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GROUPOIDS, THE PHRAGMEN-BROUWER PROPERTY, AND THE JORDAN CURVE THEOREM

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(*communicated by Frederick Cohen*)

Abstract

We publicise a proof of the Jordan Curve Theorem which relates it to the Phragmen-Brouwer Property, and whose proof uses the van Kampen theorem for the fundamental groupoid on a set of base points.

1. Introduction

This article extracts from [**Bro88**] a proof of the Jordan Curve Theorem based on the use of groupoids, the van Kampen Theorem for the fundamental groupoid on a set of base points, and the use of the Phragmen-Brouwer Property. In the process, we give short proofs of two results on the Phragmen-Brouwer Property (Propositions 4.1, 4.3). There is a renewed interest in such classical results[1], as shown in the article [**Sie**] which revisits proofs of the Schoenflies Theorem.

There are many books containing a further discussion of this area. For more on the Phragmen-Brouwer property, see [**Why42**] and [**Wil49**]. Wilder lists five other properties which he shows for a connected and locally connected metric space are each equivalent to the PBP. The proof we give of the Jordan Curve Theorem is adapted from [**Mun75**]. Because he does not have our van Kampen theorem for non-connected spaces, he is forced into rather special covering space arguments to prove his replacements for Corollary 3.5 (which is originally due to Eilenberg [**Eil37**]), and for Proposition 4.1.

The intention is to make these methods more widely available, since the 1988 edition of the book [**Bro88**] has been out of print for at least ten years, and the new edition is only just now available.

I mention in the same spirit that the results from [**Bro88**] on orbit spaces have been made available in [**BH02**]. A further example of the use of groupoid methods, this time in combinatorial group theory, is in [**Bra04**], which gives a new result combining the Kurosch theorem and a theorem of Higgins which generalises Grusko's

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[1] See also the web site on the Jordan Curve Theorem: `http://www.maths.ed.ac.uk/~aar/jordan/`

theorem. Coverings of non connected topological groups are discussed in [**BM94**]: essential use is made of the well known equivalence, for suitable X, between the categories of covering maps over X and of covering morphisms over the fundamental groupoid $\pi_1 X$. Higgins in [**Hig76**] gives a powerful normal form theorem for what he calls the *fundamental groupoid* of a graph of groups, avoiding the usual choice of a base point or a tree.

Note that we use groupoids not to give nice proofs of theorems on the fundamental *group of a space with base point*, but because we maintain that theorems in this area are about the fundamental *groupoid on a set of base points*, where that set is chosen in a way appropriate to the geometry of the situation at hand. The set of objects of a groupoid gives a spatial component to group theory which allows for more powerful and more easily understood modelling of geometry, and hence for more computational power. Indeed this was the message of the paper [**Bro67**] and even the first 1968 edition of [**Bro88**].

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2. The groupoid van Kampen theorem

We assume as known the notion of the fundamental groupoid $\pi_1 XJ$ of a topological space X on a set J: it consists of homotopy classes rel end points of paths in X joining points of $J \cap X$. We say the pair (X, J) is *connected* if J meets each path component of X. The following theorem was proved in [**Bro67**] (see also [**Bro88**, 6.7.2]).

Theorem 2.1 (van Kampen Theorem). *Let the space X be the union of open subsets U, V with intersection W, let J be a set and suppose the pairs (U, J), (V, J), (W, J) are connected. Then the pair (X, J) is connected and the following diagram of morphisms induced by inclusion is a pushout in the category of groupoids:*

$$\begin{tikzcd} \pi_1 WJ \arrow[r] \arrow[d] & \pi_1 VJ \arrow[d] \\ \pi_1 UJ \arrow[r] & \pi_1 XJ \end{tikzcd}$$

This has been generalised to unions of any number of open sets in [**BR84**]. There then has to be an assumption that (U, J) is connected for any 3-fold (and hence also 1- and 2-fold) intersection U of the sets of the cover.

3. Pushouts of groupoids

In order to apply Theorem 2.1, we need some combinatorial groupoid theory. This was set up in [**Hig05**], [**Bro88**]. We first explain here how to compute an object group $H(x)$ of a groupoid $H = G/R$ given as the quotient of a groupoid G

by a totally disconnected graph $R = \{R(x) \mid x \in \mathrm{Ob}(G)\}$ of relations: of course G/R is defined by the obvious universal property, and has the same object set as G.

Recall from [**Bro88**, 8.3.3] that:

Proposition 3.1. (a) *If G is a connected groupoid, and $x \in \mathrm{Ob}(G)$, then there is a retraction $r : G \to G(x)$ obtained by choosing for each $y \in \mathrm{Ob}(G)$ an element $\tau_y \in G(x, y)$, with $\tau_x = 1_x$.*
(b) *If further $R = \{R(y) \mid y \in \mathrm{Ob}(G)\}$ is a family of subsets of the object groups $G(y)$ of G, then the object group $(G/R)(x)$ is isomorphic to the object group $G(x)$ factored by the relations $r(\rho)$ for all $\rho \in R(y), y \in \mathrm{Ob}(G)$.*

We assume as understood the notion of free groupoid on a (directed) graph. If G, H are groupoids then their free product $G * H$ is given by the pushout of groupoids

$$\begin{tikzcd} \mathrm{Ob}(G) \cap \mathrm{Ob}(H) \arrow[r, "i"] \arrow[d, "j"'] & H \arrow[d] \\ G \arrow[r] & G * H \end{tikzcd}$$

where $\mathrm{Ob}(G) \cap \mathrm{Ob}(H)$ is regarded as the subgroupoid of identities of both G, H on this object set, and i, j are the inclusions. We assume, as may be proved from the results of [**Bro88**, Chapter 8]:

Proposition 3.2. *If G, H are free groupoids, then so also is $G * H$.*

If J is a set, then by the category of groupoids over J we mean the category whose objects are groupoids with object set J and whose morphisms are morphisms of groupoids which are the identity on J.

Proposition 3.3. *Suppose given a pushout of groupoids over J*

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "u"] \\ B \arrow[r, "v"] & G \end{tikzcd} \tag{1}$$

such that C is totally disconnected and A, B are connected. Let p be a chosen element of J. Let $r : A \to A(p)$, $s : B \to B(p)$ be retractions obtained by choosing elements $\alpha_x \in A(p, x)$, $\beta_x \in B(p, x)$, for all $x \in J$, with $\alpha_p = 1$, $\beta_p \in 1$. Let $f_x = (u\alpha_x)^{-1}(v\beta_x)$ in $G(p)$, and let F be the free group on the elements f_x, $x \in J$, with the relation $f_p = 1$. Then the object group $G(p)$ is isomorphic to the quotient of the free product group

$$A(p) * B(p) * F$$

by the relations

$$(ri\gamma)f_x(sj\gamma)^{-1}f_x^{-1} = 1 \tag{2}$$

for all $x \in J$ and all $\gamma \in C(x,x)$.

Proof. We first remark that the pushout (1) implies that the groupoid G may be presented as the quotient of the free product groupoid $A * B$ by the relations $(i\gamma)(j\gamma)^{-1}$ for all $\gamma \in C$. The problem is to interpret this fact in terms of the object group at p of G.

To this end, let T, S be the tree subgroupoids of A, B respectively generated by the elements α_x, β_x, $x \in J$. The elements α_x, β_x, $x \in J$, define isomorphisms

$$\varphi : A \to A(p) * T, \qquad \psi : B \to B(p) * S$$

where if $g \in G(x,y)$ then

$$\varphi g = \alpha_y(rg)\alpha_x^{-1}, \qquad \psi g = \beta_y(sg)\beta_x^{-1}.$$

So G is isomorphic to the quotient of the groupoid

$$H = A(p) * T * B(p) * S$$

by the relations

$$(\varphi i\gamma)(\psi j\gamma)^{-1} = 1$$

for all $\gamma \in C$. By Proposition 3.1, the object group $G(p)$ is isomorphic to the quotient of the group $H(p)$ by the relations

$$(r\varphi i\gamma)(r\psi j\gamma)^{-1} = 1$$

for all $\gamma \in C$.

Now if $J' = J \setminus \{p\}$, then T, S are free groupoids on the elements α_x, β_x, $x \in J'$, respectively. By Proposition 3.2, and as the reader may readily prove, $T*S$ is the free groupoid on all the elements α_x, β_x, $x \in J'$. It follows from [**Bro88**, 8.2.3] (and from Proposition 3.1), that $(T*S)(p)$ is the free group on the elements $r\beta_x = \alpha_x^{-1}\beta_x = f_x$, $x \in J'$. Let $f_p = 1 \in F$. Since

$$r\varphi i\gamma = ri\gamma, \qquad r\psi j\gamma = f_x(sj\gamma)f_x^{-1},$$

the result follows. □

Remark 3.4. The above formula is given in essence in van Kampen's paper [**Kam33**], since he needed the case of non connected intersection for applications in algebraic geometry. However his proof is difficult to follow, and a modern proof for the case of connected intersection was given by Crowell in [**Cro59**]. □

There is a consequence of the above computation (see [**Eil37**]) which we shall use in the next section in proving the Jordan Curve Theorem.

First, if F and H are groups, recall that we say that F is a *retract* of H if there are morphisms $\imath : F \to H$, $\rho : H \to F$ such that $\rho\imath = 1$. This implies that F is isomorphic to a subgroup of H.

Corollary 3.5. Under the situation of Proposition *3.3*, the free group F is a retract of $G(p)$. Hence if $J = \mathrm{Ob}(C)$ has more than one element, then the group $G(p)$ is not trivial, and if J has more than two elements, then $G(p)$ is not abelian.

Proof. Let $M = A(p)*B(p)*F$, and let $\imath' : F \to M$ be the inclusion. Let $\rho' : M \to F$ be the retraction which is trivial on $A(p)$ and $B(p)$ and is the identity on F. Let $q : M \to G(p)$ be the quotient morphism. Then it is clear that ρ' preserves the relations (2), and so ρ' defines uniquely a morphism $\rho : G(p) \to F$ such that $\rho q = \rho'$. Let $\imath = qi'$. Then $\rho\imath = \rho' i' = 1$. So F is a retract of $G(p)$.

The concluding statements are clear. □

We use the last two statements of the Corollary in sections 4 and 5 respectively.

4. The Phragmen-Brouwer Property

A topological space X is said to *have the Phragmen-Brouwer Property* (here abbreviated to PBP) if X is connected and the following holds: *if D and E are disjoint, closed subsets of X, and if a and b are points in $X \setminus (D \cup E)$ which lie in the same component of $X \setminus D$ and in the same component of $X \setminus E$, then a and b lie in the same component of $X \setminus (D \cup E)$.* To express this more succinctly, we say a subset D of a space X *separates* the points a and b if a and b lie in distinct components of $X \setminus D$. Thus the PBP is that: if D and E are disjoint closed subsets of X and a, b are points of X not in $D \cup E$ such that neither D nor E separate a and b, then $D \cup E$ does not separate a and b.

A standard example of a space not having the PBP is the circle $\mathbb{S}^1$, since we can take $D = \{+1\}$, $E = \{-1\}$, $a = i$, $b = -i$. This example is typical, as the next result shows. But first we remark that our criterion for the PBP will involve fundamental groups, that is will involve paths, and so we need to work with path-components rather than components. However, if X is locally path-connected, then components and path-components of open sets of X coincide, and so for these spaces we can replace in the PBP 'component' by 'path-component'. This explains the assumption of locally path-connected in the results that follow.

Proposition 4.1. *Let X be a path-connected and locally path-connected space whose fundamental group (at any point) does not have the integers $\mathbb{Z}$ as a retract. Then X has the PBP.*

Proof. Suppose X does not have the PBP. Then there are disjoint, closed subsets D and E of X and points a and b of $X \setminus (D \cup E)$ such that $D \cup E$ separates a and b but neither D nor E separates a and b. Let $U = X \setminus D$, $V = X \setminus E$, $W = X \setminus (D \cup E) = U \cap V$. Let J be a subset of W such that $a, b \in J$ and J meets each path-component of W in exactly one point. Since D and E do not separate a and b, there are elements $\alpha \in \pi_1 U(a, b)$ and $\beta \in \pi_1 V(a, b)$. Since X is path-connected, the pairs $(U, J), (V, J), (W, J)$ are connected. By the van Kampen Theorem 2.1 the following diagram of morphisms induced by inclusions is a pushout of groupoids:

$$\begin{tikzcd} \pi_1 WJ \arrow[r, "i_1"] \arrow[d, "i_2"'] & \pi_1 UJ \arrow[d, "u_1"] \\ \pi_1 VJ \arrow[r, "u_2"'] & \pi_1 XJ. \end{tikzcd}$$

Since U and V are path-connected and J has more than one element, it follows from Corollary 3.5 that $\pi_1 XJ$ has the integers $\mathbb{Z}$ as a retract. □

As an immediate application we obtain:

Proposition 4.2. *The following spaces have the PBP: the sphere $\mathbb{S}^n$ for $n > 1$; $\mathbb{S}^2 \setminus \{a\}$ for $a \in \mathbb{S}^2$; $\mathbb{S}^n \setminus \Lambda$ if Λ is a finite set in $\mathbb{S}^n$ and $n > 2$.* □

In each of these cases the fundamental group is trivial.

An important step in our proof of the Jordan Curve Theorem is to show that if A is an *arc* in $\mathbb{S}^2$, that is a subspace of $\mathbb{S}^2$ homeomorphic to the unit interval $\mathbb{I}$, then the complement of A is path-connected. This follows from the following more general result.

Proposition 4.3. *Let X be a path-connected and locally path-connected Hausdorff space such that for each x in X the space $X \setminus \{x\}$ has the PBP. Then any arc in X has path-connected complement.*

Proof. Suppose A is an arc in X and $X \setminus A$ is not path-connected. Let a and b lie in distinct path-components of $X \setminus A$.

By choosing a homeomorphism $\mathbb{I} \to A$ we can speak unambiguously of the mid-point of A or of any subarc of A. Let x be the mid-point of A, so that A is the union of sub-arcs A' and A'' with intersection $\{x\}$. Since X is Hausdorff, the compact sets A' and A'' are closed in X. Hence $A' \setminus \{x\}$ and $A'' \setminus \{x\}$ are disjoint and closed in $X \setminus \{x\}$. Also $A \setminus \{x\}$ separates a and b in $X \setminus \{x\}$ and so one at least of A', A'' separates a and b in $X \setminus \{x\}$. Write A_1 for one of A', A'' which does separate a and b. Then A_1 is also an arc in X.

In this way we can find by repeated bisection a sequence A_i, $i \geqslant 1$, of sub-arcs of A such that for all i the points a and b lie in distinct path-components of $X \setminus A_i$ and such that the intersection of the A_i for $i \geqslant 1$ is a single point, say y, of X.

Now $X \setminus \{y\}$ is path-connected, by definition of the PBP. Hence there is a path λ joining a to b in $X \setminus \{y\}$. But λ has compact image and hence lies in some $X \setminus A_i$. This is a contradiction. □

Corollary 4.4. The complement of any arc in $\mathbb{S}^n$ is path-connected.

Sketch Proof The case $n = 0$ is trivial, while the case $n = 1$ needs a special argument that the complement of any arc in $\mathbb{S}^1$ is an open arc. The case $n \geqslant 2$ follows from the above results. □

5. The Jordan Separation and Curve Theorems

We now prove one step along the way to the full Jordan Curve Theorem.

Theorem 5.1 (The Jordan Separation Theorem). *The complement of a simple closed curve in $\mathbb{S}^2$ is not connected.*

Proof. Let C be a simple closed curve in $\mathbb{S}^2$. Since C is compact and $\mathbb{S}^2$ is Hausdorff, C is closed, $\mathbb{S}^2 \setminus C$ is open, and so path-connectedness of $\mathbb{S}^2 \setminus C$ is equivalent to connectedness.

Write $C = A \cup B$ where A and B are arcs in C meeting only at a and b say. Let $U = \mathbb{S}^2 \setminus A$, $V = \mathbb{S}^2 \setminus B$, $W = U \cap V$, $X = U \cup V$. Then $W = \mathbb{S}^2 \setminus C$ and $X = \mathbb{S}^2 \setminus \{a, b\}$. Also X is path-connected, and, by Corollary 4.4, so also are U and V.

Let $x \in W$. Suppose that W is path-connected. By the van Kampen Theorem 2.1, the following diagram of morphisms induced by inclusion is a pushout of groups:

$$\begin{tikzcd} \pi_1(W,x) \arrow[r] \arrow[d] & \pi_1(U,x) \arrow[d, "i_*"] \\ \pi_1(V,x) \arrow[r, "j_*"'] & \pi_1(X,x). \end{tikzcd}$$

Now $\pi_1(X, x)$ is isomorphic to the group $\mathbb{Z}$ of integers. We derive a contradiction by proving that the morphisms i_* and j_* are trivial. We give the proof for i_*, as that for j_* is similar.

Let $f : \mathbb{S}^1 \to U$ be a map and let $g = if : \mathbb{S}^1 \to X$. Let γ be a parametrisation of A which sends 0 to b and 1 to a. Choose a homeomorphism $h : \mathbb{S}^2 \setminus \{a\} \to \mathbb{R}^2$ which takes b to 0 and such that hg maps $\mathbb{S}^1$ into $\mathbb{R}^2 \setminus \{0\}$. Then $h\gamma(0) = 0$ and $\|h\gamma(t)\|$ tends to infinity as t tends to 1. Since the image of g is compact, there is an $r > 0$ such that $hg[\mathbb{S}^1]$ is contained in $B(0, r)$. Now there exists $0 < t_0 < 1$ such that the distance from 0 to $y = h\gamma(t_0)$ is $> r$. Define the path λ to be the part of $h\gamma$ reparametrised so that $\lambda(0) = 0$ and $\lambda(1) = y$.

Define $G : \mathbb{S}^1 \times \mathbb{I} \to \mathbb{R}^2$ by

$$G(z,t) = \begin{cases} hg(z) - \lambda(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ (2-2t)hg(z) - y & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then G is well-defined. Also G never takes the value 0 (this explains the choices of λ and y). So G gives a homotopy in $\mathbb{R}^2 \setminus \{0\}$ from hg to the constant map at $-y$. So hg is inessential and hence g is inessential. This completes the proof that i_* is trivial. □

As we shall see, the Jordan Separation Theorem is used in the proof of the Jordan Curve Theorem.

Theorem 5.2 (Jordan Curve Theorem). *If C is a simple closed curve in $\mathbb{S}^2$, then the complement of C has exactly two components, each with C as boundary.*

Proof. As in the proof of Theorem 5.1, write C as the union of two arcs A and B meeting only at a and b say, and let $U = \mathbb{S}^2 \setminus A$, $V = \mathbb{S}^2 \setminus B$. Then U and V are path-connected and $X = U \cup V = \mathbb{S}^2 \setminus \{a, b\}$ has fundamental group isomorphic to $\mathbb{Z}$. Also $W = U \cap V = \mathbb{S}^2 \setminus C$ has at least two path-components, by the Jordan Separation Theorem 5.1.

If W has more than two path-components, then the fundamental group G of X contains a copy of the free group on two generators, by Corollary 3.5, and so G is non-abelian. This is a contradiction, since $G \cong \mathbb{Z}$. So W has exactly two path-components P and Q, say, and this proves the first part of Theorem 5.2.

Since C is closed in $\mathbb{S}^2$ and $\mathbb{S}^2$ is locally path-connected, the sets P and Q are open in $\mathbb{S}^2$. It follows that if $x \in \overline{P} \setminus P$ then $x \notin Q$, and hence $\overline{P} \setminus P$ is contained in C. So also is $\overline{Q} \setminus Q$, for similar reasons. We prove these sets are equal to C.

Let $x \in C$ and let N be a neighbourhood of x in $\mathbb{S}^2$. We prove N meets $\overline{P} \setminus P$. Since $\overline{P} \setminus P$ is closed and N is arbitrary, this proves that $x \in \overline{P} \setminus P$.

Write C in a possibly new way as a union of two arcs D and E intersecting in precisely two points and such that D is contained in $N \cap C$. Choose points p in P and q in Q. Since $\mathbb{S}^2 \setminus E$ is path-connected, there is a path λ joining p to q in $\mathbb{S}^2 \setminus E$. Then λ must meet D, since p and q lie in distinct path-components of $\mathbb{S}^2 \setminus E$. In fact if $s = \sup\{t \in \mathbb{I} : \lambda[0, t] \subseteq P\}$, then $\lambda(s) \in \overline{P} \setminus P$. It follows that N meets $\overline{P} \setminus P$.

So $\overline{P} \setminus P = C$ and similarly $\overline{Q} \setminus Q = C$. □

References

[Bra04] Braun, G. 'A proof of Higgins' conjecture'. *Bull. Austral. Math. Soc.* **70** (2) (2004) 207–212.

[Bro67] Brown, R. 'Groupoids and van Kampen's theorem'. *Proc. London Math. Soc.* **17** (3) (1967) 385–401.

[Bro88] Brown, R. *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid.* Ellis-Horwood, Chichester (1988); first edition McGraw Hill, (1968); revised and extended edition published as *Topology and Groupoids*, Booksurge LLC, S. Carolina, (2006).

[BH02] Brown, R. and Higgins, P. J. 'The fundamental groupoid of the quotient of a Hausdorff space by a discontinuous action of a discrete group is the orbit groupoid of the induced action'. *arXiv math.AT/0212271* (2002) 18 pages.

[BM94] Brown, R. and Mucuk, O. 'Covering groups of non-connected topological groups revisited'. *Math. Proc. Camb. Phil. Soc.* **115** (1994) 97–110.

[BR84] Brown, R. and Razak Salleh, A. 'A van Kampen theorem for unions of non-connected spaces'. *Arch. Math.* **42** (1984) 85–88.

[Cro59] Crowell, R. H. 'On the van Kampen theorem'. *Pacific J. Math.* **9** (1959) 43–50.

[Eil37] Eilenberg, S., 'Sur les espaces multicoherents II', *Fund. Math.* **29** (1937) 101–122.

[Hig05] Higgins, P. J. 'Categories and groupoids'. *Van Nostrand mathematical Studies, 1971, Reprints in Theory and Applications of Categories* **7** (2005) 1–195.

[Hig76] Higgins, P.J., 'The fundamental groupoid of a graph of groups', *J. London Math. Soc.*, **13** (1976) 145-149.

[Kam33] Kampen, E. H. Van. 'On the connection between the fundamental groups of some related spaces'. *Amer. J. Math.* **55** (1933) 261–267.

[Mun75] Munkres, J. R. *Topology: a first course.* Prentice-Hall, Englewood Cliffs (1975).

[Sie] Siebenmann, L. 'The Osgood-Schoenflies theorem revisited'. 21pp `http://hopf.math.purdue.edu/Siebenmann/Schoen-02Sept2005`.

[Why42] Whyburn, G. T. *Analytic topology*, *AMS Colloquium Publications*, Volume 28. American Mathematical Society, New York (1942).

[Wil49] Wilder, R. L. *Topology of manifolds*, *AMS Colloquium Publications*, Volume 32. American Mathematical Society, New York (1949).

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