{\mathsf{B}}$ with $\{f(x)\}$ a β-frame and so an orbit or connected component $\{U'_\beta\}$ containing it. We could map $\{U_\alpha\}$ to $\{U'_\beta\}$, but there is no reason to suppose that $\{U'_\beta\}$ will be the only such possibility, there may be many. Of course, $\{U'_\beta \mid f(x) \in U'_\beta\}$, if finite, will define a simplex of $N(\mathsf{B})$ and so we might, in that case, attempt to define the corresponding mapping from $N(\mathsf{A})$ to $N'(\mathsf{B}) = SdN(\mathsf{B})$, that is the barycentric subdivision of $N(\mathsf{B})$. This could work with care, but then the functoriality gets complicated since if we have also $g : \mathsf{B} \to \mathsf{C}$ as well, the composite of our possible $N(f)$ with $N(g)$ is not possible as the former ends at $N'(\mathsf{B})$ whilst the latter starts at $N(\mathsf{B})$. We could apply subdivision to the second map and then compose but then the composite ends up at $N''(\mathsf{C})$ not at $N'(\mathsf{C})$, which we would need for functoriality. This sort of situation is well understood in homotopy theory as it corresponds to the presence of homotopy coherence caused by the necessity of chosing an image amongst the possible vertices of a simplex. The different choices are homotopic in a 'coherent' way. It however makes the nerve construction much more complicated to use than the Vietoris one if weak morphisms are being used.

One way around the difficulty is to take geometric realisations as $|N(\mathsf{B})| \cong |N'(\mathsf{B})|$, but this defeats the purpose of working with global actions in the first place, which was to avoid topological arguments as they tended to obscure the algebraic and combinatorial processes involved. Probably the safest way is to use $V(\mathsf{A})$ when developing theoretic arguments involving weak morphisms, but using $N(\mathsf{A})$ for calculations 'up to homotopy' as $N(\mathsf{A})$ is often much smaller than $V(\mathsf{A})$.

Strong morphisms

If we now turn to strong morphisms, as any strong morphism preserves local frames, it is also a weak morphism and so we have no difficulty in inducing a $V(f)$ from a given strong morphism $f : \mathsf{A} \to \mathsf{B}$. We are thus left to see if the 'strength' of f allows us to avoid the difficulties we had above in defining a $N(f)$.

Suppose $\{U_\alpha\}$ is a vertex in $N(\mathsf{A})$ as before, then U_α is a (non-empty) local orbit in $\mathcal{G}_\alpha$. We have $f = (\eta_X, \eta_\Phi, \eta_{\mathcal{G}})$, $(\eta_{\mathcal{G}})_\alpha : \mathcal{G}_{\mathsf{A},\alpha} \to \mathcal{G}_{\mathsf{B},\eta_\Phi(\alpha)}$, and $(\eta_{\mathcal{G}})_\alpha(U_\alpha)$ is contained in a uniquely defined connected component of $\mathcal{G}_{\mathsf{B},\eta_\Phi(\alpha)}$, which we will denote by $f_*(U_\alpha)$.

Suppose now we have $\sigma = \{U_{\alpha_0}, \ldots, U_{\alpha_n}\}$ in $N(\mathsf{A})$, so there is some $x \in \bigcap \sigma$. Consider the family $f_*(\sigma) := \{f_*(U_{\alpha_0}), \ldots, f_*(U_{\alpha_n})\}$, does it have non empty intersection? On objects $(\eta_{|\mathcal{G}})_{\alpha_i}$ is just η_X, so $\eta_X x$ is an object of $f_*(U_{\alpha_i})$ for each i and so $f_*(\sigma)$ is a simplex of $N(\mathsf{B})$, with the usual proviso that repeats are removed. The construction of $f_*(\sigma)$ from σ has been made without choices and is well defined, moreover if we define $N(f)(\sigma)$ to be this $f_*(\sigma)$, $N(f)$ is a simplicial mapping and there is no problem with functoriality: $N(gf) = N(g)n(f)$. We thus have that whilst V behaves well with both types of morphsims, N behaves well only with strong morphisms.

7.4. 'Subdivision' of $\mathsf{A}(G,\mathcal{H})$

Earlier we saw that in the setting of a group G and a family $\mathcal{H} = \{H_i \mid i \in \Phi\}$ of subgroups, we could construct at least two global actions. In one of these constructions we took $\Phi_{\mathsf{A}} = \Phi$ with the discrete order. Although for this $\mathsf{A} = \mathsf{A}(G,\mathcal{H})$, the description of $N(\mathsf{A})$ was very simple (it is a generalisation of the intersection diagram we used in section 2). We noted in the discussion of the construction of $\Omega\mathsf{A}$ that this type of global action suffers from the discreteness of its coordinate system as there were few framings for curves. In fact the only curves with framings were those within a single local orbit.

This deficiency serves to highlight the importance of the order in $(\Phi_{\mathsf{A}}, \leqslant)$ and its influence on the homotopy properties of A. If A does not have enough framings of curves, paths or loops, then there will be a divergence between the properties of $\Omega\mathsf{A}$ and those of the loops on $V(\mathsf{A})$ or $N(\mathsf{A})$.

Given this difficulty, how can we change the construction of $\mathsf{A}(G,\mathcal{H})$ to gain more framings of curves? In fact, if Φ is not a singleton, this can be done in a variety of ways, of graded strength. We will look in detail at the strongest one.

The problem of framings was that, if $f : \mathsf{L} \to \mathsf{A}$ was a curve, a framing for f was a mapping $\beta : |\mathsf{L}| \to \Phi_{\mathsf{A}}$, so that $f(m) \in (X_{\mathsf{A}})_{\beta(m)}$ for each m and there was a b in Φ_{A} bigger than both $\beta(m)$ and $\beta(m+1)$, so that $f(m)$ and $f(m+1)$ were linked by some g in $(\mathcal{G}_{\mathsf{A}})_b$.

If Φ_{A} is discrete, then $b \geqslant \beta(m)$ and $b \geqslant \beta(m+1)$ implies equality of $\beta(m)$ and $\beta(m+1)$. This is not the intuition intended, but it is not the fault of 'framings', rather of the discreteness of Φ_{A}. Intuitively we expect $f(m)$ to be in two of the local orbits, so as to link previous vertices to the new ones later in the sequence. We thus need these intersections there. We could do this by replacing $\mathcal{H}$ by $\{H_i \cap H_j \mid i,j \in \Phi\}$, but then higher dimensional homotopy might perhaps suffer. It is easier to close $\mathcal{H}$ up under finite intersections as follows:

Take G, $\mathcal{H}$ as before with $\mathcal{H} = \{H_i \mid i \in \Phi\}$. Define a (new) global action $\mathsf{A}'(G,\mathcal{H})$ by

$$\begin{aligned} X = |X_{\mathsf{A}'}| &= |G|, \text{ the underlying set of } G \\ \Phi_{\mathsf{A}'} &= \text{ the set of non-empty subsets of } \Phi \\ &\quad \text{ ordered by } \supseteq, \text{ i.e. } \alpha \leqslant \beta \text{ if } \alpha \supseteq \beta; \\ (X_{\mathsf{A}'})_\alpha &= X_{\mathsf{A}'} \text{ for all } \alpha \in \Phi_{\mathsf{A}'} \\ (G_{\mathsf{A}'})_\alpha &= \bigcap_{i\in\alpha} H_i \text{ operating by left multiplication} \end{aligned}$$

with :

if $\alpha \leqslant \beta$, (so $\alpha \supseteq \beta$), then

$$(G_{\mathsf{A}'})_{\alpha\leqslant\beta} : \bigcap_{i\in\alpha} H_i \to \bigcap_{j\in\beta} H_j$$

being inclusion.

We can think of $\mathsf{A}' = \mathsf{A}'(G,\mathcal{H})$ as a 'subdivision' of $\mathsf{A}(G,\mathcal{H})$, rather like the barycentric subdivision above. The effect of this 'subdivision' on the simplicial complexes $V(\mathsf{A})$ and $N(\mathsf{A})$ is interesting:
a) Vietoris: $V(\mathsf{A}') = V(\mathsf{A})$, since any $\{x_0,\ldots,x_n\} \in V(\mathsf{A}')_n$ is there because there is some orbit $(\bigcap_{i\in\alpha} H_i)x_0$ containing it, but then $\{x_0,\ldots,x_n\} \subseteq H_i x_0$ for any $i \in \alpha$, so there are no simplices in either of the two complexes, not in the other.
b) Nerve: the relationship is more complex. We know from Dowker's theorem (above) that $|N(\mathsf{A})| \simeq |N(\mathsf{A}')|$ and, denoting by $N(\mathsf{A})'$, the barycentric subdivision of $N(\mathsf{A})$, we can relate $N(\mathsf{A})$ and $N(\mathsf{A})'$ to $N(\mathsf{A}')$. The old local orbits of A are still there in A', so we have an inclusion

$$N(\mathsf{A}) \to N(\mathsf{A}')$$

corresponding to a (strong) morphism from A to A'.

Now assume we have $\sigma \in N(\mathsf{A}')$. The vertices of σ can be totally ordered by inclusion:

$$\sigma = \{\{H_i x_i \mid i \in \alpha_0\},\ldots,\{H_i x_i \mid i \in \alpha_n\}\}.$$

with $\alpha_0 \subseteq \ldots \subseteq \alpha_n$ and $\bigcap\{H_i \mid i \in \alpha_n\} \neq \emptyset$. We therefore have an element, a in this intersection and so σ can also be written

$$\sigma = \{\{H_i a \mid i \in \alpha_0\},\ldots,\{H_i a \mid i \in \alpha_n\}\}.$$

We can assign a vertex of $N(\mathsf{A}')$ to each of the vertices of σ, by sending $\{H_i a \mid i \in \alpha\}$ to its intersection $(\bigcap H_i)a$. This will give a simplicial map from $N(\mathsf{A})'$ to $N(\mathsf{A}')$, but will often collapse simplices. (It is well behaved as a map of simplicial sets[3] , but not that nice at the simplicial complex level.) For instance, if $\mathcal{H} = \{H_1, H_2, H_3\}$ with $H_i \cap H_j = \{1\}$ if $i \neq j$, then the 2-simplex,

$$\{\{H_1\},\{H_1,H_2\},\{H_1,H_2,H_3\}\}$$

gets mapped to $\{H_1,\{1\}\}$, a 1-simplex, and there are a lot of other collapses as well.

The global action / groupoid atlas A' has the necessary property with regard to framings:

Suppose $f : \mathsf{L} \to \mathsf{A}'$ is a curve, then for $m \in \mathbb{Z}$, we have a $\beta- \in \Phi$ with a $g- : f(m) \to f(m+1)$ in $(\mathcal{G}_{\mathsf{A}})_{\beta-}$ and a $\beta+ \in \Phi$ with a $g+ : f(m) \to f(m+1)$; we take $\beta(m)$ to be $\{\beta-,\beta+\}$, or any family containing this. Of course, when we look

[3]If we put a total order on the vertices of a simplicial complex, K it naturally gives us a simplicial set, K^{simp} as we can control the notion of degenerate simplices, i.e. ones with repeated entries. We will not be using this much in this paper but we note that it can be useful when handling maps that collapse simplices as we have seen earlier. The corresponding simplicial set $V^{\mathrm{sing}}(\mathsf{A})$ is related to the bar resolution in the case that $\mathsf{A} = \mathsf{A}(G,\mathcal{H})$.

at $f(m+1)$, we can take its $\beta-$ to be the $\beta+$ of $f(m)$, so $\beta+ \geqslant b(m)$, $\beta(m+1)$ and the framing can be constructed. (Of course, the element of choice here can be avoided by replacing $\beta+$ by the family of all $\alpha \in \Phi$ such that a suitable $g+$ exists in $(\mathcal{G}_{\mathsf{A}})_\alpha$.)

If instead of L, we were mapping in higher dimensional objects, we would need, not just pairwise families, but all, as we have done. Effectively, in replacing A by A', we have introduced an object that is more 'complete' with respect to local frames than the original A. This 'completeness' allows much better exponentiation properties: we would be able to form A^{B} for any B and the result will have the 'right' sort of behaviour. The 'completeness' property required is called the 'infimum condition', (cf. Bak, [2–4]).

A groupoid atlas A satisfies the *infimum condition* if given any non-empty finite subset $U \subseteq |\mathsf{A}|$, the set

$$\{\alpha \in \Phi_{\mathsf{A}} : U \text{ is a local frame in } \alpha\}$$

is empty or has an initial element in the order induced from Φ_{A}.

Example 7.9. $\mathsf{A}'(G, \mathcal{H})$ *is infimum (i.e. satisfies the infimum condition).*

If $U = \{x_0, \dots, x_n\}$ *is any finite set of elements of* G*, let*

$$\alpha = \{H \in \Phi : x_1, \dots, x_n \in Hx_0\}.$$

If α *is non-empty, then* $\alpha \in \Phi_{\mathsf{A}'}$ *and* $x_0, \dots, x_n$ *is in* $(\bigcup_\alpha H)x_0$*. This* α *is thus the initial element required, or is empty.* □

Problem/Question 7.10. *Given any groupoid atlas,* A*, find a groupoid atlas* A'*, which satisfies the infimum condition, comes together with a strong morphism,*

$$\mathsf{A} \to \mathsf{A}'$$

and, if possible, is universal with these properties.

Remarks on the problem.

It seems that $\mathsf{A}'(G, \mathcal{H})$ will be the solution for $\mathsf{A}(G, \mathcal{H})$. Presumably other single domain global actions will be 'completed' in a similar way.

We plan to return to the infimum condition later in this sequence of papers.

The passage from $\mathsf{A}(G, \mathcal{H})$ to $\mathsf{A}'(G, \mathcal{H})$ really corresponds to an operation on $\mathcal{H}$, giving Φ itself more structure and closing $\mathcal{H}$ up under intersections. If we extend the notation $\mathsf{A}(G, \mathcal{H})$ to include families $\mathcal{H}$ with additional structure, then a convenient notation is $\mathsf{A}(G, \bar{\mathcal{H}})$ for what we have denoted $\mathsf{A}'(G, \mathcal{H})$, thus emphasising that it is the 'closure' of $\mathcal{H}$, that is used. This construction of $\mathsf{A}(G, \bar{\mathcal{H}})$, with two additional conditions, is closely related to Volodin's definition of K-groups of rings with extra structure, [19]. We will assume $\mathcal{H}$ is already 'closed' in this way:

Suppose as before that G is a group. Let $\mathcal{H}$ be a family of subgroups of G indexed by $(\Phi_{\mathsf{A}}, \leqslant)$, where

(i) $H_\alpha = H_\beta$ if and only if $\alpha = \beta$;
(ii) $\alpha \leqslant \beta$ if and only if $H_\alpha \subseteq H_\beta$;
(iii) there is some $* \in \Phi_{\mathsf{A}}$ with $H_* = \{1\}$.

We assume $\mathcal{H}$ is closed under arbitrary intersections, and that if H_α and H_β are contained in some $H_{\gamma'}$, $\gamma' \in \Phi_{\mathsf{A}}$, then the subgroup generated by H_α and H_β, denoted $\langle H_\alpha, H_\beta\rangle$, is itself some H_γ for $\gamma \in \Phi_{\mathsf{A}}$. (Thus the order structure of $(\Phi_{\mathsf{A}}, \leqslant)$ is almost a lattice, but a top element need not exist.) In this case $\mathsf{A}(G, \mathcal{H})$ is called a *Volodin model.*

8. Calculations of fundamental groups - some easy examples.

Our examples will all be single domain global actions, i.e. the local actions are all based on a single set, $(X_{\mathsf{A}})_\alpha = X_{\mathsf{A}}$ for all $\alpha \in \Phi_{\mathsf{A}}$. They will all be of the form $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$, where G is a group and $\mathcal{H} = \{H_i \mid i \in \Phi\}$ is a family of subgroups (see section 1).

Example 8.1. *(already considered in example 2.5)*

$$\begin{aligned} G = S_3 &= \langle a, b \mid a^3 = b^2 = (ab)^2 = 1\rangle, \ \textit{so } a = (1,2,3),\ b = (1,2);\\ H_1 &= \langle a\rangle = \{1,\ (1,2,3),\ (1,3,2)\},\\ H_2 &= \langle b\rangle = \{1,\ (1,2)\};\\ \mathcal{H} &= \{H_1, H_2\} \end{aligned}$$

The intersection diagram given in our earlier look at this example is in fact the nerve $N(\mathsf{A})$ having 5 vertices and 6 edges. The other complex $V(\mathsf{A})$ is almost as simple. It has 6 vertices corresponding to the 6 elements of S_3, and each orbit yields a simplex
$H_1 = \{1, a, a^2\}$ gives a 2-simplex (and 3 1 -simplices),
$H_1 b = \{b, ab, a^2 b\}$ also gives a 2-simplex;
$H_2 = \{1, b\}$ yields a 1-simplex, as do its cosets $H_2 a$ and $H_2 a^2$.

We thus have $V(\mathsf{A})$ has two 2-simplices joined by 1-simplices at the vertices, (see below).

As $N(\mathsf{A})$ is a connected graph with 5 vertices and 6 edges, we know $\pi_1 N(\mathsf{A})$ is free on 2 generators. (The number of generators is the number of edges outside a maximal tree.) This same rank can be read of equally easily from $V(\mathsf{A})$ as that complex is homotopically equivalent to a bouquet of 2 circles, (i.e. a figure eight). The generators can be identified with words in the free product $H_1 * H_2$ (one choice being shown in example 2.5) and relate to the kernel of the natural homomorphism from $H_1 * H_2$ to S_3.

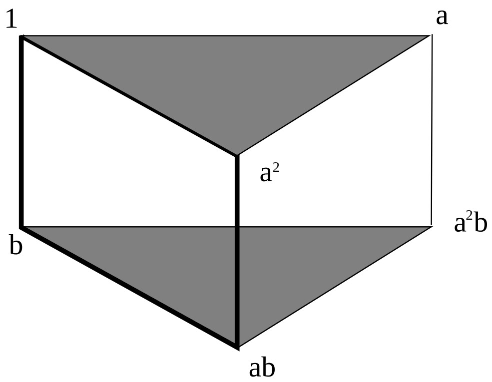


Figure 1: $V(\mathsf{A}(S_3, \{\langle a\rangle, \langle b\rangle\}))$

The heavy line in the figure corresponds to a loop at 1 given by

$$1 \xrightarrow{b} b \xrightarrow{a} ab \xrightarrow{b} a^2 \xrightarrow{a} 1$$

and the word is $abab \in C_2 * C_3$.

The reason that this happens in clear. Starting at 1, each part of the loop corresponds to a left multiplication either by an element of $H_1 \cong C_3$ or of $H_2 \cong C_2$. We thus get a word in $H_1 * H_2 \cong C_2 * C_3$. As the loop also finishes at 1, we must have that the corresponding word must evaluate to 1 when projected down into S_3.

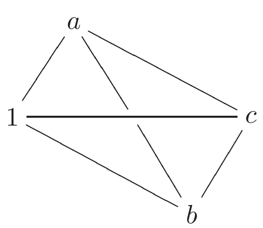
In more complex examples, the interpretation of $\pi_1(V(\mathsf{A}), 1)$ will be the same, but sometimes when G has more elements, $N(\mathsf{A})$ may be easier to analyse than $V(\mathsf{A})$. The important idea to retain is that the two complexes give the same information, so either can be used or both together.

Some of the limitations of the information encoded by $\pi_1(\mathsf{A})$ are illustrated by the next two examples.

Example 8.2. *$G = K_4$, the Klein 4 group, $\{1, a, b, c\} \cong C_2 \times C_2$, so $a^2 = b^2 = c^2 = 1$ and $ab = c$;*
$\mathcal{H} = \{H_a, H_b, H_c\}$ where $H_a = \{1, a\}$, etc. Set $\mathsf{A}_{K4} = \mathsf{A}(K_4, \mathcal{H})$.

The cosets are $H_a, H_a b, H_b, H_b a, H_c, H_c a$ each with two elements so

$V(\mathsf{A}_{K4}) \cong$ the 1-skeleton of $\Delta[3]$



$N(\mathsf{A}_{K4})$ *is "prettier":*

Labelling the cosets from 1 to 6 in the order given above, we have 6 vertices, 12 1-simplices and 4 2-simplices. For instance $\{1, 3, 5\}$ *has the identity in the intersection,* $\{1, 4, 6\}$ *gives* $H_a \cap H_b a \cap H_c a$*, so contains* a *and so on. The picture is of the shell of an octahedron with 4 of the faces removed.*

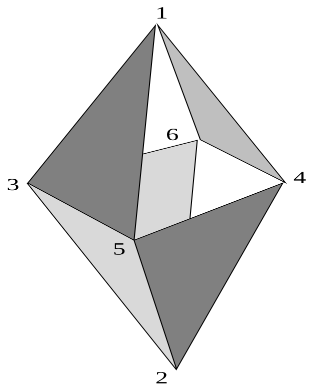


Figure 2: $N(\mathsf{A}_{K4})$

From either diagram it is clear that $\pi_1 \mathsf{A}_{K4}$ *is free of rank 3.*

Again explicit representations for elements are easy to give.

Using $V(\mathsf{A})$ *and the maximal tree given by the edges 1a, 1b and 1c, a typical generating loop would be*

$$1 \to a \to b \to 1,$$

i.e., $(1, a, b, 1)$ *as the sequence of points. There is a strong representative for this, namely*

$$1 \xrightarrow{a} a \xrightarrow{c} b \xrightarrow{b} 1$$

and up to shifts, this is the only strong representative. □

In general any based path at 1 in an $\mathsf{A}(G, \mathcal{H})$ will yield a word in $\sqcup\mathcal{H}$, the free product of the family $\mathcal{H}$. Whether or not that representative is unique depends on whether or not there are complicated intersections and "nestings" between the subgroups in $\mathcal{H}$, since for instance, if H_i is a subgroup of H_j, then if $f(n) \to f(n+1)$ using $g \in H_i$, it could equally well be taken to be $g \in H_j$. The characteristic of the single domain global actions of form $\mathsf{A}(G, \mathcal{H})$ is that since $X_{\mathsf{A}} = G$, there is only one possible element linking each $f(n)$ to the next $f(n+1)$ namely $f(n+1)f(n)^{-1}$. We thus have a strong link between

$$\Gamma(\mathsf{A}(G, \mathcal{H})) \text{ and } \underset{\cap}{\sqcup}\mathcal{H},$$

the 'amalgamated product' of $\mathcal{H}$ over its intersections, and an analysis of homotopy classes will prove (later) that

$$\pi_1(\mathsf{A}(G,\mathcal{H}),1) \cong \operatorname{Ker}(\bigsqcup_{\cap}\mathcal{H} \to G),$$

since a based path $(g_1, g_2, \cdots, g_n)$ ends at 1 if and only if the product $g_1 \cdots g_n = 1$. These identifications will be investigated more fully (and justified) shortly.

Remark 8.3. *Many aspects of these $\mathsf{A}(\mathcal{G},\mathcal{H})$ are considered in the paper by Abels and Holz [1]. In particular the above identification of $\pi_1(N(\mathsf{A}(G,\mathcal{H})))$ in terms of the kernel of the evaluation morphism is Corollary 2.5 part (b) (page 318). Their proof uses covering space techniques. We will explore other aspects of their paper later.*

Example 8.4. *The number of subgroups in $\mathcal{H}$ clearly determines the dimension of $N(\mathsf{A})$, when $\mathsf{A} = \mathsf{A}(G,\mathcal{H})$. Here is another 3 subgroup example.*

Take $q8 = \{1, i, j, k, -1, -i, -j, -k\}$ to be the quaternion group, so $i^4 = j^4 = k^4 = 1$, and $ij = k$. Set $H_i = \{1, -1, i, -i\}$ etc., so $H_i \cap H_j = H_i \cap H_k = H_j \cap H_k = \{1, -1\}$ and let $\mathcal{H} = \{H_i, H_j, H_k\}$,
and $\mathsf{A}_{q8} = \mathsf{A}(q8,\mathcal{H})$.

Then $N(\mathsf{A}_{q8})$ is, as above in Example 8.2, a shell of an octahedron with 4 faces missing. Note however that $V(\mathsf{A}_{q8})$ has 8 vertices and, comparing with $V(\mathsf{A}_{K4})$, each edge of that diagram has become enlarged to a 3-simplex. It is still feasible to work with $V(\mathsf{A}_{q8})$ directly, but $N(\mathsf{A}_{q8})$ gives a clearer indication that

$$\pi_1(\mathsf{A}_{q8}, 1) \text{ is free of rank 3.}$$

Example 8.5. *Consider next the symmetric group, S_3, given by the presentation*

$$S_3 := \langle x_1, x_2 \mid x_1^2 = x_2^2 = 1, (x_1x_2)^3 = 1\rangle$$

Take $H_1 = \langle x_1\rangle$, $H_2 = \langle x_2\rangle$ so both are of index 3. Each coset intersects two cosets in the other list giving a nerve of form (see below): so $\pi_1(\mathsf{A}(S_3,\mathcal{H}))$ is infinite cyclic.

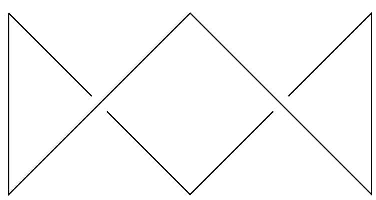


Figure 3: $N(\mathsf{A}(S_3,\mathcal{H}))$

Example 8.6. *The next symmetric group, S_4, has presentation*

$$S_4 := \langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = 1, (x_1x_2)^3 = (x_2x_3)^3 = 1, (x_1x_3)^2 = 1\rangle.$$

Take $H_1 = \langle x_1, x_2\rangle$, $H_2 = \langle x_2, x_3\rangle$, $H_3 = \langle x_1, x_3\rangle$. H_1 and H_2 are copies of S_3, but H_3 is isomorphic to the Klein 4 group, K_4. Thus there are 4 + 4 + 6 cosets in all.

There are 36 pairwise intersections and each edge is in two 2-simplices. Each vertex is either at the centre of a hexagon or a square, depending on whether it corresponds to a coset of H_1, H_2 or of H_3. There are 24 triangles, and $N(\mathsf{A}(S_4,\mathcal{H}))$ is a surface. Calculation of the Euler characteristic gives 2, so this is a triangulation of S^2, the two sphere. It is almost certainly the dual of the 'permutahedron'.(Thanks to Chris Wensley for help with the calculation using GAP.)

The fundamental group of $\mathsf{A}(S_4,\mathcal{H})$ is thus trivial and using the result mentioned above

$$S_4 \cong \bigsqcup_{\cap} H_i,$$

the coproduct of the subgroups amalgamated over the intersection..

Problems/Questions 8.7.

1) *Taking $S_n := \langle x_1, \cdots, x_{n-1} \mid x_i^2 = 1, i = 1, \cdots, n-1,$ all $(x_i x_{i+1})^3 = 1,$ and $(x_i x_j)^2 = 1$ if $|i-j| \geqslant 2\rangle, \mathcal{H} = \{H_i \mid i = 1, \cdots, n-1\}$ where H_i is generated by all the x_j* except *x_i, investigate if $N(\mathsf{A}(S_n,\mathcal{H}))$ is an $(n-2)$- sphere. (It is known to be of the homotopy type of a bouquet of S^{n-2}s cf. proof in Abels and Holz's paper [1] for Tits systems, S_n being a Coxeter group.)*
2) *There may be a link between $N(\mathsf{A}(G,\mathcal{H}))$ and various lattices of subgroups, classifying spaces for families of subgroups (Lück et al, Dwyer, ...). This needs investigation.*
3) *Examine other classes of groups e.g. generalised Coxeter groups relative to parabolics (cf. Abels and Holz, [1]), the triangle and von Dyck groups, $\Delta(\ell,m,n)$ and $D(\ell,m,n)$ and graph products of groups including graph groups.*

9. Single domain global actions I

In this section we will continue the study of single domain global actions including a discussion of their path spaces and fundamental groups. Certain facets of this study must wait until we have a theory of covering spaces for global actions/groupoid atlases.

We have so far examined in detail only those single domain global actions of the form $\mathsf{A} = \mathsf{A}(G,\mathcal{H})$ where G is a group and $\mathcal{H}$ is a family of subgroups of G. There are two obvious variants:

(i) Usually so far we have let $\mathcal{H} = \{H_i \mid i \in \Phi\}$ be a family of subgroups without any order structure on Φ. More generally one can take Φ to have a reflexive relation on it mirroring the intersection and subgroup relations between subgroups in $\mathcal{H}$. This seems to make little difference to invariants such as $N(\mathsf{A})$. Of course, this is related to the discussion of 'subdivision' above.

(ii) If K is a subgroup of G then we can form a variant of $\mathsf{A}(G, \mathcal{H})$ mod K. We will write this as $\mathsf{A} = \mathsf{A}((G, K), \mathcal{H})$. It is specified by

$$|X_{\mathsf{A}}| = G/K, \text{ the set of right cosets of } K \text{ in } G,$$
$$(X_{\mathsf{A}})_\alpha = X_{\mathsf{A}} \text{ for all } \alpha \in \Phi$$

(where Φ may be as in (i) above)

$$H_i \curvearrowright X_{\mathsf{A}} \text{ by left multiplication}$$

so the local orbits are of the form $H_i x K$.
We will see later that *all* connected single domain global actions have this general form, up to isomorphism.

We now turn to the investigation of the fundamental groups of single domain global actions (of this simple form). First we look at some results on paths under fixed end point homotopies.

9.1. The based paths on $\mathsf{A}(G, \mathcal{H})$

Suppose A is a general global action with Φ_{A} as coordinate system, $(X_{\mathsf{A}})_\alpha$ as local sets, $\alpha \in \Phi_{\mathsf{A}}$ and $(G_{\mathsf{A}})_\alpha$ as local groups. We assume A is connected and that a base point $a_0 \in \mathsf{A}$ has been chosen. In our earlier discussion we saw that a path in A based at a_0 is given by a curve

$$f : L \to \mathsf{A}$$

and thus by a sequence of points $f(n) \in X_{\mathsf{A}}$ such that

(i) there is a stabilisation pair (N_f^-, N_f^+), i.e., for $n \leqslant N_f^-$, $f(n) = a_0$, and for $n \geqslant N_f^+$, $f(n) = f(N_f^+)$;
(ii) there is a sequence of $\beta_n \in \Phi_{\mathsf{A}}$ with arrows $g_{\beta_n} : f(n) \to f(n+1)$ in the groupoid $(\mathcal{G}_{\mathsf{A}})_{\beta_n}$ (and as A is a global action, we assume $g_{\beta_n} \in (G_{\mathsf{A}})_{\beta_n}$), and $g_{\beta_n} f(n) = f(n+1)$.

(Furthermore the g_{β_n} are assumed to stabilise to the identity arrows for $n \geqslant n_f^+$, or $n \leqq n_f^-$.)

As we are considering weak curves, the particular g_β used are not really in question. If $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$, then this does not matter as there will be a unique g_β satisfying $g_{\beta_n} f(n) = f(n+1)$, namely $g_{\beta_n} = f(n+1) f(n)^{-1}$. In a relative case, the g_β will be determined up to multiplication by elements of K, and in the case of interrelationships between the H_i, the same element may be considered as an element of different H_i. In complete generality, we can thus say little about the elements g_β. Because of this, we will initially assume $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$, and may therefore base it at 1. We also assume A is connected.

We can thus consider f to be represented by a word in $\sqcup H_i$. As f starts at 1, f looks like

$$1 \xrightarrow[g_1]{} f(1) \xrightarrow[g_2]{} f(2) \xrightarrow[g_3]{} \cdots \longrightarrow f(N_f^+)$$

(We have reindexed to set N_f^- to 0.) The partial word $(g_m, \cdots, g_1)$ determines $f(m)$ since $g_m \cdots g_1 = f(m)$. We can thus think of f as being this list of elements $(g_m, \cdots, g_1)$ and hence as an element of $\sqcup H_i$. We will need to examine the loops (in which $f(N_f^+) = 1$), but can examine homotopy of based paths independently of what value is taken by $f(N_f^+)$.

A fixed end point homotopy from

$$f \leftrightarrow (g_N, \cdots, g_1)$$

to

$$f' \leftrightarrow (g'_{N'}, \cdots, g'_1)$$

is given by a map of $L \times L$ that stabilises in both directions. It can therefore be thought of as a sequence of "moves" on fs each corresponding to an elementary homotopy,

$$\begin{aligned} h : L \times L &\to \mathsf{A} \\ h(m, 0) &= f_0(m) = f(m) \end{aligned}$$

with

$$\begin{aligned} h(m, 1) &= f_1(m) \\ h(m, n) &= h(m, 0) \quad n \leqslant 0 \end{aligned}$$

and

$$h(m, n) = h(m, 1) \quad n \geqslant 1.$$

We then can visualise h as being given by a "ladder"

$$\begin{tikzcd} 1 \arrow[r, "g_1"] \arrow[d, equal] & f(1) \arrow[r, "g_2"] \arrow[d, "h_1"'] & f(2) \arrow[r] \arrow[d, "h_2"'] & {} \\ 1 \arrow[r, "g'_1"'] & f'(1) \arrow[r, "g'_2"'] & f'(2) \arrow[r] & {} \end{tikzcd} \qquad \text{etc.}$$

where $h_i f(i) = f'(i)$ and the square

$$\begin{tikzcd} f(i-1) \arrow[r, "g_i"] \arrow[d, "h_{i-1}"'] & f(i) \arrow[d, "h_i"] \\ f'(i-1) \arrow[r, "g_i"'] & f'(i) \end{tikzcd}$$

"commutes", so for some $H_{\alpha_i}, \alpha_i \in \Phi$,

a) $g'_i, g_i, h_{i-1}, h_i \in H_{\alpha_i}$;
b) $g'_i h_{i-1} = h_i g_i$.

This of course means that $h_i \in H_{\alpha_i} \cap H_{\alpha_{i+1}}$

Remark 9.1. *For convenience we assumed $f(0) = 1$, i.e., N_f^- can be taken to be 0, but the discussion of ripple homotopies earlier shows that shifting f to left or right keeps within the fixed end point homotopy class and, of course, does not change the representing word in $\sqcup H_i$.*

Returning to elementary homotopies, we can read off a new based path from the diagram above namely

$$1 \xrightarrow{g_1} f(1) \xrightarrow{h_1} f'(1) \xrightarrow{g_2'} f'(2) \xrightarrow{g_3'} \cdots$$

and so a representing word $(g'_{N'}, \cdots, g'_2, h_1, g_1)$. This process can be repeated to get

$$1 \xrightarrow{g_1} f(1) \xrightarrow{g_2} f(2) \xrightarrow{h_2} f'(2) \xrightarrow{g_3'} \cdots$$

so we can track the homotopy from f to f' *within* the representing words by moves that transfer

$$(\cdots, g_i' h_{i-1}, g_{i-1}, \cdots)$$

to

$$(\cdots, g_i', h_{i-1}, g_{i-1}, \cdots)$$

and thus to

$$(\cdots, g_i', h_{i-1} g_{i-1}, \cdots)$$

which of course equals

$$(\cdots, g_i', g_{i-1}' h_{i-2}, \cdots).$$

In other words homotopy between based paths corresponds exactly to passing between representing words in $\sqcup H_i$ by the usual moves that give the amalgamation over the (pairwise) intersections. We thus have proved

Proposition 9.2. *If $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$, then*

$$|\Gamma(\mathsf{A}, 1)|/_\sim \cong \underset{\cap}{\sqcup} \mathcal{H},$$

the amalgamated coproduct of the groups in $\mathcal{H}$. □

Remarks 9.3.

(i) As the usual construction of universal covering spaces in topology is the analogue, there, of the left hand side of this isomorphism, it is natural to expect the right hand side, the amalgamated coproduct, to play that role here. We will look at coverings separately, and in some detail, shortly so here it suffices to note that the end point map

$$\Gamma(\mathsf{A}, 1) \to \mathsf{A}$$

induces a map

$$|\Gamma(\mathsf{A}, 1)|/_\sim \to G$$

which interprets as the natural evaluation of a word $(g_m, \cdots, g_1)$ *to the product,* $g_m \cdots g_1$*, i.e., to the natural homomorphism*

$$\underset{\cap}{\sqcup}\mathcal{H} \to G,$$

induced by the universal colimit-property of the amalgamated coproduct and the inclusions of the subgroups H_i *into* G.

(ii) *We note for future examination that* $\Gamma(\mathsf{A}, 1)$ has a global action/groupoid atlas structure *and it is natural to expect that the quotient by fixed end point homotopies will inherit a similar structure, but that we have not yet described the construction of colimits, and in particular, quotients, in this setting. In the particular case above* $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$*, it is easily seen that the amalgamated coproduct carries a global action structure:*
There are inclusions

$$i_\alpha : H_\alpha \to \underset{\cap}{\sqcup}\mathcal{H}$$

and writing

$$\tilde{G} = \underset{\cap}{\sqcup}\mathcal{H}, \qquad \tilde{\mathcal{H}} = \{i_\alpha(H_\alpha) : \alpha \in \Phi\},$$

we can construct, $\tilde{\mathsf{A}} = \mathsf{A}(\tilde{G}, \tilde{\mathcal{H}})$.
The map

$$\tilde{\mathsf{A}} \to \mathsf{A}$$

is a regular morphism of global actions. (Left as an exercise!)

Given the analogy between the above and the topological case, it is no surprise that restricting attention to the loops at 1 in A, the defining equation

$$\pi_1(\mathsf{A}, 1) = \pi_0(\Omega\mathsf{A}) = |\Omega(\mathsf{A}, 1)|/_\sim$$

gives:

Corollary 9.4. *If* $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$,

$$\pi_1(\mathsf{A}, 1) \cong \ Ker(\underset{\cap}{\sqcup}\mathcal{H} \to G).$$

□

This result in this context was found by A. Bak. Given the identification of $\pi_1(\mathsf{A}, 1)$ with $\pi_1(V(\mathsf{A}), 1)$ and the Dowker theorem identifying this with $\pi_1(N(\mathsf{A}), 1)$, it can be seen to be a version of a result of Abels and Holz, [1]. They, in turn, relate it to earlier results of Behr, [6], and Soulé, [17], and mention applications of a related result given by Tits, [18]. The proof given above has the advantage of being very elementary and "constructive"!

Problem/Question 9.5. *Compare the use of* $N(\mathsf{A})$ *as a simplicial complex with* $N^{simp}(\mathsf{A})$*, as simplicial set. The action of* G *(which we will look at next) gives* $N(\mathsf{A})/G$ *is a simplex, but Abels and Holz identify* $\pi_1(N^{simp}(\mathsf{A})/G)$ *as being* $\underset{\cap}{\sqcup}\mathcal{H}$.

The comparison of $V(\mathsf{A})$ with the bar resolutions of H_i and G studied in Abels and Holz, [1], also needs examining in detail (and greater generality).

9.2. Group actions on $N(\mathsf{A})$

Further information on $N(\mathsf{A})$ and $V(\mathsf{A})$ may be obtained by exploiting the natural action of G on these simplicial complexes. This leads to a connection of these single domain global actions not only with the work of Abels and Holz, but with related work on complexes of groups by Corson, Haefliger and others, [7, 9–11, 13, 14].

Again G will be a group, $\mathcal{H} = \{H_i \mid i \in \Phi\}$ a family of subgroups and $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$ the corresponding single domain global action. We will assume that A is connected so G is generated by the union of the H_is. Recall that $N(\mathsf{A})$ is the simplicial complex given by the nerve of the covering, $\mathfrak{H}$, of G by left cosets of the H_i, $H_i \in \mathcal{H}$.

The group G acts on $N(\mathsf{A})$ by right translation. A typical n-simplex of $N(\mathsf{A})$ is of the form

$$\sigma = \{H_{\alpha_0}x_0, \cdots, H_{\alpha_n}x_n\}$$

where

$$\bigcap \sigma = \cap_{i=0}^{n} \ H_{\alpha_i}x_i \neq \emptyset.$$

If $g \in G$, we can consider $\sigma.g = \{H_{\alpha_0}x_0g, \cdots, H_{\alpha_n}x_ng\}$.

If $y \in \cap_{i=0}^{n} H_{\alpha_i}x_i = \bigcap \sigma$, then $y.g \in \bigcap \sigma.g$ so

$$\sigma.g \in N(\mathsf{A}).$$

This is clearly a group action. It is "without inversion" (Haefliger) or "regular" (Abels and Holz,[1]) in as much as if $\sigma.g = \sigma$, then $H_{\alpha_i}x_ig = H_{\alpha_i}x_i$, since the x_i used in a given σ are all distinct. This implies that the orbit space of $N(\mathsf{A})$ is also a complex.

Proposition 9.6. *If $\sigma = \{H_{\alpha_0}x_0, \cdots, H_{\alpha_n}x_n\}$ is an n-simplex of $N(\mathsf{A})$ then for any $a \in \bigcap \sigma$*

$$\sigma a^{-1} = \{H_{\alpha_0}, \cdots, H_{\alpha_n}\}$$

Moreover any finite subset J of Φ corresponds to a unique G-orbit of $N(\mathsf{A})$ and vice versa.

Proof. As $a \in \bigcap \sigma$, there are elements $h_{\alpha_i} \in H_{\alpha_i}$ with $a = h_{\alpha_i}x_i$ for $i = 0, 1, \cdots, n$. Thus

$$H_{\alpha_i}x_ia^{-1} = H_{\alpha_i}$$

and $\sigma a^{-1} = \{H_{\alpha_0}, \cdots, H_{\alpha_n}\}$. The orbit of σ is thus determined by the indices of the subgroups, $H_i \in \mathcal{H}$, used in it. The orbit of σ then corresponds to the finite subset $J_\sigma = \{\alpha_0, \cdots, \alpha_n\}$ of Φ and conversely. $\square$

Corollary 9.7. *For $\sigma = \{H_{\alpha_j} x_j : j = 0, \cdots, n\}$ as above, and $a \in \bigcap \sigma$,*

$$Stab_G(\sigma) = a^{-1} \left(\bigcap \{H_j : j \in J_\sigma\} \right) a.$$

Proof. Write $\sigma_0 = \{H_j : j \in J_\sigma\} \in N(\mathsf{A})$, then $\sigma_0 a = \sigma$ so $g \in Stab_G \sigma$ if and only if $\sigma_0 a g = \sigma_0 a$ i.e. if $\sigma_0 a g a^{-1} = \sigma_0$ which just says $aga^{-1} \in Stab_G \sigma_0$. However $Stab_G \sigma_0$ is clearly equal to $\bigcap \sigma_0$, which completes the proof. □

Corollary 9.8. *The space of orbits $N(\mathsf{A})/G$ is a simplex of dimension card$(\Phi) - 1$.* □

Examples 9.9.

1) *$G = S_3, H_1 = \{1, (1\ 2\ 3), (1\ 3\ 2)\}, H_2 = \{1, (1\ 2)\}$. The nerve $N(\mathsf{A})$ in this case is the graph given in example 2.5 with vertices*

$$\begin{array}{ccccc} & H_1 & & H_1 b & \\ & & & & \\ H_2 & & H_2 a & & H_2 a^2 \end{array}$$

(where, as there, $a = (1\ 2\ 3)$, $b = (1\ 2)$). The action is given by: a fixes H_1 and $H_1 b$ and permutes the cosets of H_2 in the obvious way; b permutes H_1 and $H_1 b$ and *$H_2 a$ and $H_2 a^2$, but fixes H_2 (of course). On 1-simplices*

$$a \in H_1 \cap H_2 a \quad so \quad H_1 a^{-1} \cap H_2 a a^{-1} = H_1 \cap H_2 \neq \emptyset$$

and so on. It is thus easy to see that $N(\mathsf{A})/S_3 \cong \Delta[1]$.
Of more interest are the examples:

2) *$G = K_4 = \{1, a, b, c\}$, $N(\mathsf{A}_{K4})$ is the octahedral shell with 4 faces removed. Using the same notation as before: a fixes 1 and 2, permutes 3 and 4, and also 5 and 6, so in the diagram in example 8.2, a corresponds to a rotation through $180°$ about the vertical axis. Similarly for b and c, but about the two horizontal axes. The orbit space is $\Delta[2]$ as this example has 3 subgroups.*

3) *$G = q8$, $N(\mathsf{A}_{q8})$ has the action of $q8$ via the quotient homomorphism to K_4 and the action outlined before in 2. Of course, $N(\mathsf{A}_{q8})/q8$ is again a 2-simplex.*
Our final two examples are

4) *S_3 with $H_1 = \langle (1\ 2) \rangle$, $H_2 = \langle (2\ 3) \rangle$, so $N(\mathsf{A})$ is a hexagon (empty) and, of course, the S_3-action collapses this down to a 1-simplex.*

and

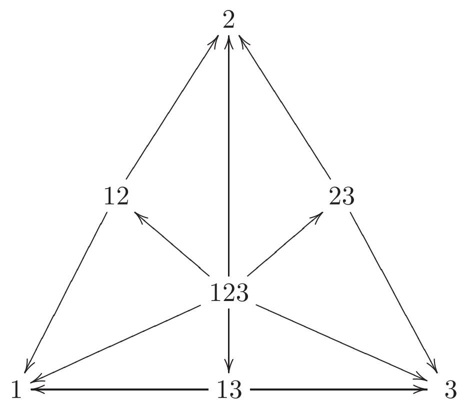
5) *S_4 with three subgroups,*
$H_1 = \langle (1, 2), (2, 3) \rangle$,
$H_2 = \langle (2, 3), (3, 4) \rangle$
and
$H_3 = \langle (1, 2), (3, 4) \rangle$.
The nerve was found earlier to be a triangulation of S^2. The action can be specified, but will not be given here and, of course, the quotient is $\Delta[2]$.

This situation is a simple form of a general one considered by Haefliger (cf. [7, 13, 14]) and Corson (cf. [9–11]). They consider a simplicial complex (or more generally a simplicial cell complex, cf. Haefliger, [13] or a scwol (small category without loops) cf. Bridson and Haefliger, [7]) on which a group G acts without inversion. Then $\tilde{X}/G$ is also a simplicial (cell) complex. Their work uses complexes of groups, a notion generalising that of graphs of groups as in Bass-Serre theory. We will give definitions shortly, but first need to introduce some more detailed notation and terminology relating to barycentric subdivisions.

If K is a simplicial complex, we can encode the information in K in a simply way by considering K as a partially ordered set. The elements of this partially ordered set are the elements of S_K, the set of simplices of K ordered by inclusion. The barycentric subdivision of K is then just the (categorical) nerve of the poset $(S_K, \subseteq)$ as noted earlier. We will follow Haefliger [13] in orienting the edges of K' in the following way:

The vertices of $K'(= Sd(K))$ are the simplices of K. An (unoriented) edge of K' consists of a pair (σ, τ) with either $\sigma \subset \tau$ or $\tau \subset \sigma$. If a is an edge of K' contained in a simplex σ of K, then the *initial point* $i(a)$ of a is the barycentre of σ (i.e. σ as a vertex of K') and its *terminal point*, $t(a)$, is the barycentre of some smaller simplex, τ. We write $i(a) = \sigma$, $t(a) = \tau$ and so have $a = (\tau, \sigma)$, with $\tau \subset \sigma$. (This is perhaps the opposite order from that which seems natural, but it avoids considering dual posets later.)

Example 9.10. *For the 2-simplex, considered as the simplicial complex of non-empty subsets of* $\{1, 2, 3\}$, *this gives*



Although it is usual to consider partially ordered sets as categories, because his complexes <u>*are*</u> *more general than mere simplicial complexes, Haefliger introduces a specific construction of a small category associated to K (cf. [13]).*

Define a category $C(K)$ *with set of objects* S_K*, the set of vertices of the barycentric subdivision* K' *of* K *and with arrows* $\mathrm{Arr}(C(K)) = E_k \sqcup S_k$*, the set of edges of* K' *together with* S_K*. (Of course, the vertices are considered as identity arrows at themselves.) Two edges* a *and* b *are considered composable if* $i(a) = t(b)$ *and the* *composite is $c = ba$ such that a, b, c form the boundary of a 2-simplex in K'.*

$$\begin{tikzcd} & \cdot \\ \cdot \arrow[ur, "c"] \arrow[r, "b"'] & \cdot \arrow[u, "a"'] \end{tikzcd}$$

This category $C(K)$ is an example of a *small category without loops* as introduced by Haefliger [7, 13]. We shall consider a small category, χ, to consist of a set, $V(\chi)$, of vertices or objects (denoted here by Greek letters, τ, σ, etc.) and a set $E(\chi)$ of edges (denoted by Latin letters, a, b, ...), together with maps

$i : E(\chi) \to V(\chi)$, the initial vertex or source map,
$t : E(\chi) \to V(\chi)$, the terminal vertex or target map

and a composition

$E^{(2)}(\chi) \to E(\chi)$,

where $E^{(2)}(\chi) = \{(a, b) \in E(\chi) \times E(\chi) : i(a) = t(b)\}$, together with associativity of composition and the rules $i(ba) = i(b)$, $t(ba) = t(a)$ for ba, the composite of a and b.

The small category χ is a small category without loops, or scwol, if for all a in $E(\chi)$, $i(a) \neq t(b)$.

Remark 9.11. *Haefliger's definition of a small category without loops in [7] (p.521) is optimised for the statement of the no loops condition, but omits to define composition of an arbitrary arrow with a vertex. This is handled correctly (p.573) in an appendix. This does not influence the later development.*

For the moment we will move attention back to K and the definition of a complex of groups.

9.3. Complexes of groups

A complex of groups $G(K)$ on K is $(K, G_0, \psi_a, g_{a,b})$ given by

1) a group G_σ for each simplex σ of K;
2) an injective homomorphism
$$\psi_a : G_{i(a)} \to G_{t(a)}$$
for each edge $a \in E_K$ of the barycentric subdivision of K;
3) for two composable edges a and b in E_K, an element $g_{a,b} \in G_{t(a)}$ is given such that
$$g_{a,b}^{-1} \psi_{ba}(\_) g_{a,b} = \psi_a \psi_b$$
and such that the "cocycle condition"
$$g_{a,cb} \psi_a(g_{b,c}) = g_{ab,c} g_{a,b}$$
holds.
(If the dimension of K is less than 3, this condition is void.)

Almost generic example: developable complexes of groups.

Suppose we have a simplicial complex $\tilde{X}$ with a right G action which is "without inversion", i.e., if $\sigma.g = \sigma$ then $xg = x$ for all vertices x of σ. Write $X = \tilde{X}/G$ for the quotient complex. We will specify a complex of groups $G(X)$ on X:
Set $p : \tilde{X} \to X$ to be the quotient mapping.

For a simplex σ of X, pick a $\tilde{\sigma} \in X$ with $p(\tilde{\sigma}) = \sigma$, we say $\tilde{\sigma}$ is the *chosen lift* of σ, and set

$$\begin{aligned} G_\sigma &= G_{\tilde{\sigma}}, \text{ the stability subgroup of } \tilde{\sigma}, \\ &= \{g : \tilde{\sigma}g = \tilde{\sigma}\}. \end{aligned}$$

For each $a \in E_X$ with $i(a) = \sigma$, let $\tilde{a}$ be the edge in $\tilde{\sigma}$ whose projection is a, i.e. $p(\tilde{a}) = a$ and $i(\tilde{a}) = \tilde{\sigma}$. Then there is some $h_a \in G$ with $t(\tilde{a}.h_a) = \tilde{\tau}$ where $\tilde{\tau}$ is the chosen lift of $\tau = t(a)$. (If $t(\tilde{a}) = \tilde{\tau}$ already, we agree to take H_a to be the identity of G.)

Define

$$\psi_a : G_{i(a)} \to G_{t(a)}$$

by

$$\psi_a(g) = h_a^{-1} g h_a \quad \text{for } g \in G_{i(a)}.$$

Given two composable edges a and b define

$$g_{a,b} = h_{ba}^{-1} h_b h_a.$$

Verification of conditions

(Although easy to do, this helps the intuition:)

(i) Suppose $g \in G_{i(a)}$, then $\tilde{a} = (\tilde{\tau} h_a^{-1}, \tilde{\sigma})$ or $\tilde{a}.h_a = (\tilde{\tau}, \tilde{\sigma}.h_a)$. As $\tilde{\sigma}g = \tilde{\sigma}$, and $\tilde{\tau} h_a^{-1} \subset \tilde{\sigma}$, we have

$$\tilde{\tau} h_a^{-1} g = \tilde{\tau} h_a^{-1}$$

and $h_a^{-1} g h_a \in G_{t(a)}$, i.e. $\psi_a(g) \in G_{t(a)}$.

(ii) Suppose a, b are composable: $i(ba) = i(b)$, $t(ba) = t(a)$, then

$$\psi_b : G_{i(b)} \to G_{t(b)} = G_{i(a)}$$

$$\psi_b(g) = h_b^{-1} g h_b.$$

Similarly $\psi_a \psi_b(g) = h_a^{-1} h_b^{-1} g h_b h_a$,
whilst

$$\psi_{ba}(g) = h_{ba}^{-1} g h_{ba}.$$

It is clear that $g_{a,b}$ as defined above does the job.

(iii) cocycle condition:

$$g_{a,cb} \psi_a(g_{b,c}) = h_{cba}^{-1} h_{cb} h_a . h_a^{-1} h_{cb}^{-1} h_c h_b h_a$$

$$g_{a,cb}.g_{a,b} = h_{cba}^{-1}h_c h_{ba}.h_{ba}^{-1}h_b h_a,$$

so it does check out correctly.

In the case of a single domain global action $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$ where $\mathcal{H} = \{H_1, \cdots, H_n\}$ with $H_i < G$, then $N(\mathsf{A})/G \cong \Delta^{n-1}$. Suppose $\sigma \in S_{\Delta^{n-2}}$ then if $\sigma = \{\alpha_1, \cdots, \alpha_r\}$, we can always choose $\tilde{\sigma} = \{H_{\alpha_1}, \cdots, H_{\alpha_r}\}$. If a is an edge of $Sd(\Delta^{n-1})$ then for $i(a) = \sigma$ and $t(a) = \tau, \tilde{\tau} \subset \tilde{\sigma}$ and hence

$$G_\tau = G_{\tilde{\tau}} = \bigcap\{H_i \mid i \in \tilde{\tau}\},$$
$$G_\sigma = G_{\tilde{\sigma}} = \bigcap\{H_i \mid i \in \tilde{\sigma}\},$$

so there is no need to have $h_a \neq 1$. Because of this, ψ_a is simply an inclusion of a subgroup and $g_{a,b}$ can be chosen to be 1. Thus single domain global actions yield simplices of groups of a particularly simple kind. This does *not* imply that the more general case is irrelevant to global actions, merely that single domain global action are "untwisted".

Problem/Question 9.12. *Are there 'twisted' variants that also arise from global actions?*

Given a complex of groups, both Corson and Haefliger show how to construct a universal covering complex and a fundamental group which yields the given complex of groups, provided certain fairly mild restrictions are satisfied.

9.4. Fundamental group(oid) of a complex of groups.

Let $G(K) = (K, G_\sigma, \psi_a, g_{a,b})$ as before.
Let $E_K^\pm$ be the set of edges of K' with an orientation

$$a^+ = a, \quad a^- = a \text{ with the opposite orientation}$$
$$\text{so } i(a^-) = t(a^+) \text{ etc.}$$

First define $FG(K)$ to be the group generated by

$$\bigsqcup\{G_\sigma : \sigma \in V_K\} \cup E_K^\pm$$

subject to the relations

- the relations of each G_σ,
- $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$,
- $\psi_a(g) = a^- g a^+$ for $g \in G_{i(a)}$,
- $(ba)^+ g_{ab} = b^+ a^+$ for composable a, b.

The image of G_σ in $FG(K)$ will be denoted $\bar{G}_\sigma$.

Haefliger defines $\pi_1(G(K), \sigma_0)$ in two equivalent ways:

Definition 9.13. *Version 1. If $\sigma_0, \sigma_1 \in V_K$, the vertices of K, a $G(K)$-path c from σ_0 to σ_1 is a sequence $(g_0, e_1, g_1, \cdots, e_n, g_n)$, where $(e_1, \cdots, e_n)$ is an edge path in K' from $i(e_1) = \sigma_0$ to $t(e_n) = e_1$ and $e_i \in E_K^{\pm}$, $i = 1, \cdots, n$ and where $g_K \in G_{t(e_k)} = G_{i(e_{k+1})}$.*

Such a $G(K)$-path, c, represents $g_0 e_1 \cdots e_n g_n \in FG(K)$. Two such paths from σ_0 to σ_1 are said to be homotopic if they represent the same element of $FG(K)$. We set $\pi_1(G(K), \sigma_0, \sigma_1)$ equal to the subset of $FG(K)$ represented by $G(K)$-paths from σ_0 to σ_1. When $\sigma_0 = \sigma_1$, we write

$$\pi_1(G(K), \sigma_0) = \pi_1(G(K), \sigma_0, \sigma_0).$$

This is a subgroup of $F(G)$ and is called the fundamental group of $G(K)$.

Version 2. Assume K is connected and pick a maximal tree T in the 1-skeleton of $Sd(K) = K'$. Let $N(T)$ be the normal subgroup of $FG(K)$ generated by $\{a^+ : a \in T\}$, then

$$\pi_1(G(K), T) \cong FG(K)/N(T),$$

and hence has a presentation:

- *generators*	$\sqcup G_\sigma \sqcup E_K$	
- *relations :*	- $g_1 \cdot g_2 = g_1 g_2$	*within any particular G_σ*
	- $\psi_\alpha(g) = \alpha^{-1} g \alpha$	$g \in G_{i(\alpha)}$,
	- $(\beta\alpha) g_{\alpha,\beta} = \beta.\alpha$	*if $\alpha, \beta \in E_K$ are composable*
	- $\alpha = 1$	*if $\alpha \in T$.*

Example 9.14. *Suppose $\mathsf{A} = \mathsf{A}(G, \mathcal{H})$, $\tilde{K} = N(\mathsf{A})$, $\mathcal{H} = \{H_1, \cdots, H_n\}$ so $K = \Delta^{n-1}$. Pick the maximal tree with edges radiating out from the vertex $\{H_1\}$, e.g. if $n = 3$, we get figure 4*

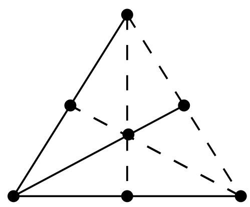


Figure 4: Barycentric Subdivision of Δ^2 with the chosen maximal tree shown.

There is an obvious collapse of Δ^{n-1} to T. We have already noted that all the $g_{a,b}$ are trivial in these examples so we can prove (inductively via the collapsing order) that if a is any edge in $Sd(\Delta^{n-1})$, the fact that $\alpha = 1$ for $\alpha \in T$ implies that $a = 1$ in $\pi_1(G(K), 1)$. Thus $\pi_1(G(K), 1)$ has a presentation with

- *generators*	$\sqcup G_\sigma$	
- *relations :*	- $g_1 \cdot g_2 = g_1 g_2$	*within any particular G_σ*
	- $\psi_\alpha(g) = g$	*for $g \in G_{i(\alpha)}$*

As $G_\sigma = \bigcap\{H_i \mid i \in \sigma\}$, we have

$$\pi_1(G(K), 1) \cong \underset{\cap}{\sqcup} H_i,$$

the coproduct of the H_i amalgamated over the intersection.

It is noticeable that there is, as before, a homomorphism

$$\pi_1(G(K), 1) \to G$$

with kernel $\pi_1 N(\mathsf{A})$. □

Thus for single domain global actions, the fundamental group is the same as the fundamental group of the corresponding complex of groups.

10. Coverings of global actions.

To complete the study of the various interpretations of $\pi_1(\mathsf{A}, a_0)$ we need to consider covering maps of global actions and the analogues of the universal covering.

10.1. Covering maps

Definition 10.1. *Let $p : \mathsf{B} \to \mathsf{A}$ be a morphism of global actions. We say that p is a covering map if it satisfies the* unique local frame lifting property*. Explicitly, for any given local frame $x_0, \ldots, x_n$ in A and any $y_0 \in p^{-1}(x_0)$ in B, there exists a unique local frame $y_0, \ldots, y_n$ in B such that $p(y_i) = x_i$ for all i.*

In the context of groupoids, the definition of covering is based on the Stars of the objects (cf. Brown [8]). In the global action case, we have notions of local and global Stars:

Let C be an arbitrary global action. Let $\alpha \in \Phi_{\mathsf{C}}$ and $x \in X_\alpha$, then $\mathrm{Star}_\alpha(x) = \{\sigma x \mid \sigma \in G_\alpha\}$. If $x \in X$, then set $\mathrm{Star}_C(x) = \cup\{\mathrm{Star}_\alpha(x) \mid \alpha \in \Phi_{\mathsf{C}}$ and $x \in X_\alpha\}$.

Proposition 10.2. *Let $p : \mathsf{B} \to \mathsf{A}$ be a covering of global actions. Then the following two conditions are satisfied.*

1) *if $x_1, x_2 \in X_{\mathsf{B}}$ are such that $p(x_1) = p(x_2)$, $x_1 \neq x_2$, and there is some $\beta \in \Phi_{\mathsf{B}}$ with $x_1, x_2 \in X_\beta$ then $Star_\beta(x_1) \cap\ Star_\beta(x_2) = \emptyset$;*
2) *if $x \in X_{\mathsf{B}}$ then $p \downharpoonright_{Star_{\mathsf{B}}(x)}: Star_{\mathsf{B}}(x) \to Star_{\mathsf{A}}(p(x))$ is a bijection.*

(In fact 2) implies 1))

The proof is omitted.

Note that the converse of this result is false. Consider for example the global action B with 3 points, say $\{a, b, c\}$ and 3 local sets $\{a, b\}, \{a, c\}, \{b, c\}$. The cyclic group C_2 acts on each local set in the obvious way. Let A be a global action with the same underlying set but with only one local set on which C_3 acts. Then the identity map between the underlying sets induces a morphism $\mathsf{B} \to \mathsf{A}$ which satisfies the conditions of the last proposition but this map is not a covering map of global actions.

We say p is *regular* if $\Phi_{\mathsf{B}} = \Phi_{\mathsf{A}}$ and p is a regular morphism with $p = (p_\Phi, p_G, p_X)$ such that $p_\Phi : \Phi_{\mathsf{B}} \to \Phi_{\mathsf{A}}$ is the identity map.

Proposition 10.3. *Any covering is weakly isomorphic to a regular covering.*

The proof is omitted, as we will nor be using the result here.

The construction and theory of coverings is very similar to that in the classical topological theory. We will assume A is connected and when necessary that a base point $a_0 \in X_{\mathsf{A}}$ has been chosen.

10.2. Path lifting and homotopy lifting

Suppose $p : \mathsf{B} \to \mathsf{A}$ is a covering map of global actions, suppose $f : \mathsf{L} \to \mathsf{A}$ is a path and (N_f^-, N_f^+) is a stabilisation pair for f. Set $x_0 = f(N_f^-)$.

Lemma 10.4. *Given any $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\tilde{f} : \mathsf{L} \to \mathsf{B}$ with the same stabilisation pair (N_f^-, N_f^+) such that $p\tilde{f} = f$ and $\tilde{f}(N_f^-) = \tilde{x}_0$.*

Proof. (For simplicity of notation we assume $N_f^- = 0$, and write $N_f^+ = N$.) Suppose $f = (f(0), f(1), \cdots, f(N))$. There is an $\alpha_1 \in \Phi_{\mathsf{A}}$ with $f(0), f(1) \in (X_{\mathsf{A}})_{\alpha_1}$, and a $g_1 \in (G_{\mathsf{A}})_{\alpha_1}$ such that $g_1 f(0) = f(1)$. Thus $x_1 = f(1) \in \mathrm{Star}_{\mathsf{A}}(x_0)$. As $|_{\mathrm{Star}_{\mathsf{B}}(\tilde{x}_0)}$ is a bijection between $\mathrm{Star}_{\mathsf{B}}(\tilde{x}_0)$ and $\mathrm{Star}_{\mathsf{A}}(x_0)$, there is a *unique* $\tilde{x}_1$ with $p(\tilde{x}_1) = x_1$ and a $\tilde{g}_{\alpha_1}$ with $\tilde{x}_1 = \tilde{g}_{\alpha_1}\tilde{x}_0$. A simple use of induction completes the proof. ☐

Again suppose $f_0, f_1 : L \to \mathsf{A}$ are paths, but in addition suppose $h : \mathsf{L} \times \mathsf{L} \to \mathsf{A}$ is a (fixed end point) homotopy from f_0 to f_1. (This, of course, implies that f_0, f_1 stabilise to points x_0 and x_N say.) Again let $\tilde{x}_0 \in p^{-1}(x_0)$ and then we have

Lemma 10.5. *The two lifts $\tilde{f}_0, \tilde{f}_1$ of f_0, f_1 are homotopic by a homotopy $\tilde{h}$ lifting h. In particular $\tilde{f}_0(N) = f_1(N)$.*

Proof. We can assume h stabilises outside a square. We may assume this square is actually a 1 by 1 square as the general case follows by induction. Initially we will need

$$\begin{tikzcd} x_0 \arrow[r, "g_1'"] & x_1' \\ x_0 \arrow[u, equal] \arrow[r, "g_1"'] & x_1 \arrow[u, "h_1"'] \end{tikzcd}$$

but in general,

$$\begin{tikzcd} x'_i \arrow[r] & x'_{i+1} \\ x_i \arrow[u] \arrow[r] & x_{i+1} \arrow[u] \end{tikzcd}$$

Since h is a morphism of global actions, any such square must end up within a single local patch of A and so can be lifted. "Uniqueness" ensures that it glues to any lifts constructed earlier in the process in the obvious way. The only statement left unproved is the last.

We end up with $\tilde{x}'_N$ and $\tilde{x}_N$ lying over x_N and the right hand side of the homotopy giving us a path from $\tilde{x}_N$ to $\tilde{x}'_N$ which maps down (via p) to the identity path from x_N to itself. "Uniqueness" of path lifting then shows this must be the identity path at $\tilde{x}_N$ and so $\tilde{x}_N = \tilde{x}'_N$ as required. □

Corollary 10.6. *If $p : \mathsf{B} \to \mathsf{A}$ is a covering and A is connected then all the fibres $p^{-1}(x)$, $x \in \mathsf{A}$ have the same cardinality.*

Proof. If $x_0, x_1 \in \mathsf{A}$, let $f : \mathsf{L} \to \mathsf{A}$ be a path from x_0 to x_1. Now pick $\tilde{x}_0 \in p^{-1}(x_0)$, and lift f to $\tilde{f}$ joining $\tilde{x}_0$ to some uniquely determined $\tilde{x}_1 \in p^{-1}(x_1)$. This assignment is a bijection since the reverse path is also uniquely determined. □

10.3. The $\pi_1(\mathsf{A}, a_0)$-action

This gives a way of associating to each covering $p : \mathsf{B} \to \mathsf{A}$, a set $F_{a_0} (= p^{-1}(a_0))$ together with an action of $\pi_1(\mathsf{A}, a_0)$. Alternatively the covering may be thought of as a collection of fibres indexed by the elements of X_A and then we get an action of $\Pi_1\mathsf{A}$ on B by "deck transformations" over A.

Theorem 10.7. *Suppose $p : \mathsf{B} \to \mathsf{A}$ is a covering map, and $b_0 \in p^{-1}(a_0)$ then the induced maps*

$$p_*; \Pi_1\mathsf{B} \to \Pi_1\mathsf{A}$$

and

$$p_* : \pi_1(\mathsf{B}, b_0) \to \pi_1(\mathsf{A}, a_0)$$

are monomorphisms.

Proof. The induced maps take $\omega \in \Pi_1\mathsf{B}$ with $w = [f]$ to $p_*(\omega) = [pf] \in \Pi_1\mathsf{A}$. The result is just an immediate consequence of unique path lifting together with the lifting of homotopies. (The proof is easy and follows that given in many elementary homotopy texts.) □

Now suppose $a_0 \in \mathsf{A}$ is the chosen base point, that B is connected and we have $b_0, b_1 \in p^{-1}(a_0)$. Choose a path class γ from b_0 to b_1 in $\Pi_1\mathsf{B}$ which thus gives an isomorphism

$$u : \pi_1(\mathsf{B}, b_0) \to \pi_1(\mathsf{B}, b_1)$$

by conjugation $u(\omega) = \gamma^{-1}\omega\gamma$ within $\Pi_1\mathsf{B}$. As in the topological case we get an inner automorphism ν of $\pi_1(\mathsf{A}, a_0), \nu(\omega) = p_*(\gamma)^{-1}\omega p_*(\gamma)$, and of course $p_*(\gamma)$ is a loop since $p(b_0) = p(b_1)$. We thus have

Proposition 10.8. *The images of $\pi_1(\mathsf{B}, b_0)$ and $\pi_1(\mathsf{B}, b_1)$ are conjugate subgroups of $\pi_1(\mathsf{A}, a_0)$.* □

Path lifting then shows that any conjugate subgroup of $\pi_1(\mathsf{B}, b_0)$ in $\pi_1(\mathsf{A}, a_0)$ can arise in this way. Just lift any conjugating element to a path in B.

The theory of coverings of global actions follows the same general development as the classical topological one (cf. Massey [15]) or the groupoid one (cf. Brown, [8]). For instance one can easily prove the following results.

Proposition 10.9. *Let $p : \mathsf{C} \to \mathsf{B}$ and $q : \mathsf{B} \to \mathsf{A}$ be morphisms of global actions.*

1. *If p and q are covering maps, so is qp.*
2. *If p and qp are covering maps and p is epi, then q is a covering.*

Proposition 10.10. *If $p : \mathsf{B} \to \mathsf{A}$ and $q : \mathsf{C} \to \mathsf{A}$ are coverings and $f : \mathsf{B} \to \mathsf{C}$ is a morphism over A, then f is also a covering.*

The proof of the following result is in fact easier than in the classical case. It uses the unique path lifting property of coverings.

Proposition 10.11. *Let $p : \mathsf{B} \to \mathsf{A}$ be a covering and $a_0 \in X_\mathsf{A}$, $b_0 \in X_\mathsf{B}$ such that $p(b_0) = a_0$, let $f : \mathsf{C} \to \mathsf{A}$ be a morphism with $f(c_0) = a_0$ and suppose that C is connected. Then f lifts to a morphism $\tilde{f} : \mathsf{C} \to \mathsf{B}$, with $\tilde{f}(c_0) = b_0$, if and only if $f_*(\pi_1(\mathsf{C}, c_0)) \subseteq p_*(\pi_1(\mathsf{B}, b_0))$.* □

Remark 10.12. *Given the similarity of the development to the classical and groupoid cases, it should be clear that all of the above goes across to the context of groupoid atlases. There are also strong lifting properties for strong coverings.*

We fix a base point a_0 in A and denote for simplicity $\pi_1\mathsf{A} = \pi_1(\mathsf{A}, a_0)$.

The action of $\pi_1\mathsf{A}$ on $p^{-1}(a_0)$ extends to a functor from the category of coverings over A to that of $\pi_1\mathsf{A}$-sets, i.e., sets with $\pi_1\mathsf{A}$-actions. Explicitly if $b_0 \in p^{-1}(a_0)$ and $\gamma \in \pi_1(\mathsf{A}, a_0), \gamma = [f]$ say; lift γ to a path in B starting at b_0. Its other end will be $b_0^\gamma \in p^{-1}(a_0)$.

If $g : (\mathsf{B}, p) \to (C, q)$ is a morphism in the category of coverings over A, then g restricts to a map $p^{-1}(a_0) \to q^{-1}(a_0)$. Uniqueness of path lifting then shows that

$$g(b_0^\gamma) = g(b_0)^\gamma$$

as hoped for.

Writing Cov/A for the category of coverings of A, we will get

$$\mathrm{Cov}/\mathsf{A} \to \pi_1\mathsf{A}\text{-Sets}.$$

If (B, p) is a covering global action of A, we will write $\mathrm{Aut}_{\mathsf{A}}(\mathsf{B}, p)$ for its automorphism group (group of covering or deck transformations) within Cov/A.

The functor above gives a homomorphism

$$\mathrm{Aut}_{\mathsf{A}}(\mathsf{B}, p) \to \mathrm{Aut}_{\pi_1\mathsf{A}\text{-Sets}}(p^{-1}(a_0))$$

as is easily checked.

If $\varphi : p^{-1}(a_0) \to p^{-1}(a_0)$ is an automorphism of $\pi_1\mathsf{A}$-sets then the isotropy subgroup of any point $b_0 \in p^{-1}(a_0)$ is the same as that of $\varphi(b_0)$, i.e., from our earlier discussion, it is easily seen to be $p_*(\pi_1(\mathsf{B}, b_0))$. Thus as a $\pi_1\mathsf{A}$-set, $p^{-1}(a_0)$ is isomorphic to the "coset space" $\pi_1\mathsf{A}/p_*\pi_1\mathsf{B}$. By 10.11 one has that the automorphism φ can be realised by a deck transformation and so $\mathrm{Aut}_{\mathsf{A}}(\mathsf{B}, p)$ and $\mathrm{Aut}_{\pi_1\mathsf{A}\text{-Sets}}(p^{-1}(a_0))$ are isomorphic.

Corollary 10.13. *If* $\pi_1\mathsf{B}$ *is trivial (i.e. the covering global action is simply connected) then* $\mathrm{Aut}_{\mathsf{A}}(\mathsf{B}, p) \cong \pi_1(\mathsf{A}, a_0)$*, i.e.* (B, p) *is a universal covering.* □

10.4. The Galois-Poincaré theorem for global actions

To complete the triple description of $\pi_1\mathsf{A}$, we need to show that a simply connected covering exists. In fact we will show more, namely that given any conjugacy class of subgroups of $\pi_1\mathsf{A}$, we can find a covering (B, p) corresponding to that conjugacy class (i.e. $\{p_*\pi_1(\mathsf{B}, p) : b \in p^{-1}(a_0)\}$ gives exactly the given conjugacy class). This and generalities on $\pi_1\mathsf{A}$-sets will then establish that

$$\mathrm{Cov}/_{\mathsf{A}} \to \pi_1\mathsf{A}\text{-Sets}$$

is an equivalence of categories, which is the Galois-Poincaré correspondence theorem in this context.

As before let A be connected and pick a base point $a_0 \in X_{\mathsf{A}}$. Let H be a subgroup of $\pi_1(\mathsf{A}, a_0)$.

We have a set $\Gamma\mathsf{A}$ of based paths and a projection

$$p : \Gamma\mathsf{A} \to \mathsf{A}$$

given by $p(\omega) = e^1(\omega)$. Define an equivalence relation $\sim_H$ on $\Gamma\mathsf{A}$ by $f \sim f'$ if $p(f) = p(f')$ and $[f][f']^{-1} \in H$, where the composition is viewed as taking place within the fundamental groupoid $\Pi_1\mathsf{A}$ and H as a subgroup of the vertex group at a_0.

Let X_{A_H} denote the set of equivalence classes, $\langle f\rangle$, of based paths under $\sim_H$.

The function $p : X_{\Gamma\mathsf{A}} \to X_{\mathsf{A}}$ clearly induces one, $p_H : X_{\mathsf{A}_H} \to X_{\mathsf{A}}$ given by $p_H\langle f\rangle = e^1(f)$. We will give A_H a global action structure. Take

$$\Phi_{\mathsf{A}_H} = \Phi_{\mathsf{A}}$$

For $\alpha \in \Phi_{\mathsf{A}_H}$, $(X_{\mathsf{A}_H})_\alpha = \{\omega \in X_{\mathsf{A}_H} \mid p_H(\omega) \in (X_{\mathsf{A}})_\alpha\}$ and $(G_{\mathsf{A}_H})_\alpha = (G_{\mathsf{A}})_\alpha$. The action of $(G_{\mathsf{A}_H})_\alpha$ on $(X_{\mathsf{A}_H})_\alpha$ is as follows:

Let $f \in \Gamma\mathsf{A}$ (and we as usual assume $N_f^- = 0$ and $N_f^+ = n$, say), then $p(f) = f(n)$. Suppose $f \in (X_{\mathsf{A}_H})_\alpha$, so $f(n) \in (X_\mathsf{A})_\alpha$ and let $\sigma \in (G_\mathsf{A})_\alpha$. Define a path σ_n^f by

$$\sigma_n^f(m) = \begin{cases} f(n) & \text{if } m \leqslant n, \\ \sigma.f(n) & \text{if } m \geqslant n+1 \end{cases}$$

and set

$$\sigma.f = f * \sigma_n^f.$$

Thus

$$\sigma.f(m) = \begin{cases} f(m) & \text{if } m \leqslant n \\ \sigma.f(n) & \text{if } m \geqslant n+1 \end{cases}$$

It is easily checked that this gives an action on equivalence classes by

$$\sigma.\langle f\rangle = \langle \sigma.f\rangle$$

since it just adds one extra "link" to the path. Clearly $p_H\sigma.\langle f\rangle = \sigma.p_H\langle f\rangle = \sigma.f(n)$, A_H is a global action and p_H a regular morphism of global actions.

We can now prove that (A_H, p_H) is a covering of A.

Suppose that $x_0, \dots, x_n$ is a local frame in A and that $\omega_0 = \langle f_0\rangle$ is an element in A_H such that $p_H(\omega_0) = x_0$. Since $x_0, \dots, x_n$ is a local frame, there exists $\alpha \in \Phi_\mathsf{A} = \Phi_{\mathsf{A}_H}$ and $g_i \in G_\alpha$ such that $x_i = g_i x_0$. For each i take $\omega_i = g_i\omega_0$. It is clear by definition, that $\omega_0, \dots, \omega_n$ is a local frame in A_H and that $p_H(\omega_i) = x_i$. Moreover, it is the unique local frame with this property. This proves that (A_H, p_H) is a covering.

We note that $p_H^{-1}(a_0)$ is the set of $\sim_H$ equivalence classes of loops at a_0. If we look at the equivalence relation $\sim_H$, it is clearly made up of two parts:

(i) if $f \sim f'$ (i.e. fixed end point homotopic) then clearly $[f][f']^{-1} = 1_{a_0} \in \Pi_1\mathsf{A}$, and so $f \sim_H f'$ whatever H is chosen;

(ii) if f is a loop at a_0 then $f \sim_H 1_{a_0}$ if $[f] \in H$.

Together these imply that $p^{-1}(a_0) \cong G/H$ where we have written G for $\pi_1(\mathsf{A}, a_0)$.

Since (A_H, p_H) is a covering of A, there is a G-action on $p^{-1}(a_0)$ making this a G-set isomorphism.

Any path φ at a_0 in A will lift to a path given a choice of initial point. Fix $\tilde{a}_0$ to be the class of the constant path at a_0, so $\tilde{a}_0 \in \mathsf{A}_H$ and $p_H(\tilde{a}_0) = a_0$. If $\tilde{\varphi}$ is the lift of φ starting at $\tilde{a}_0$ then $\tilde{\varphi}(n)$ is the element of A_H represented by the partial path from $\varphi(0)$ to $\varphi(n)$. If $n > N_\varphi^+$ and φ is a loop at a_0 representing an element of $H \leqslant \pi_1(\mathsf{A}, a_0)$ then the end point of $\tilde{\varphi}$ is the point of A_H represented by the path from $\varphi(0)$ to $\varphi(n)$, i.e., $\langle\varphi\rangle$, but $[\varphi] \in H$ so $\langle\varphi\rangle = \langle\tilde{a}_0\rangle$ and φ lifts to a loop in A_H. Conversely any loop in A_H at a_0 is the lift of a loop at a_0 which represents an element of H.

A similar argument implies that if $\bar{\varphi}$ and $\bar{\varphi}'$ are homotopic loops at $\tilde{a}_0$ in A_H then they are lifts of homotopic loops at a_0 in A which then of course represent the same element of H. We thus have

Proposition 10.14. *The induced homomorphism*

$$p_{H*} : \pi_1(\mathsf{A}_H, \tilde{a}_0) \to \pi_1(\mathsf{A}, a_0)$$

is a monomorphism with image, H.

Proof. The proof is by direct calculation using the explicitly defined lifts of paths and homotopies. □

Remark 10.15. *Much of the above would work for strong paths, but the proof that the strong version of (A_H, p_H) is a covering would seem to depend on a local condition which is in some way analogous to "semi locally simply connected". This would say that small strong loops were strongly null-homotopic. Here by small we mean*

$$a \xrightarrow{g} b \xrightarrow{g'} a.$$

This is clearly satisfied for many examples.

Problem/Question 10.16. *Adapt the above discussion to handle strong coverings and /or groupoid atlases.*

To summarise:

Theorem 10.17. *Given $(\mathsf{A}, a_0), \mathsf{A}$ connected, and $H \leqslant \pi_1(\mathsf{A}, a_0)$ then there is a connected covering space (A_H, p_H) with*

$$p_{H_*}(\pi_1(\mathsf{A}_H, \tilde{a}_0)) = H.$$

In particular corresponding to $H = 1$, the trivial subgroup of $\pi_1(\mathsf{A}, a_0)$, one has a simply connected covering space $(\tilde{\mathsf{A}}, p)$. □

Of course by an earlier result $\mathrm{Aut}_{\mathsf{A}}(\tilde{\mathsf{A}}, p) \cong \pi_1(\mathsf{A}, a_0)$.

Now set $G = \pi_1(\mathsf{A}, a_0)$. Any G-set X can be decomposed as a disjoint union of "connected" G-sets. Here "connected" merely means single orbit or transitive G-sets. These all have form G/H and we can note that $p_H^{-1}(a_0) \cong G/H$ as G-sets. Using disjoint unions of (A_H, p_H)s for various subgroups H will yield a covering space (B, p) with $p^{-1}(a_0) \cong X$ as G-sets. It is then more or less routine to check that

$$\mathrm{Cov}/\mathsf{A} \leftrightarrows G\text{-Sets}$$

is an equivalence of categories.

Thus we have three descriptions of $\pi_1(\mathsf{A}, a_0)$ for a connected global action A:

(i) equivalence (homotopy) classes of loops at a_0,

(ii) $\pi_0(\Omega\mathsf{A})$,

(iii) $\mathrm{Aut}_{\mathsf{A}}(\tilde{\mathsf{A}}, p)$ and thus the group G in the above equivalence.

11. Single domain global actions II.

In this section we will examine general single domain global actions and their coverings.

11.1. General single domain global actions and $\mathsf{A}(G,\mathcal{H})$s

Suppose $\mathsf{A} = (X_\mathsf{A}, \Phi_\mathsf{A}, G_\mathsf{A})$ is a single domain global action. We thus have that Φ_A is a set with a reflexive relation $\leqslant$ defined on it, then

$$G_\mathsf{A} : \Phi_\mathsf{A} \to \mathsf{Groups}$$

can be considered as a (generalised) functor and we can form its colimit $G = \operatorname{colim} G_\mathsf{A}$. Each $G_\alpha, \alpha \in \Phi_\mathsf{A}$ acts on G by left multiplication via its image in G. (Note: G_α need not be isomorphic to a subgroup of G, but other than that one has virtually the situation of $\mathsf{A}(G,\mathcal{H})$.)

Define a global action G with $|\mathsf{G}| = G$

$$\begin{aligned} \Phi_\mathsf{G} &= \Phi_\mathsf{A} \\ (G_\mathsf{G})_\alpha &= G_\alpha \\ (X_\mathsf{G})_\alpha &= G, \end{aligned}$$

so G is a single domain global action.

If H is any subgroup of G, we can form a quotient global action G/H with $|\mathsf{G}/H| = |\mathsf{G}|/H$, the set of right cosets $\Phi_{\mathsf{G}/H} = \Phi_\mathsf{G}$, etc, so G/H is again a single domain global action.

It is clear that

$$\pi_0(\mathsf{G}) = \pi_0(\mathsf{G}/H) = 1.$$

Theorem 11.1. *Any connected single domain global action is regularly isomorphic to some* G/H.

Proof. Let A be a connected single domain global action and let $a_0 \in \mathsf{A}$ be a chosen basepoint. Let G_A be the global action constructed above from colim $(G_\mathsf{A} : \Phi_\mathsf{A} \to \mathsf{Groups})$.

The group G acts on A and also, of course, on $|\mathsf{G}_\mathsf{A}|$. Define a function

$$p : |\mathsf{G}_\mathsf{A}| \to |\mathsf{A}|$$

(using the base point) sending $\omega \in G_\mathsf{A}$ to $\omega.a_0$, i.e., read the word ω off from the right acting on a_0, inductively. The possible ambiguities in the word are due to cases of $\alpha \leqslant \beta$ and the compatibility condition ensures this does not matter.

Since $g_\alpha \cdot \omega$ gets sent to $(g_\alpha \cdot \omega a_0) = g_\alpha(\omega.a_0)$, this defines a regular morphism of global actions.

Let $H_\mathsf{A} = p^{-1}(a_0)$, which is the stabiliser of a_0 in G_A.

Clearly

$$\mathsf{G}_{\mathsf{A}}/H_{\mathsf{A}} \cong \mathsf{A}.$$

□

The only difference therefore between single domain global actions of the form $\mathsf{A}((G,K),\mathcal{H})$ as introduced in section 9 and the general case is that the H_i may not be subgroups and may have interrelations between them.

Examples 11.2.

1. *As before take* $S_3 = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1\rangle$, $H_1 = \langle a\rangle$, $H_2 = \langle b\rangle$ *to get* $\mathsf{A}(S_3, \{\langle a\rangle, \langle b\rangle\})$. *Then the colimit group is* $C_3 * C_2$ *and, of course, the stabiliser of 1 in* G_{A} *is merely* $Ker(C_3 * C_2 \to S_3)$, *that is* $\pi_1(\mathsf{A}(S_3, \{\langle a\rangle, \langle b\rangle\})$. *The quotient map*
$$p : C_3 * C_2 \to S_3$$
is that with kernel the normal closure of $(ab)^2$ *and writing* K *for that kernel (and thus for* π_1*), we have*
$$\mathsf{A}(S_3, \{\langle a\rangle, \langle b\rangle\}) \cong \mathsf{A}((C_3 * C_2, K)\{\langle a\rangle, \langle b\rangle\}).$$
A similar picture emerges with the other examples.
2. *For* $\mathsf{A}_{K4} = \mathsf{A}(K_4, \{\langle a\rangle, \langle b\rangle, \langle c\rangle\})$, *the colimit group is* $C_2 * C_2 * C_2 = C_2^{(3)}$ *and the stabiliser is the normal closure of abc. This normal subgroup has rank 3. Thus* A_{K4} *has a second description as* $\mathsf{A}((C_2^{(3)}, K), \mathcal{H})$, *where* $\mathcal{H} = \{\langle a\rangle, \langle b\rangle, \langle c\rangle\}$, *these subgroups being here subgroups of* $C_2^{(3)}$, *not of* K_4. *Of course,* $K \cong \pi_1(\mathsf{A}_{K4})$.
3. *The only change for* A_{q8} *is that the colimit group is* $(C_4 * C_4 * C_4)/\{1, -1\}$, *the free product with amalgamation.*
4. *Taking* S_3 *again, but with presentation* $\langle x_1, x_2 \mid x_1^2 = x_2^2 = 1, (x_1x_2)^3 = 1\rangle$, $H_1 = \langle x_1\rangle$, $H_2 = \langle x_2\rangle$, *gives colimit group* $C_2 * C_2$. *The stabiliser of 1 / fundamental group is free on* $(x_1x_2)^3$. *We again get a second description as a 'relative'* $\mathsf{A}((G,K),\mathcal{H})$.
5. *For our final example,* S_4 *with presentation* $\langle x_1, x_2, x_3 \mid x_1^2, i = 1, 2, 3, (x_1x_2)^3 = 1 = (x_2x_3)^3, (x_1x_3)^2 = 1\rangle$ *is* S_4 *itself and the given description is the one we have found earlier through the general process.*

11.2. Coverings of single domain global actions

If $\mathsf{A} \cong \mathsf{G}_{\mathsf{A}}/H_{\mathsf{A}}$ as above then for $\tilde{\mathsf{A}}$, its universal or simply connected covering, $\tilde{\mathsf{A}}$ is also a single domain global action and as the diagram $G_{\tilde{\mathsf{A}}} = G_{\mathsf{A}}$,

$$\tilde{\mathsf{A}} \cong \mathsf{G}_{\mathsf{A}}/H_{\tilde{\mathsf{A}}}.$$

Clearly there is a diagram

$$\begin{tikzcd} \mathsf{G}_{\tilde{\mathsf{A}}}/H_{\tilde{\mathsf{A}}} \arrow[r, "\cong"] \arrow[d] & \tilde{\mathsf{A}} \arrow[d] \\ \mathsf{G}_{\mathsf{A}}/H_{\mathsf{A}} \arrow[r, "\cong"] & \mathsf{A} \end{tikzcd}$$

so $H_{\tilde{\mathsf{A}}} \subseteq H_{\mathsf{A}}$.

It remains to relate $H_{\tilde{\mathsf{A}}}$ more closely to H_{A}.

If $\alpha \in \Phi_{\tilde{\mathsf{A}}} = \Phi_{\mathsf{A}}$, then set $(G'_{\mathsf{A}})_\alpha =$ image $((G_{\mathsf{A}})_\alpha \to G)$.

Let

$$\mathcal{H}_A = \{H \leqslant |G_{\mathsf{A}}| \mid \text{ for all } \sigma \in |G_{\mathsf{A}}|,\ \sigma H \sigma^{-1} \cap (G'_{\mathsf{A}})_\alpha = \sigma H_{\mathsf{A}} \sigma^{-1} \cap (G'_{\mathsf{A}})_\alpha \\ \text{for all } \alpha \in \Phi\}$$

and let $H_{\tilde{\mathsf{A}}} = \bigcap\{H \in \mathcal{H}\}$. Then $H_{\tilde{\mathsf{A}}} \in \mathcal{H}_{\mathsf{A}}$ and is minimal. Of course $H_{\tilde{\mathsf{A}}} \triangleleft H_{\mathsf{A}}$ and $H_{\mathsf{A}}/H_{\tilde{\mathsf{A}}} \cong \pi_1(\mathsf{A})$. Thus as a corollary of the main classification theorem for coverings we get:

Proposition 11.3. *General connected coverings of* A *correspond bijectively to intermediate groups between* $H_{\tilde{\mathsf{A}}}$ *and* H_{A}. □

Remark 11.4. *If* $\pi_0(\mathsf{A})$ *is not trivial, i.e.* A *is not connected, then it is important to remember that* $\tilde{\mathsf{A}}$ *only covers the connected component of the basepoint.*

Again we turn to our examples to see what these results give there.

In each case we have a description of A as $\mathsf{A}((G_{\mathsf{A}}, K_{\mathsf{A}}), \mathcal{H})$ and as our initial situation had

$$K_{\mathsf{A}} = \mathrm{Ker}(p : G \to X_{\mathsf{A}}),$$

where G is the colimit group of the original system, we have

$$K_{\mathsf{A}} \cong \pi_1(\mathsf{A}, 1)$$

in each case. This implies that $K_{\tilde{\mathsf{A}}}$ is trivial. We could also deduce this from the description of $\mathcal{H}_{\mathsf{A}}$ with $K_{\tilde{\mathsf{A}}} = \bigcap \mathcal{H}_{\mathsf{A}}$. ($K_{\mathsf{A}}$ is normal in G_{A} so $\sigma K_{\mathsf{A}} \sigma^{-1} \cap (G'_{\mathsf{A}})_\alpha = K_{\mathsf{A}} \cap (G'_{\mathsf{A}})_\alpha$. In each example this is trivial, so $\mathcal{H}_{\mathsf{A}}$ contains the trivial group and hence has that group as its intersection.)

We can describe the simply connected covering of A in each case:

1. $\mathsf{A} = \mathsf{A}(S_3, \{\langle a\rangle, \langle b\rangle\})$, $\tilde{\mathsf{A}} = \mathsf{A}(C_3 * C_2, \{\langle a\rangle, \langle b\rangle\})$, where as before $\langle a\rangle$ is to be interpreted *in context* as a subgroup of the corresponding group.

2. $\mathsf{A}_{K4} = \mathsf{A}(K_4, \{\langle a\rangle, \langle b\rangle, \langle c\rangle\})$, $\tilde{\mathsf{A}}_{K4} = \mathsf{A}(C_3^{(3)}, \{\langle a\rangle, \langle b\rangle, \langle c\rangle\})$.

3. $\mathsf{A}_{q8} = \mathsf{A}(q8, \{\langle i\rangle, \langle j\rangle, \langle k\rangle\})$, $\tilde{\mathsf{A}}_{q8} = \mathsf{A}(G_{\mathsf{A}}, \{\langle i\rangle, \langle j\rangle, \langle k\rangle\})$, where G_{A} has presentation

$$\langle i, j, k \mid i^4 = 1, i^2 = j^2 = k^2\rangle.$$

4. $\mathsf{A} = \mathsf{A}(S_3, \mathcal{H})$, with $\mathcal{H} = \{H_1, H_2\}$ with each H_i generated by a transposition, $\tilde{\mathsf{A}} = \mathsf{A}(C_2^{(2)}, \{\langle x_1\rangle, \{\langle x_2\rangle\}\})$.

5. The case $\mathsf{A}(S_4, \mathcal{H})$ with $\mathcal{H} = \{H_1, H_2, H_3\}$, as above (example 8.6), is already simply connected, so is its own simply connected cover.

In the next section we examine a more complex example, namely the elementary matrix group, $\mathrm{E}_n(R)$ of a ring R, which forms the connected component of the identity in the global action $\mathsf{GL}_n(R)$.

12. The Steinberg Group and Coverings of $\mathsf{GL}_n(R)$

A particularly important example of a single domain global action is the General Linear Global Action $\mathsf{GL}_n(R)$. We saw, example 2.4, that $\pi_0(\mathsf{GL}_n(R))$ was the set of right cosets of the group $\mathrm{GL}_n(R)$ modulo the subgroup $\mathrm{E}_n(R)$ of elementary matrices and hence was identifiable as being $K_1(n, R)$. We asked "*is* $K_2(n, R) \cong \pi_0(\mathsf{GL}_n(R))$*?*" It is to this question we now turn.

12.1. The Steinberg group $\mathrm{St}_n(R)$

The usual approach to $K_2(n, R)$ is via the Steinberg group $\mathrm{St}_n(R)$. We earlier, example 2.4, introduced the notation $\varepsilon_{ij}(r)$ for the elementary matrix with

$$\varepsilon_{ij}(r)_{k,l} = \begin{cases} 1 & \text{if } k = l, \\ r & \text{if } (k,l) = (i,j), \\ 0 & \text{otherwise .} \end{cases}$$

Here, of course, $(i, j) \in \Delta$, the set of non-diagonal positions in an $n \times n$ array. Elementary matrices satisfy certain standard relations and the Steinberg group is obtained by considering the group having generators $x_{ij}(r)$, abstracting the elementary matrices, and having as relations just these standard, almost universal, relations. More precisely (and a standard reference is Milnor's notes, [16]), $\mathrm{St}_n(R)$ is given by generators $x_{ij}(r)$, $r \in R$, $i, j = 1, 2, \ldots, n$, $i \neq j$, which are subject to the relations:

St1 $\quad x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a + b)$;

St2 $\quad [x_{i,j}(a), x_{k,\ell}(b)] = \left\{ \begin{array}{ll} 1 & \text{if } i \neq \ell, j \neq k \\ x_{i,\ell}(ab) & i \neq \ell, j = k \end{array} \right.$

These are called the *Steinberg relations.*

There is an epimorphism

$$\varphi : \mathrm{St}_n(R) \to \mathrm{E}_n(R)$$

given by mapping $x_{i,j}(a)$ to $\varepsilon_{i,j}(a)$. The second (unstable) K-group, $K_2(n, R)$ is then defined to be $Ker\ \varphi$.

12.2. A more detailed look at $\mathsf{GL}_n(R)$

We will construct a global action analogous to $St_n(R)$ but for this we need to understand $\mathsf{GL}_n(R)$ better. Early in this paper, §2.4, p.106, we introduced the elementary matrix groups, $\mathrm{GL}_n(R)_\alpha$. Recall that we let Δ be the set of off-diagonal positions in an $n \times n$ array and called a subset $\alpha \subseteq \Delta$ *closed* if it corresponded to a transitive relation, i.e. if $(i, j) \in \alpha$ and $(j, k) \in \alpha$, then $(i, k) \in \alpha$. The general linear global action $\mathsf{GL}_n(R)$ then had coordinate system Φ, the set of closed subsets of Δ ordered by inclusion. The underlying set of $\mathsf{GL}_n(R)$ was the general linear group $\mathrm{GL}_n(R)$ and for $\alpha \in \Phi$, $\mathrm{GL}_n(R)_\alpha$ was the group of elementary matrices generated by the $\varepsilon_{i,j}(r)$ with $(i, j) \in \alpha$.

Clearly this single domain global action is of the form $\mathsf{A}(G, \mathcal{H})$. To form its connected covering (which will cover the connected component of 1, that is, will cover the sub-global action, $\mathrm{E}_n(R)$, determined by the elementary matrices), we need to take the colimit of the $\mathrm{GL}_n(R)_\alpha$. Clearly to examine this colimit we need to see what the maximal elements of Φ are, and to examine the corresponding $\mathrm{GL}_n(R)_\alpha$.

Lemma 12.1. *If $\alpha \in \Phi$ is maximal, then it is a total order on $\{1, \ldots, n\}$.*

Proof. Transitivity follows from closedness. If $(i, j) \in \alpha$, then $(j, i) \notin \alpha$, since no diagonal elements are in α, but as α is maximal, one or other of (i, j) and (j, i) must be in α - otherwise we could add it in! □

Lemma 12.2. *Let $T_n(R)$ be the group of upper triangular $n \times n$ matrices over R. If $\alpha \in \Phi$ is maximal, then*

$$\mathrm{GL}_n(R)_\alpha \cong T_n(R).$$

Proof. Pick an order isomorphism, f between α and the total order $1 < 2 < \ldots < n$. Map the generator $\varepsilon_{i,j}(r)$ to $\varepsilon_{f(i),f(j)}(r)$. This extends to the required isomorphism from $\mathrm{GL}_n(R)_\alpha$ to $T_n(R)$. □

12.3. The Steinberg global action $\mathsf{St}_n(R)$

It is known (cf. Milnor's notes [16]) that $T_n(R)$ has a presentation given by the $x_{k,\ell}(r)$, with $1 \leqslant k < \ell \leqslant n$, and with the Steinberg relation (restricted to those indices (k, ℓ) with $k < \ell$) between them. Let $\mathrm{St}_n(R)_\alpha$ be the group given by generators $x_{k,\ell}(r)$ with $(i, j) \in \alpha$ and with the corresponding Steinberg relations, then

$$colim_{\alpha \in \Phi} \mathrm{St}_n(R)_\alpha = \mathrm{St}_n(R),$$

so if we define a global action as below, it will be connected.

Definition 12.3. *Let $\mathsf{St}_n(R)$ be the global action having $\mathrm{St}_n(R)$ as its underlying set, Δ, above, as its coordinate system and, for $\alpha \in \Delta$, $\mathrm{St}_n(R)_\alpha$ as the corresponding local group.*

The isomorphism $\mathrm{GL}_n(R)_\alpha \cong T_n(R)$ for maximal α together with the fact that $T_n(R) = \mathrm{St}_n(R)_{\alpha_0}$ for $\alpha_0 = \{(i,j) : i < j\}$ gives that there is an isomorphism $\varphi_\alpha : \mathrm{St}_n(R)_\alpha \xrightarrow{\cong} \mathrm{GL}_n(R)_\alpha$ for maximal α compatible with the inclusions into $\mathrm{St}_n(R)$ and $\mathrm{GL}_n(R)$ and the homomorphism φ introduced earlier.

Two maximal α can be linked with each other by a zig-zag where intermediate maximal elements (total orders) differ by the transposition of two elements only. The corresponding isomorphisms Φ_α agree on intersections of the corresponding groups thus giving an isomorphism

$$\mathrm{St}_n(R) = colim_{\alpha\in\Phi}\,\mathrm{St}_n(R)_\alpha \xrightarrow{\cong} colim_{\alpha\in\Phi}\,\mathrm{GL}_n(R)_\alpha = \widetilde{\mathrm{GL}_n(R)}.$$

The resulting map is then easily shown to be φ. It remains to analyse this map a little more.

The kernel,

$$H_{\mathsf{A}} = \mathrm{Ker}(\mathrm{St}_n(R) \simeq colim_{\alpha\in\Phi}\,\mathrm{GL}_n(R)_\alpha \to \mathrm{E}_n(R)),$$

is central (again see Milnor's notes), hence

$$\sigma H_{\mathsf{A}}\sigma^{-1} \cap Im\,\mathrm{GL}_n(R)_\alpha = H_{\mathsf{A}} \cap \mathrm{GL}_n(R)_\alpha = \{1\}$$

as no element of $\mathrm{GL}_n(R)_\alpha$ vanishes when mapped into $\mathrm{GL}_n(R)$ as the mapping is an inclusion. Thus the family, whose minimal element we need, contains the trivial subgroup! Hence that must be the minimal element. □

We have proved

Theorem 12.4. *The simply connected universal covering of $\mathsf{GL}_n(R)$ is isomorphic to $\mathsf{St}_n(R)$. The covering map is given on elements by the evaluation mapping*

$$\varphi : \mathrm{St}_n(R) \to \mathrm{GL}_n(R).$$

□

Corollary 12.5. *The second K-group $K_2(n,R)$ is isomorphic to $\pi_1(\mathsf{GL}_n(R))$.* □

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