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SEPARABLE MORPHISMS OF SIMPLICIAL SETS

DIMITRI CHIKHLADZE

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Abstract

We show that the class of separable morphisms in the sense of G. Janelidze and W. Tholen in the case of Galois structure of second order coverings of simplicial sets due to R. Brown and G. Janelidze coincides with the class of covering maps of simplicial sets.

Introduction

Separable morphisms were introduced in [**3**] by A. Carboni and G. Janelidze for lextensive categories. In the way of [**3**] one can consider separable morphism in a lextensive category $Fam(A)$, the category of families of objects in a category A. What we call Γ_1 below can be seen as a special case of this situation, a characterization of separable morphisms for which is given by Theorem 2.1.

G. Janelidze and W. Tholen [**8**] defined separable morphisms in a category C for a given pointed endofunctor of C. Given an adjunction $I, H : X \leftrightarrows C$ one can consider separable morphisms with respect to the induced monad. Then, in a special case, an appropriate adjunction $I, H : Sets \leftrightarrows Fam(A)$ between $Fam(A)$ and the category of sets gives the same notion of separable morphism in a lextensive category $Fam(A)$ as [**3**].

Definition 1.1 in this paper is essentially of [**8**], the difference being that in place of an adjunction we consider a Galois structure, which together with a pair of adjoint functors $I, H : X \leftrightarrows C$ consists of specified classes of morphisms F and F', called fibrations, in C and X respectively (see G. Janelidze [**7**], the earlier reference is G. Janelidze [**6**]), and we require separable morphisms to be fibrations.

Our purpose is to describe the class of separable morphisms for the Galois structure introduced by R. Brown and G. Janelidze in [**2**] (Γ_2 below). Theorem 2.4 states that for this Galois structure separable morphisms are exactly the Kan fibrations which are covering maps of simplicity sets.

1. Separable morphisms

In this section C is a finitely complete category. Let X be a full reflective subcategory of C with the inclusion $H : X \to C$. Suppose the reflection $I : C \to X$ and its unit $\eta : 1 \to HI$ are chosen in a such way that the counit is an identity $IH = 1$.

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Let F and F' be pullback stable classes of morphisms in C and X respectively, such that $I(F) \subset F'$ and $H(F') \subset F$.

Throughout this paper by Galois structure we mean the data $\Gamma = (C, X, H, I, \eta, F, F')$, with C, X, H, I, η, F and F' as above. Morphisms from F and F' will be called fibrations.

We will say that a morphism $f : A \to B$ of the category C is a trivial covering or a cartesian morphism with respect to the Galois structure Γ if it is a fibration, and the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HIA \arrow[d, "HIf"] \\ B \arrow[r, "\eta_B"] & HIB \end{tikzcd} \qquad (*)$$

is a pullback (see [**5**], 3.1).

Definition 1.1. *A fibration $h : A \to B$ from F is called a separable morphism with respect to the Galois structure $\Gamma = (C, X, H, I, \eta, F, F')$ if the diagonal $\Delta = \langle 1_A, 1_A\rangle : A \to A \times_B A$ is a cartesian morphism with respect to Γ.*

Consider an example. Suppose A is a category with a terminal object t. Let $Fam(A)$ be the category of families of objects in A. A morphism $(m, \lambda) : (A_\lambda)_{\lambda\in\Lambda} \to (A'_{\lambda'})_{\lambda'\in\Lambda'}$ consists of a map $m : \Lambda \to \Lambda'$ and morphisms $\alpha_\lambda : A_\lambda \to A'_{f(\lambda)}$ for all $\lambda \in \Lambda$. Let:

$C = Fam(A)$;

$X = Sets$ is the category of sets;

$H : X \to (A_x)_{x\in X}$ where $A_x = t$ for all x.

$I : (A_\lambda)_{\lambda\in\Lambda} \to \Lambda$; I is a left adjoint to H with the obvious unit η;

F and F' are the classes of all morphisms in C and X respectively.

It is straightforward that a morphism $(m, \lambda) : (A_\lambda)_{\lambda\in\Lambda} \to (A'_{\lambda'})_{\lambda'\in\Lambda'}$ is cartesian with respect to this Galois structure if and only if all $\alpha_\lambda : A_\lambda \to A'_{f(\lambda)}$ are isomorphisms.

2. Separable morphisms for simplicial sets

We will consider two Galois structures; first Γ_1, in which:

$C = Sets^{\Delta^{op}}$ is the category of simplicial sets;

$X = Sets$ is the category of sets;

$H : X \to C$ is the canonical inclusion;

$I : C \to X$ is the functor sending a simplicial set to the set of its connected components; it is a left adjoint to the inclusion H with the obvious unit η;

F and F' are the classes of all morphisms in C and X respectively.

Note that this structure is a special case of the example from the section 1, when A is the category of connected simplicial sets. Then, it is easy to see that cartesian morphisms with respect to Γ_1 are exactly the trivial coverings of simplicial sets in the usual sense.

Theorem 2.1. *A morphism of simplicial sets is a separable morphism with respect to the Galois structure* Γ_1 *if and only if for each commutative diagram*

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d] & A \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, shift left] \arrow[ur, shift right] & B \end{tikzcd} \qquad (**)$$

the diagonal morphisms are equal.

Proof. Observe that an injection $m : A \to B$ is cartesian with respect to Γ_1 (i.e. trivial covering of simplicial sets) if and only if each connected component of B either is contained in the image of A under m, or has no intersection with it. We will use this observation for $\Delta : A \to A \times_B A$ (here and further $A \times_B A$ denotes the pullback of a typical $h : A \to B$ by itself; Δ is the diagonal $\Delta = \langle 1_A, 1_A \rangle : A \to A \times_B A$). In this case, let D denote the image of A under Δ; it consists of all simplices of the form (x, x) in $A \times_B A$.

Let in a diagram (**) h be a separable morphism with respect to Γ_1, what is to say that Δ is cartesian with respect to Γ_1. If x_1 and x_2 are the n-simplices of A corresponding to the diagonal morphisms of (**), then $h(x_1) = h(x_2)$, and the pair (x_1, x_2) is an n-simplex of $A \times_B A$. The upper horizontal morphism in the diagram (**) gives a vertex a of A, and by commutativity of (**) (a, a) is a vertex of (x_1, x_2). So, the connected component containing the simplex (x_1, x_2) has an intersection with D, but then by separability of Δ it is completely contained in D. It follows that $x_1 = x_2$.

Now suppose that for the fixed $h : A \to B$ in each commutative diagram (**) the diagonals are equal. We will prove that $\Delta : A \to A \times_B A$ is cartesian with respect to Γ_1.

Let K be a connected component of $A \times_B A$ the intersection of which with D is not empty. K contains a vertex of the form (a, a), $a \in A[0]$. If (d_1, d_2) $(d_1, d_2 \in A[0])$ is another vertex of K, then there is a path from (a, a) to (d_1, d_2) in $A \times_B A$.

Here we observe, if (e_1, e_2) is a 1-simplex in $A \times_B A$ "connecting" (a', a') with $({d_1}', {d_2}')$, then there is a commutative diagram

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d] & A \arrow[d] \\ \Delta[1] \arrow[r] \arrow[ur, shift left] \arrow[ur, shift right] & B \end{tikzcd}$$

wherein the upper horizontal morphism is determined by a' and the diagonals by e_1 and e_2. Since these diagonals are equal $e_1 = e_2$, yielding also ${d_1}' = {d_2}'$. Applying this argument consecutively to 1-simplices in the chain connecting (a, a) with (d_1, d_2) we get $d_1 = d_2$. Thus, each vertex of K is in D.

Suppose now a pair (x_1, x_2), with $x_1, x_2 \in A[n], n \geqslant 1$, is an n-simplex of K. Since the vertices of K are in D we can construct a commutative diagram of the form (**), wherein the diagonals correspond to x_1 and x_2. These diagonals are equal, so

$x_1 = x_2$. We have proved that K is completely contained in D showing that Δ is cartesian with respect to Γ_1. $\square$

Consider now a Galois structure Γ_2, introduced in [**2**]:
$C = Sets^{\Delta^{op}}$ is the category of simplicial sets;
X is the category of groupoids;
$H : X \to C$ is the canonical inclusion called the nerve functor;
$I = \pi_1 : C \to X$ is the fundamental groupoid functor, which is a left adjoint to H with the obvious unit η;
F and F' are the classes of Kan fibrations ([**5**], p. 65) in C and X respectively.

Definition 2.2. *A morphism of simplicial sets $h : A \to B$ is called a covering map if there exists a surjection $p : E \to B$ and the pullback of h along p is a trivial covering of simplicial sets.*

This definition is a special case of Definition 4.1 in [**7**] for the Galois structure Γ_1. However, as mentioned in A.3.9(iii), [**1**] by F. Borceux and G. Janelidze, it is equivalent to the definition of covering map of simplicial sets given in [**5**] by P. Gabriel and M. Zisman, where a morphism of simplicial sets $h : A \to B$ is said to be a covering map if and only if for each commutative square

$$\begin{tikzcd} \Delta[0] \arrow[r, "v"] \arrow[d, "i"'] & A \arrow[d, "h"] \\ \Delta[n] \arrow[r, "u"] & B \end{tikzcd} \qquad (***)$$

there exists a unique morphism $x : \Delta[n] \to A$ with $hx = u$ and $xi = v$; x satisfying these equations is called a diagonal fill-in of the square (***).

Lemma 2.3. *An injection of simplicial sets is cartesian with respect to Γ_2 if and only if it is cartesian with respect to Γ_1.*

Proof. A trivial covering of simplicial sets is a Kan fibration. Also, it is not difficult to see that for the Galois structure Γ_2, a square (*) is a pullback if h is a trivial covering of simplicial sets. The "if" part of the Lemma is proved.

Conversely, if $m : A \to B$ is cartesian with respect to Γ_2, then, by definition, it is a Kan fibration. In particular, the inverse image of each connected component K of B is either empty or surjectively mapped on K by m. If in addition to this m is injective, then clearly it is a trivial covering of simplicial sets. $\square$

Theorem 2.4. *A Kan fibration of simplicial sets is separable with respect to Γ_2 if and only if it is a covering map.*

Proof. By Lemma 2.3 the diagonal $\Delta : A \to A \times_B A$ is cartesian with respect to Γ_2 if and only if it is cartesian with respect to Γ_1. Then, Theorem 2.4 will follow from Theorem 2.1 if we note that in a commutative diagram (***) where h is a Kan fibration a diagonal fill-in exists. $\square$

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Dimitri Chikhladze
`d.chikhladze@gmail.com`

Department of Mathematics
Tbilisi State University
2 University St., 0143 Tbilisi,
Georgia