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ALGEBRAIC K-THEORY OF FREDHOLM MODULES AND KK-THEORY

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(*communicated by Jonathan Rosenberg*)

Abstract

Kasparov KK-groups $KK(A, B)$ are represented as homotopy groups of the Pedersen-Weibel nonconnective algebraic K-theory spectrum of the additive category of Fredholm (A, B)-bimodules for A and B, respectively, a separable and σ-unital trivially graded real or complex C^*-algebra acted upon by a fixed compact metrizable group.

dedicated to Hvedri Inassaridze

Introduction

In noncommutative topology and differential geometry some of the useful and powerful tools are methods of algebraic K-theory, Kasparov's KK-theory, spectra and so on. Therefore a comprehensive study of relationships between them may be considered as an interesting task. The main goal of this paper is characterization of Kasparov KK-groups as algebraic K-groups of an additive category. On first view, calculation of the KK-theory by algebraic K-theory seems to be highly improbable, as algebraic K-theory and KK-theory are independent, highly nontrivial, theories, having almost no connections with each other. The key is thus to find suitable objects which make sense for both algebraic K-theory and KK-theory. In this paper we concentrate on the additive C^*-category $\mathrm{Rep}(A, B)$, namely the category of Fredholm modules, where A is a separable and B is a σ-unital real or complex C^*-algebra with action of a fixed compact second countable group. Our main result claims the natural isomorphisms

$$\mathbb{K}_n^a(\mathrm{Rep}(A; B)) \simeq KK_{n-1}(A; B),$$

where $\mathbb{K}_n^a$ denote the algebraic K-functors isomorphic to Quillen's K-functors in nonnegative dimensions, and isomorphic to Pedersen-Weibel K-functors in negative dimensions.

There are already several papers dedicated to interpretations of KK-theory, each with their own advantages. Let us point out in brief some fundamental papers of this

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sort which are sources of further research. These are G. Kasparov's interpretation of KK-theory in terms of extensions of C^*-algebras [**24**]; J. Cuntz's result in [**6**], based on the homotopy category of C^*-algebras; N. Higson's approach, considering KK-theory as an universal enveloping additive category of the category of separable C^*-algebras [**6**]; the interpretation of the KK-theory with the aid of the category of C^*-algebras and asymptotic homomorphisms, due to A. Connes and N. Higson [**4**], and R. Meyer's and R. Nest's joint paper [**25**], where Kasparov category KK turned into a triangulated category.

Let us shortly review some known results in KK-theory established by methods of topological K-theory and spectra.

In [**29**] E. K. Pedersen and C. Weibel showed that values on finite CW-complexes X of the homology theory associated with the nonconnective algebraic K-theory spectrum of a unital ring R may be interpreted as the algebraic K-groups (up to a shift in dimension) of a suitably constructed additive category $\mathcal{C}_{O(X)}(R)$. According to this result, in [**31**] J. Rosenberg showed that

$$K_1(\mathcal{C}_{O(X)}(R)) \simeq KK(C(X), R)$$

for a unital C^*-algebra R. He also constructed algebraic KK-theory spectra, denoted by $\mathbf{KK}(A, B)$, having the property

$$\pi_0(\mathbf{KK}(A, B)) \simeq KK(A, B).$$

The similar question for nonzero dimensions has been left open in that paper.

In [**12**], there have been constructed $\mathbb{KK}^{\mathrm{top}}(C_0(X), B)$-spectra, related to Kasparov groups $KK_n(C_0(X), B)$, which were used to construct the splitting assembly map

$$A : \mathbb{KK}^{\mathrm{top}}(C_0(X), B) \to \mathbb{K}^{top}(C^*(X)),$$

where $C^*(X)$ is the C^*-algebra of the coarse space X. A large number of results on the Novikov conjecture can be included under this scheme [**12**].

More general approaches to non-equivariant KK-theory as homotopy groups of a spectrum can be found in the following papers. These are, [**13**], where T. G. Houghton-Larsen and K. Thomsen, utilizing spaces of C^*-extensions, have constructed KK-theory spectra; and [**26**], where P. Mitchener, using methods of topological K-theory, symmetric spectra and C^*-categories has defined KK-theory spectra, too.

Let us make few remarks concerning our approach. In [**16**], [**19**] we have calculated topological and Karoubi-Villamayor K-groups of the C^*-category $\mathrm{Rep}(A, B)$ which are related to KK-groups by the equalities

$$K_n^t(\mathrm{Rep}(A, B)) = K_n^{\mathrm{KV}}(\mathrm{Rep}(A, B)) = KK_{n-1}(A, B), \quad n \geqslant 0, \tag{0.1}$$

where A and B are G-C^*-algebras, separable and σ-unital respectively. Thus the additive C^*-category $\mathrm{Rep}(A, B))$ is a good object for our purposes at first sight.

In the article [**17**] it was announced that an isomorphism similar to 0.1 for algebraic K-groups also holds, as well. The present paper is an attempt to explain in detail the results announced in [**17**]. As a consequence construction of a nonconnective algebraic KK-theory spectrum $\mathbb{KK}^{\mathrm{alg}}(A, B) = \mathbb{K}(\mathrm{Rep}(A, B))$ arises, where the right hand side is the Pedersen-Weibel nonconnective algebraic K-theory spectrum of the idempotent-complete additive category $\mathrm{Rep}(A,B)$. This spectrum has the property

$$\pi_n(\mathbb{KK}^{\mathrm{alg}}(A,B)) \simeq KK_{n-1}(A,B),$$

for all $n \in \mathbb{Z}$. In a further paper we hope to apply the algebraic KK-theory spectra, in particular, to problems related to the Novikov conjecture. Our approach is mainly based on the author's unpublished preprint [**20**].

1. The main Theorem and an outline of the proof

The purpose of this section is to summarize some of the concepts that are needed for the formulation of the main theorem, and then to give an outline of the proof. First, we recall the definitions of a C^*-category and an idempotent-compete C^*-category; and in Subsection 1.1 an idempotent-complete C^*-category $\mathrm{Rep}(A,B)$ is constructed that we will need later on.

Let A be a category such that for any pair (a,b) of objects in A, the set $\hom(a,b)$ is equipped with the structure of a Banach space in such a way that composition is a continuous k-bilinear map. Such a category is said to be a Banach category over k, or simply a Banach category. A Banach category A is called a C^*-category if it is equipped with a family of anti-linear maps $* : \hom(a,b) \to \hom(b,a)$ for any $a, b \in \mathrm{Ob}(A)$ such that

1. $(f^*)^* = f$;
2. $(fg)^* = g^*f^*$, if fg exists;
3. $\|f^*\| = \|f\|$;
4. $\|f^*f\| = \|f\|^2$, if k is the complex numbers; and $\|f\|^2 \leqslant \|f^*f + g^*g\|^2$, if k is the real numbers.
5. For any morphism $f : a \to b$ in A the morphism f^*f is a positive element of the C^*-algebra $\hom(a,a)$.

Let A and B be C^*-categories. A functor $\mathcal{F} : A \to B$ is said to be a $*$-functor if

- $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$;
- $\mathcal{F}(\lambda f) = \lambda\mathcal{F}(f)$;
- $\mathcal{F}(f^*) = \mathcal{F}^*$,

where $\lambda \in k$, and f and g are morphisms in A. (cf. [**3**], [**16**], [**18**]).

We say that a $*$-functor is faithful if it is injective on both objects and morphisms. Any $*$-functor is norm-nonincreasing. Moreover, a faithful $*$-functor preserves norms [**3**].

The category $\mathcal{H}(k)$ of separable Hilbert spaces and bounded linear maps has a natural structure of a C^*-category. There exists a faithful $*$-functor from every C^*-category into $\mathcal{H}(k)$.

Let A be a C^*-category and $I \subset \hom A$. Put $\hom_I(a,b) = \hom(a,b) \cap I$. Then I is called a left ideal if $\hom_I(a,b)$ is a linear subspace of $\hom(a,b)$ and $f \in \hom_I(a,b)$, $g \in \hom(b,c)$ imply $gf \in \hom_I(a,c)$. A right ideal is defined similarly. I is a two-sided ideal if it is both a left and a right ideal. An ideal I is closed if $\hom(a,b)_I$

is closed in $\hom(a,b)$ for each pair of objects. A closed two-sided ideal I is called a C^*-ideal if $I = I^*$. Every C^*-ideal determines an equivalence relation on the morphisms of A: $f \sim g$ if $f - g \in I$; the set of equivalence classes A/I can be made into a C^*-category in a unique way by requiring that the canonical map $f \mapsto \hat{f}$ gives rise to a $*$-functor $A \to A/I$. Arguing as for C^*-algebras, one can show that every closed ideal is a C^*-ideal [**3**]. Among the C^*-categories containing I as a C^*-ideal there exists a universal one, the so called multiplier C^*-category. We will denote by $M(I)$ the multiplier C^*-category of a C^*-ideal I [**3**].

Let A be an additive category. *Idempotent completion* of A is an additive category $\hat{A}$ whose objects have the form (a,q), where a is an object in A and q is an idempotent in $\hom(a,a)$, and a morphism $f : (a,q) \to (a',q')$ is a morphism $f : a \to a'$ such that $fq = q'f = f$. There is a natural functor $A \to \hat{A}$, defined by assignments $a \mapsto (a, 1_a)$ and $f \mapsto f$. An additive category B is said to be *idempotent-complete* if there are an additive category A, and an additive functor $F : B \to \hat{A}$ which is an equivalence of categories.

Note that for an additive C^*-category A the category $\hat{A}$ is not necessarily a C^*-category. Below we will adapt the above construction to the case of additive C^*-categories.

Recall that a projection p in a C^*-category is a morphism with the properties $p^* = p$ and $p^2 = p$, i. e., a projection is a selfadjoint idempotent.

Let A be an additive C^*-category. Consider the additive C^*-category $\tilde{A}$ with objects of the form (a,p), where $a \in \mathrm{Ob}(A)$ and $p \in \hom(a,a)$ is a projection. A morphism from (a,p) to (b,q) is a morphism $f : a \to b$ in A such that $fp = qf = f$. Composition of morphisms is the same as in A. The sum is given by $(a,p) \oplus (b,q) = (a \oplus b, p \oplus q)$, and the norm of morphisms is inherited from A [**16**]. There is a natural functor $\nu : \tilde{A} \to \hat{A}$ defined by identity maps on objects and morphisms.

Let us show the following simple lemma.

Lemma 1.1. *Let A be an additive C^*-category. Then $\tilde{A}$ is an idempotent-complete C^*-category.*

Proof. Consider the natural additive functor $\nu : \tilde{A} \to \hat{A}$, which is, of course, faithful. Let us show that ν is a full functor. Indeed, if $q \in \hom(a,a)$ is an idempotent then

$$p = ((2q^* - 1)(2q - 1) + 1)^{\frac{1}{2}} \cdot q \cdot ((2q^* - 1)(2q - 1) + 1)^{-\frac{1}{2}}$$

is a projection and the pairs $(a;q)$ and $(a;p)$ are isomorphic in $\hat{A}$ via the morphism

$$p((2q^* - 1)(2q - 1) + 1)^{\frac{1}{2}} q.$$

□

Now, we define examples of additive C^*-categories which are used in the remaining part of paper.

1.1. On additive C^*-categories $Rep(A,B)$ and $\mathrm{Rep}(A,B)$

Let $\mathcal{H}_G(B)$ be the additive C^*-category of countably generated right Hilbert B-modules equipped with a B-linear, norm-continuous G-action over a fixed compact second countable group G [**23**]. Note that the compact group acts on the morphisms by the following rule: for $f : E \to E'$ the morphism $gf : E \to E'$ is defined by the formula $(gf)(x) = g(f(g^{-1}(x)))$.

The category $\mathcal{H}_G(B)$ contains the class of compact B-homomorphisms [**23**]. Denote it by $\mathcal{K}_G(B)$. Known properties of compact B-homomorphisms imply that $\mathcal{K}_G(B)$ is a C^*-ideal [**3**] in $\mathcal{H}_G(B)$.

Objects of the category $Rep(A, B)$ are pairs of the form (E, φ), where E is an object in $\mathcal{H}_G(B)$ and $\varphi : A \to \mathcal{L}(E)$ is an equivariant $*$-homomorphism. A morphism $f : (E, \phi) \to (E', \phi')$ is a G-invariant morphism $f : E \to E'$ in $\mathcal{H}_G(B)$ such that

$$f\phi(a) - \phi'(a)f \in \mathcal{K}_G(E, E')$$

for all $a \in A$. The structure of a C^*-category is inherited from $\mathcal{H}_G(B)$. It is easy to see that $Rep(A, B)$ is an additive C^*-category, not idempotent-complete.

Now, we are ready to construct our main C^*-category, that is $\mathrm{Rep}(A, B)$. Objects of it are triples (E, ϕ, p), where (E, ϕ) is an object and $p : (E, \phi) \to (E, \phi)$ is a morphism in $Rep(A, B)$ such that $p^* = p$ and $p^2 = p$. A morphism $f : (E, \phi, p) \to (E', \phi', p')$ is a morphism $f : (E, \phi) \to (E', \phi')$ in $Rep(A, B)$ such that $fp = p'f = f$. In detail, f must satisfy

$$f\phi(a) - \phi'(a)f \in \mathcal{K}(E, F) \text{ and } fp = p'f = f. \tag{1.1}$$

So, by definition

$$\mathrm{Rep}(A, B) = \widetilde{Rep(A, B)}.$$

The structure of a C^*-category on $\mathrm{Rep}(A, B)$ comes from the corresponding structure on $Rep(A, B)$.

Let S_G denote the category of trivially graded separable C^*-algebras over k with an action of the compact second countable group G and equivariant $*$-homomorphisms. Functors $\mathbb{K}^a_n$ are defined by

$$\mathbb{K}^a_n(A) = \pi_n\mathbb{K}(A), \quad n \in \mathbb{Z},$$

where $\mathbb{K}(A)$ is the Pedersen-Weibel nonconnective algebraic K-theory spectrum [**28**] of an idempotent-complete additive category A. Functors $\mathbb{K}^t_n$ are the topological K-functors on idempotent-complete additive C^*-categories, defined by Karoubi [**22**],[**21**]. For simplicity, Kasparov's groups $KK^{-n}_G(A, B)$ will be denoted by $KK_n(A, B)$.

Now, we present our main result in the following theorem.

Theorem 1.2. *Let B be a σ-unital trivially graded C^*-algebra with an action of a second countable compact group G. There are natural isomorphisms*

$$\mathbb{K}^a_n(\mathrm{Rep}(-; B)) \simeq \mathbb{K}^t_n(\mathrm{Rep}(-; B)) \simeq KK_{n-1}(-; B) \tag{1.2}$$

of functors on the category S_G, for all $n \in \mathbb{Z}$.

Outline of proof. Theorem 1.2 is a consequence of the argument presented below.

A family $\mathbb{H} = \{H_n\}_{n\in\mathbb{Z}}$ of contravariant functors from S_G to the category of abelian groups and homomorphisms is said to be a *stable cohomology theory* on the category S_G if

1. $\mathbb{H}$ *has the weak excision property.* Namely, for any exact *proper* sequence

$$0 \to I \to B \to A \to 0$$

(which means that the involved epimorphism admits an equivariant completely positive contractive section) of algebras in S_G there exists a natural homomorphism $\delta_n : H_n(I) \to H_{n-1}(A)$, for any $n \in \mathbb{Z}$, such that the resulting natural sequence of abelian groups (extending in both directions)

$$\cdots \to H_n(A) \to H_n(B) \to H_n(I) \xrightarrow{\delta_n} H_{n-1}(A) \to \cdots$$

is exact.

2. $\mathbb{H}$ *is stable.* This means that if $e_A : A \to A \otimes \mathcal{K}$ is a homomorphism defined by the map $a \mapsto a \otimes p$, where p is a rank one projection in $\mathcal{K}$, then $H_n(e_A) : H_n(A \otimes \mathcal{K}) \to H_n(A)$ is an isomorphism, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable Hilbert space $\mathcal{H}$ over k, with the trivial action of the group G.

Denote by $C_k(S^1)$ the C^*-algebra of continuous complex functions on the standard unit circle S^1 of modulus one complex numbers, in case k is the field of complex numbers; while if k is the field of reals, let it be the subalgebra of the former consisting of functions invariant under the conjugation defined by the map $f(z) \mapsto \overline{f(\bar{z})}$. It is clear that any continuous complex function $f : S^1 \to \mathbb{C}$ may be represented in the form

$$f(z) = \frac{f(z) + \overline{f(\bar{z})}}{2} + i\frac{(-if(z)) + (\overline{-if(\bar{z})})}{2},$$

which means that $C_{\mathbb{C}}(S^1)$ is the complexification of the real C^*-algebra $C_{\mathbb{R}}(S^1)$.

Let $\mathbf{T}_k$ be the Toeplitz C^*-algebra—the universal C^*-algebra over k generated by an isometry v. There is a conjugation on $\mathbf{T}_{\mathbb{C}}$ defined by the equality $\bar{v} = v$ on the generator v of $\mathbf{T}_{\mathbb{C}}$. According to the universal property of the Toeplitz algebra, one gets a natural homomorphism $\mathbf{T}_{\mathbb{R}} \to \mathbf{T}_{\mathbb{C}}$ so that the induced homomorphism $\mathbf{T}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbf{T}_{\mathbb{C}}$ is an isomorphism, so that there is a short exact sequence

$$0 \to \mathcal{K}_k \to \mathbf{T}_k \xrightarrow{t} C_k(S^1) \to 0.$$

where $\mathcal{K}_k$ is the C^*-algebra of compact operators on a separable Hilbert space over k.

Let $\tau : C_k(S^1) \to k$ be the homomorphism given by $f \mapsto f(1)$. Denote by $\mho_k$ the kernel of τ. It is clear that $\mho_{\mathbb{R}}$ is naturally isomorphic to the algebra $C_0^{\mathbb{R}}(i\mathbb{R})$ defined in [**5**], and $\mho_{\mathbb{C}}$ is isomorphic to $\Omega_{\mathbb{C}}$, where $\Omega_k = \{f : I = [0,1] \to k \mid f(0) = f(1) = 0\}$. The actions of G on the algebras considered above are trivial.

It follows from the Higson's theorem (see [**10**], and [**15**] for the real case) that the functor H_n is homotopy invariant for any $n \in \mathbb{Z}$. Now, the proofs of the following proposition and theorem coincide (up to trivial changes) with the proofs of suitable results in [**5**].

Proposition 1.3. *Let $\mathbb{H}$ be a stable cohomology theory and let $g : \mathbf{T}_k \to k$ be the homomorphism defined by $v \mapsto 1$. Then the homomorphism*

$$H_n(\mathrm{id}_A \otimes g) : H_n(A) \xrightarrow{\cong} H_n(A \otimes \mathbf{T}_k)$$

is an isomorphism, for any $A \in S_G$ and $n \in \mathbb{Z}$.

Theorem 1.4. *Let $\mathbb{H}$ be a stable cohomology theory, $\mho A = A \otimes \mho_k$ and $\Omega A = A \otimes \Omega_k$. Then there are natural isomorphisms*

$$H_{n+1}(A) \simeq H_n(\mho A) \quad \text{and} \quad H_{n-1}(A) \simeq H_n(\Omega A). \tag{1.3}$$

for any $A \in S_G$ and $n \in \mathbb{Z}$.

As a consequence we have the following working principle: *two stable cohomology theories are isomorphic if and only if they are isomorphic in some fixed dimension.* Thus in the remaining sections of the paper we show that the families of functors in Theorem 1.2 are stable cohomology theories and they are isomorphic when $n = 0$.

In more detail, in section 2 we study an interpretation of algebraic and topological K-theories of C^*-categories. Our definition is an adaptation to our cases of some arguments from [**1**], [**9**], [**30**]. Let A be a C^*-category and let I be a C^*-ideal in A; let a and a' be objects in A. We write $a \leqslant a'$ if there exists a morphism $s : a \to a'$ such that $s^*s = 1_a$ (such a morphism is said to be an *isometry*). Denote by $\mathcal{L}(a)$ (resp. by $\mathcal{I}(a)$) the C^*-algebra $\hom_A(a,a)$ (resp $\hom_I(a,a)$). We have a well-defined inductive system of abelian groups $\{\mathbb{K}_n^a(\mathcal{L}(a)), \sigma_{aa'}\}_a$ and $\{\mathbb{K}_n^t(\mathcal{I}(a)), \sigma_{aa'}\}_a$. We suppose that

$$\mathbf{K}_n^a(A) = \varinjlim{}_a \mathbb{K}_n^a(\mathcal{L}(a)) \quad \text{and} \quad \mathbf{K}_n^a(I) = \varinjlim{}_a \mathbb{K}_n^a(\mathcal{I}(a))). \tag{1.4}$$

Thanks to the results of A. Suslin and M. Wodzicki on the excision property of algebraic K-groups on C^*-algebras [**33**], the right hand side of the second equation is well-defined. Algebraic K-groups obtained in this way are naturally isomorphic to Quillen's K-groups $\mathbb{K}_n^Q(A)$ when $n \geqslant 0$; and are isomorphic to the Pedersen-Weibel K-groups in negative dimensions. Note that a new interpretation of algebraic K-groups implies existence of a simple flexible technical tool. Namely, any element of an algebraic K-group of an additive category may be represented as an element of an algebraic K-group of the endomorphism algebra of an object, and such an interpretation is unique up to a manageable equivalence. Throughout the paper, this principle will be used repeatedly.

In section 2, according to the excision property of algebraic K-groups on the category of C^*-algebras [**33**], we establish the excision property for a short exact sequence associated to a C^*-ideal in an additive C^*-category (see Proposition 2.4). In section 3 this property is used to prove Theorem 3.1 about the excision property of functors

$$\{\mathbf{K}_n^a(\mathrm{Rep}(-; B))\}_{n \in Z}. \tag{1.5}$$

In addition to the excision property, proof of Theorem 3.1 uses two nontrivial results. These are theorem 3.5 and Theorem 3.8.

In section 4, the stability property of the functors 1.5 will be shown.

Now, since the family of Kasparov's functors $KK_n(-; B),\ n \in \mathbb{Z}$ is a stable cohomology theory [**7**], the proof of Theorem 1.2 boils down to showing the isomorphism

$$\mathbb{K}_0(\mathrm{Rep}(A, B)) \simeq KK_1(A, B),$$

which is done in section 5.

Remark 1.5. Discussions for topological K-groups are omitted, because they literally coincide with the considered case of algebraic K-groups.

Remark 1.6. The main result (with essential changes in definitions and theorems) is also true for a locally compact group G. We hope to discuss this case independently, not in this paper.

2. Some remarks on the algebraic K-theory of additive C^*-categories

We will use the Pedersen-Weibel interpretation of algebraic K-groups [**28**], denoted here by $\mathbb{K}_n^a$, instead of Quillen's definition of algebraic K-groups in [**30**] which was given through homotopy groups of certain space. In this section we review some properties of algebraic K-groups of idempotent-complete additive categories, based on Pedersen-Weibel's nonconnective spectra (in this context there are defined negative K-groups, too). Then, we reinterpret algebraic K-groups of idempotent-complete additive C^*-categories and, with the aid of results from [**33**], generalize them to C^*-ideals in additive C^*-categories. This material plays auxiliary role in this paper.

In the following lemma we list some simple properties of algebraic K-groups which suffice for our purposes.

Lemma 2.1. *Let A be an idempotent-complete small additive category. Then*

1. *if $A = A_1 \times A_2$, then $\mathbb{K}_n^a(A) = \mathbb{K}_n^a(A_1) \times \mathbb{K}_n^a(A_2)$, $n \in \mathbb{Z}$;*
2. *if $\{A_\alpha\}$ is a direct system of full additive subcategories in A such that $\varinjlim A_\alpha = A$. Then $\mathbb{K}_n^a(A) = \varinjlim \mathbb{K}_n^a(A_\alpha)$, $n \in \mathbb{Z}$.*

In the remaining part of this section we give an interpretation of the groups $\mathbb{K}_n^a(A)$, $n \in \mathbb{Z}$ for additive C^*-categories. This interpretation will convenient for our purposes.

Let a be an object in an idempotent-complete C^*-category A. Consider the full sub-C^*-category A_a in A consisting of all those objects a' of A which admit an isometry $s : a' \to a^{\oplus n}$. It is clear that A_a is an idempotent-complete C^*-category equivalent to the category $\mathcal{P}(\mathcal{L}(a))$ of f. g. projective modules over $\mathcal{L}(a)$.

Now, consider a direct system of abelian groups for A based on subcategories A_a. It is evident that if there exists an isometry $s : a' \to a$ then one has a natural additive inclusion $*$-functor (not depending on s) $i_{a'a} : A_{a'} \to A_a$ and thus we have the direct system $\{A_a, i_{a'a}\}_{(obA,\leqslant)}$ of idempotent-complete C^*-categories. Because of the continuity property of algebraic K-groups (property (2) in Lemma 2.1) and the isomorphism of categories $A = \varinjlim A_a$, one has an isomorphism

$$\mathbb{K}_n^a(A) = \varinjlim \mathbb{K}_n^a(A_a)$$

$n \in \mathbb{Z}$.

This suggests that $\mathbb{K}_n^a(A)$ can be interpreted in the form

$$\mathbb{K}_n^a(A) = \varinjlim \mathbb{K}_n^a(\mathcal{L}(a)). \tag{2.1}$$

Below it is done in detail.

2.1. Algebraic K-functors of C^*-ideals

Let us make some comments on the results of A. Suslin and M. Wodzicki in algebraic K-theory before we introduce our view on algebraic K-theory of C^*-ideals. One of their main results is, by Proposition 10.2 in [**33**], that C^*-algebras have the factorization property $(\mathbf{TF})_{\mathrm{right}}$. Thus any C^*-algebra possesses the property $\mathbf{AH_Z}$. These results have many useful consequences in algebraic K-theory of C^*-algebras, which are listed below. (Recall that if A is a C^*-algebra and A^+ is the C^*-algebra obtained by adjoining a unit to A, then

$$K_n^a(A) = \ker(K_n^a(A^+) \to K_n^a(k)), \tag{2.2}$$

$n \in \mathbb{Z}$).

1. K_i^a is a covariant functor from the category of C^*-algebras and $*$-homomorphisms to the category of abelian groups for any $i \in \mathbb{Z}$;
2. For every unital C^*-algebra R containing the C^* algebra A as a two-sided ideal, the canonical map $K_n^a(A) \to K_n^a(R, A)$ is an isomorphism;
3. The natural embedding into the upper left corner $A \hookrightarrow M_k(A)$ induces, for every natural n, an isomorphism $K_n^a(A) \simeq K_n^a(M_k(A))$;
4. Any extension of C^*-algebras

   $$0 \to I \to B \to A \to 0$$

   induces a functorial two-sided long exact sequence of algebraic K-groups

   $$\cdots \to K_{i+1}^a(A) \to K_i^a(I) \to K_i^a(B) \to K_i^a(A) \to \cdots \qquad (i \in \mathbb{Z}). \tag{2.3}$$
5. Let A be a C^*-algebra and let u be a unitary element in a unital C^*-algebra containing A as a closed two-sided ideal. Then the inner automorphism $ad(u) : A \to A$ induces the identity map of algebraic K-groups.

Below we define algebraic K-groups for C^*-ideals. These groups possess all properties similar to those represented above.

Let A be an additive C^*-category and let J be its closed C^*-ideal. Let $\mathcal{L}_A(a) = \hom_A(a, a)$ and $\mathcal{L}_A(a, J) = \mathcal{L}(a)_A \cap J$ for any object $a \in A$. The latter is a closed ideal in the C^*-algebra $\mathcal{L}_A(a)$. Let us write $a \leqslant a'$ if there is an isometry $v : a \to a'$, i.e $v^*v = 1_a$. The relation "$a \leqslant a$" makes the set of objects of A into a directed system. Any isometry $v : a \to a'$ defines a $*$-homomorphism of C^*-algebras

$$\mathrm{Ad}(v) : \mathcal{L}_A(a) \to \mathcal{L}_A(a')$$

by the rule $x \mapsto vxv^*$. It maps $\mathcal{L}_A(a, J)$ to $\mathcal{L}_A(a', J)$.

Let $v_1 : a \to a'$ and $v_2 : a \to a'$ be two isometries. Then $\mathrm{Ad}v_1$ and $\mathrm{Ad}v_2$ induce the same homomorphisms

$$\mathrm{Ad}_*v_1 = \mathrm{Ad}_*v_2 : K_n^a(\mathcal{L}_A(a)) \to K_n^a(\mathcal{L}_A(a'))$$

and

$$\mathrm{Ad}_*v_1 = \mathrm{Ad}_*v_2 : K_n^a(\mathcal{L}_A(a, J)) \to K_n^a(\mathcal{L}_A(a', J)).$$

The similar result for topological K-theory is in [**9**]. This means that the homomorphism $\nu_*^{aa'} = K_n^a(\nu^{aa'})$ is independent of choosing an isometry $\nu^{aa'} : a \to a'$.

Therefore one has a directed system $\{K_n^a(\mathcal{L}_A(a,J)),\nu_*^{aa'})\}_{a,a'\in obA}$ of abelian groups, for all $n\in\mathbb{Z}$.

Definition 2.2. Let A be an additive C^*-category and let J be its closed C^*-ideal. Define

$$\mathbf{K}_n^a(A,J)=\varinjlim K_n^a(\mathcal{L}_A(a,J)) \quad \text{and} \quad \mathbf{K}_n^a(J)=\mathbf{K}_n^a(M(J),J) \tag{2.4}$$

where $M(J)$ is the so called *multiplier* C^*-category of J [**18**].

Lemma 2.3. *Let J be a C^*-ideal in an additive C^*-category A. Then*

1. $\mathbf{K}_n^a(J)=\mathbf{K}_n^a(A,J)$*;*
2. *if A' is a cofinal subcategory in A then* $\mathbf{K}_n^a(A')\simeq\mathbf{K}_n^a(A)$*;*
3. *if A is idempotent-complete, then* $\mathbf{K}_n^a(A)\simeq\mathbb{K}_n^a(A)$.

Proof. 1. The natural $*$-functor $\rho: A\to M(J)$ induced from the identity on J by the universal property of $M(J)$ obviously preserves the relation "$\leqslant$". This implies that there is a natural morphism of directed systems

$$\{K_n^a(\mathcal{L}_A(a,J)),\nu_*^{aa'})\}_{a,a'\in \mathrm{ob}A}\xrightarrow{\{\rho_n^a\}}\{K_n^a(\mathcal{L}_{M(J)}(a,J)),\nu_*^{aa'})\}_{a,a'\in \mathrm{ob}M(J)},$$

where the homomorphism $\rho_n^a: K_n^a(\mathcal{L}_A(a,J))\to K_n^a(\mathcal{L}_{M(J)}(a,J))$ is induced by the above $*$-functor $\rho: A\to M(J)$. In view of the isomorphism $K_n(A)\to K_n(R,A)$ [**33**] one concludes that the homomorphism ρ_n^a is an isomorphism for all $n\in\mathbb{Z}$ $a\in A$. This morphism is cofinal, since if "$a\leqslant a'$" in $M(J)$ then "$a\leqslant a\oplus a'$" in A and "$a'\leqslant a\oplus a'$" in $M(J)$. Therefore, $\{\rho_n^a\}$ is an isomorphism of direct systems.

2. This is an easy consequence of Definition 2.2.

3. This results from comparison of Definition 2.2 and isomorphism 2.1. ☐

Now, we prove the excision property of algebraic K-theory which will be used in the next section.

Proposition 2.4. *Let A be an additive C^*-category and let J be a C^*-ideal in A. Then the two-sided sequence of algebraic K-groups*

$$...\to\mathbf{K}_{n+1}^a(A/J)\to\mathbf{K}_n^a(J)\to\mathbf{K}_n^a(A)\to\mathbf{K}_n^a(A/J)\to...,\tag{2.5}$$

$n\in\mathbb{Z}$, is exact.

Proof. Consider the exact sequence of C^*-algebras

$$0\to\mathcal{L}(a,J)\to\mathcal{L}(a,A)\to\mathcal{L}(a,A)/\mathcal{L}(a,J)\to 0.$$

By the excision property of algebraic K-theory on C^*-algebras [**33**], one has a two-sided long exact sequence of algebraic K-groups

$$\begin{aligned}\cdots\to K_n^a(\mathcal{L}(a,A)/\mathcal{L}(a,J))\to&\\ K_{n-1}^a(\mathcal{L}(a,J))\to K_{n-1}^a(\mathcal{L}(a,A))&\to K_{n-1}^a(\mathcal{L}(a,A)/\mathcal{L}(a,J))\to\cdots\end{aligned}\tag{2.6}$$

According to the exact sequence 2.6 and the fact that directed colimits preserve exactness, one obtains the following long exact sequence of abelian groups:

$$\cdots \to \varinjlim K^a_{n+1}(\mathcal{L}(a,B)/\mathcal{L}(a,J)) \to \\ \mathbf{K}^a_n(J) \to \mathbf{K}^a_n(B) \to \varinjlim K^a_n(\mathcal{L}(a,B)/\mathcal{L}(a,J)) \to \cdots \quad (2.7)$$

There is a natural morphism of directed systems

$$\{\omega_a\} : \{K^a_n(\mathcal{L}(a,A)/\mathcal{L}(a,J)\} \to \{K^a_n(\mathcal{L}(a,A/J)\}$$

so that ω_a is the identity map for any object a in A. It is clear that this morphism is cofinal. Thus the induced homomorphism

$$\omega : \varinjlim K^a_n(\mathcal{L}(a,A)/\mathcal{L}(a,J)) \to \varinjlim K^a_n(\mathcal{L}(a,A/J)) = \mathbf{K}^a_n(A/J)$$

is an isomorphism. □

3. Excision Property of $\mathbb{K}^a_n((Rep(-;B))$

In this section we will prove that the contravariant functors $\mathbb{K}^a_n((\mathrm{Rep}(-;B))$, $n \in \mathbb{Z}$, have the weak excision property. The similar result for topological K-theory, in a particular case, has been proved in [**9**].

Remark 3.1. From now on for convenience of calculations the functors $\mathbf{K}^a_n((Rep(-;B))$ are considered instead of $\mathbf{K}^a_n((\mathrm{Rep}(-;B))$ and $\mathbb{K}^a_n((\mathrm{Rep}(-;B))$. Since $Rep(A;B)$ is a cofinal full subcategory in $\mathrm{Rep}(A;B)$, by Lemma 2.3 (2),(3) these functors are isomorphic.

For any closed invariant ideal J in a separable C^*-algebra A from S_G there is a C^*-ideal $D(A,J;B)$ in $Rep(A;B)$ which is defined in the following manner. Let (E,ϕ) and (E',ϕ') be objects in $Rep(A,B)$. A morphism $\alpha : (E,\phi) \to (E',\phi')$ in $Rep(A,B)$ is in $D(A,J;B)$ if

$$\alpha\phi(x) \in \mathcal{K}((E,\phi),(E'\phi')) \ \text{ and } \ \phi'(x)\alpha \in \mathcal{K}((E,\phi)), \text{ for all } x \in A.$$

The space of all morphisms from (E,ϕ) to (E',ϕ') in the C^*-ideal $D(A,J;B)$ will be denoted by $D_{\phi,\phi'}(A,J;E,E';B)$ (if $(E',\phi') = (E,\phi)$ then it is also denoted by $D_\phi(A,J;E;B)$) (cf. [**9**]) .

Theorem 3.2. *Let B be a σ-unital C^*-algebra and let $0 \to I \to A \xrightarrow{p} A/I \to 0$ be a proper sequence of separable C^*-algebras in S_G. Then the sequence of groups*

$$... \to \mathbf{K}^a_n(Rep(A/J,B)) \to \mathbf{K}^a_n(Rep(A,B)) \to \mathbf{K}^a_n(Rep(J,B)) \xrightarrow{\partial} \\ \xrightarrow{\partial} \mathbf{K}^a_{n-1}(Rep(A,B)) \to ... \quad (3.1)$$

is exact, for all $n \in \mathbb{Z}$.

Proof. Consider the short exact sequence of C^*-categories and of a C^*-ideal

$$0 \to D(A,J;B) \to Rep(A,B) \to Rep(A,B)/D(A,J;B) \to 0.$$

According to Proposition 2.4, one has a two-sided long exact sequence

$$\begin{aligned} ... \to \mathbf{K}^a_n(D(A,J;B)) \to \mathbf{K}^a_n(Rep(A,B)) \to \\ \to \mathbf{K}^a_n(Rep(A,B)/D(A,J;B)) \xrightarrow{\partial} \mathbf{K}^a_{n-1}(D(A,J;B)) \to ... \end{aligned} \quad (3.2)$$

of abelian groups. According to Theorems 3.5 and 3.8, replacements

$$\mathbf{K}^a_n(Rep(A;B)/D(A,J;B)) \text{ by } \mathbf{K}^a_n(Rep(J;B))$$

and

$$\mathbf{K}^a_n(D(A,J;B) \text{ by } \mathbf{K}^a_n(Rep(A/J;B)),$$

ensure exactness of the two-sided long exact sequence (3.1). Thus it suffices to prove Theorems 3.5 and 3.8, which will be done in the next part of this section. □

3.1. On the Isomorphism $\mathbf{K}^a_n(Rep(A;B)/D(A,J;B)) \approx \mathbf{K}^a_n(Rep(J;B))$

Let (E,ϕ) be an object in $Rep(A,B)$ and let $j : J \to A$ be the natural equivariant inclusion. There is a $*$-functor induced by the natural inclusion j

$$\mathrm{j} : Rep(A;B) \to Rep(J;B) \quad (3.3)$$

defined by the assignments $(E,\phi) \mapsto (E,\phi j)$ and $x \mapsto x$.

The following trivial lemma is used in the next proposition.

Lemma 3.3. *Let A and B be additive C^*-categories and let $F : A \to B$ be an additive $*$-functor. Then F is a $*$-isomorphism if and only if F is bijective on objects and the induced $*$-homomorphisms of C^*-algebras $F_a : \mathcal{L}(a) \to \mathcal{L}(f(a))$ are $*$-isomorphisms for all objects a in A.*

The following proposition is a slight generalization of the similar result in [**9**]).

Proposition 3.4. *The canonical $*$-functor 3.3 maps $D(A,J;B)$ to $D(J,J;B)$ and the induced $*$-functor*

$$\xi : Rep(A;B)/D(A,J;B) \to Rep(J;B)/D(J,J;B) \quad (3.4)$$

is an isomorphism of C^*-categories.

Proof. (cf. [**9**]) By lemma 3.3 it suffices to show that for any object (E,ϕ) the $*$-homomorphism of C^*-algebras

$$\xi_{J,\phi} : D_\phi(A;E;B)/D_\phi(A,J,E;B) \to D_{\phi\cdot j}(J,E;B)/D_{\phi\cdot j}(J,J,E;B)$$

is a $*$-isomorphism. It is easy to show that $\xi_{J,\phi}$ is a monomorphism. To show that $\xi_{J,\phi}$ is an epimorphism, take $x \in D_{\phi\cdot j}(J,E;B)$ and let E_1 be the G-C^*-algebra in $\mathcal{L}(E)$ generated by $\phi(J) \cup \mathcal{K}(E)$; let E_2 be the separable G-C^*-algebra generated by all elements of the form $[x,\phi(y)]$, $y \in J$; and let $\mathcal{F}$ be the G-invariant separable linear space generated by x and $\phi(A)$. One has

- $E_1 \cdot E_2 \subset \mathcal{K}(E)$, because $\phi(b)[\phi(a),x] \sim [\phi(ba),x] \in \mathcal{K}(E)$, $a \in A$, $b \in J$,
- $[\mathcal{F},E_1] \subset E_1$, because $[x,\phi(J)] \subset \mathcal{K}(E)$ and $[\phi(A),\phi(J)] \subset \phi(J)$.

From a technical theorem by Kasparov it follows that there exists a positive G-invariant operator X such that

1. $X \cdot \phi(J) \subset \mathcal{K}(E)$;
2. $(1-X) \cdot [\phi(A), x] \subset \mathcal{K}(E)$;
3. $[x, X] \in \mathcal{K}(E)$.

Since $[(1-X)x, \phi(a)] = (1-X)[x, \phi(a)] - [X, \phi(a)]x$, it follows from (2) and (3) that $(1-X)x \in D_\phi(A, E; B)$. In addition, it follows from (2) that $Xx \in D_{\phi\cdot j}(J, J, E; B)$, and so that the image of $(1-X)x$ in $D_{\phi\cdot j}(J, E; B)/D_{\phi\cdot j}(J, J, E; B)$ coincides with the image of x. $\square$

Now, we prove the following

Theorem 3.5. *Let A be a separable C^*-algebra and let B be a σ-unital C^*-algebra. Let J be a closed ideal in A. There exists an essential isomorphism*

$$\mathbf{K}^a_n(Rep(A, B)/D(A, J; B)) \approx \mathbf{K}^a_n(Rep(J, B)) \tag{3.5}$$

Proof. According to Proposition 3.4, it suffices to show that the homomorphism

$$\mathbf{K}^a_*(Rep(J; B)) \to \mathbf{K}^a_*(Rep(J; B)/D(J, J; B))$$

is an isomorphism. The exact sequence

$$\begin{aligned}... \to \mathbf{K}^a_n(D(J, J; B)) \to \mathbf{K}^a_n(Rep(J, B)) \to \\ \to \mathbf{K}^a_n(Rep(J, B)/D(J, J; B)) \xrightarrow{\partial} \mathbf{K}^a_{n-1}(D(J, J; B)) \to ...\end{aligned} \tag{3.6}$$

shows that it suffices to show $\mathbf{K}^a_*(D(J, J; B)) = 0$. According to Kasparov's stabilization theorem, one concludes that the full subcategory $Rep_{H_B}(J; B)$ on all objects of the form (H_B, φ) is a cofinal subcategory in $Rep(J; B)$, where H_B is Kasparov's universal Hilbert B-module [**23**]. Note that the canonical isometry

$$i_1 : H_B \to H_B \oplus H_B$$

in the first summand is in $D_{\phi,\phi\oplus 0}(J; H_B, H_B \oplus H_B; B)$. So it induces inner homomorphism

$$\mathrm{ad}(i_1) : D_\phi(J, J; H_B; B) \to D_{\phi\oplus 0}(J, J; H_B \oplus H_B; B).$$

Consider the sequence of $*$-homomorphisms

$$D_\phi(J, J; H_B; B) \to D_{\phi\oplus\phi}(J, J; H_B \oplus H_B; B) \subset D_{\phi\oplus 0}(J, J; H_B \oplus H_B; B), \tag{3.7}$$

where the inclusion is given by $x \mapsto x$. If the first arrow is induced by the inclusion $\iota_1 : H_B \to H_B \oplus H_B$ into the first summand, then the composite is $\mathrm{ad}(i_1)$. If the first arrow is induced by the inclusion $\iota_2 : H_B \to H_B \oplus H_B$ into the second summand, one obtains a homomorphism λ. Since $K^a_n(\mathrm{ad}(\iota_1)) = K^a_n(\mathrm{ad}(\iota_2))$, one has $K^a_n(\mathrm{ad}(i_1)) = K^a_n(\lambda)$. On the other hand, the homomorphism λ is the composite of $*$-homomorphisms of C^*-algebras

$$D_\phi(J, J; H_B; B) \to D_0(J, J; H_B; B) \to D_{\phi\oplus 0}(J, J; H_B \oplus H_B; B),$$

defined by the assignments

$$x \mapsto x \text{ and } x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}.$$

Note that one has $D_0(J, J; H_B; B) \simeq M(J \otimes \mathcal{K}_G)$. It is a well-known fact that the latter algebra has trivial algebraic K-groups. If we apply K-functors then the homomorphism corresponding to λ will be zero. Now, as a consequence we have the following. Let $\alpha \in K_n^a(D_\phi(J, J; H_B; B))$ represent an element in $\mathbf{K}_n^a(D(J, J; B))$. Since

$$\alpha = K_n^a(\mathrm{ad}(i_1)(\alpha)) = K_n^a(\lambda(\alpha)) = 0,$$

one concludes that the class of α in $\mathbf{K}_*^a(D(J, J; B))$ is zero. Therefore $\mathbf{K}_*^a(D(J, J; B)) = 0$. □

3.2. On the Isomorphism $\mathbf{K}_n^a(Rep(A/J; B)) \simeq \mathbf{K}_n^a(D(A, J; B))$

Let

$$0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0$$

be a proper exact sequence and let $\sigma : A/J \to A$ be a completely positive and contractive (equivariant) section.

Let (E, ϕ) be an object in $Rep(A; B)$. A $*$-homomorphism

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} : A/J \to \mathcal{L}(E \oplus E'))$$

will be called a *σ-dilation* for ϕ if $\psi_{11}(a) = \phi(\sigma(a))$, where E' is a right Hilbert B-module. By generalized Stinespring's theorem there exists a σ-dilation for ϕ, where $\sigma : A/J \to A$ is a completely positive and contractive section [**23**].

Lemma 3.6. *Let ψ be a σ-dilation for ϕ. Then*

1. $\psi_{12}(a^*) = \psi_{21}(a)^*$*;*
2. *for any $a, b \in A/J$ there exists a $j \in J$ such that $\psi_{12}(a)\psi_{21}(b) = \phi(j)$;*
3. *$\psi_{12}(a)x$ and $x\psi_{21}(a)$ are compact morphisms for any $a \in A/J$ and $x \in D_\phi(A, J; B)$;*
4. *Let $\phi : A/J \to \mathcal{L}(E)$ be a $*$-homomorphism and let $\psi : A/J \to \mathcal{L}(E \oplus E')$ be a σ-dilation for ϕq. There exists a $*$-homomorphism $\varphi : A/J \to \mathcal{L}(E')$ such that*

$$\psi = \begin{pmatrix} \phi & 0 \\ 0 & \varphi \end{pmatrix}.$$

Proof. The case (1) is trivial, because ψ is a $*$-homomorphism.

The case (2). Since ψ is σ-dilation for ϕ, one has $\phi \cdot (\sigma(ab) - \sigma(a) \cdot \sigma(b)) = \psi_{12}(a) \cdot \psi_{21}(b)$. But $j = \sigma(ab) - \sigma(a) \cdot s(b) \in J$. Therefore $\psi_{12}(a) \cdot \psi_{21}(b) = \phi(j)$.

The case (3). If $x \in D_\phi(A, J; B)$ then, by definition, $x\phi(j)$ and $\phi(j)x$ are compact morphisms for any $j \in J$. According to cases (1) and (2), one has $x\psi_{12}(a) \cdot \psi_{12}^*(a)x^* = x\phi(j')x^*$ for some $j' \in J$. Thus $x\psi_{12}(a) \cdot \psi_{12}(a^*)x^*$ is a compact morphism. Therefore $x\psi_{12}(a)$ and $\psi_{21}(a)x$ $(= (x^*\psi_{12}(a^*))^*)$ are compact morphisms, too.

The case (4). Since ψ is a σ-dilation for ϕq,

$$\psi = \begin{pmatrix} \phi & \psi_{12} \\ \psi_{21} & \varphi \end{pmatrix}.$$

According to case (2), for any $a, b \in A/J$ there exists $j \in J$ such that $\psi_{12}(a)\psi_{21}(b) = \phi q(j) = 0$. Applying the case (1), one has $\psi_{12}(a)\psi_{12}^*(a) = 0$. Therefore $\psi_{12}(a) = 0$ and $\psi_{21}(a) = 0$ for all $a \in A/J$ and φ is a $*$-homomorphism. □

The following lemma is used in Theorem 3.8.

Lemma 3.7. *Let (E, ϕ) be an object in Rep(A, B) and let $\psi : A/J \to \mathcal{L}(E \oplus E')$ be a σ-dilation for ϕ. Then a map $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x' = \begin{pmatrix} x_{11} & 0 & x_{12} \\ 0 & 0 & 0 \\ x_{21} & 0 & x_{22} \end{pmatrix}$ defines a $*$-monomorphism*

$$\rho : M_2(D_\phi(A, J, E \oplus E; B)) \to D_{\psi \cdot q \oplus \phi}(A, J, E \oplus E' \oplus E; B). \tag{3.8}$$

Proof. It suffices to show that $x' \in D_{\psi \cdot q \oplus \phi}(A, J, E \oplus E' \oplus E; B)$. By assumption one has

$$(\phi(a) \oplus \phi(a))x - x(\phi(a) \oplus \phi(a)) \in \mathcal{K}(E \oplus E),$$

for any $a \in A$, and

$$(\phi(b) \oplus \phi(b))x \in \mathcal{K}(E \oplus E),$$

$x(\phi(b) \oplus \phi(b)) \in \mathcal{K}(E \oplus E)$ for any $b \in J$. It implies that

$$\phi(a)x_{mn} - x_{mn}\phi(a) \in \mathcal{K}(E), \quad \phi(b)x_{mn} \in \mathcal{K}(E), \quad x_{mn}\phi(b) \in \mathcal{K}(E),$$

$a \in A$ and $b \in J$. Then $(\psi(q(a)) \oplus \phi(a)) \cdot x' - x' \cdot (\psi(q(a)) \oplus \phi(a)) =$

$$= \begin{pmatrix} x_{11}\psi_{11}(q(a)) - \psi_{11}(q(a))x_{11} & x_{11}\psi_{12}(p(a)) & x_{12}\phi(a) - \psi_{11}(q(a))x_{12} \\ \psi_{21}(q(a))x_{11} & 0 & \psi_{21}(q(a))x_{12} \\ x_{21}\psi_{11}(q(a)) - \phi(a)x_{21} & x_{21}\psi_{12}(q(a)) & x_{22}\phi(a) - \phi(a)x_{22} \end{pmatrix}.$$

By Lemma 3.6 (3), the morphisms $\psi_{21}(q(a))x_{11}$, $x_{11}\psi_{12}(q(a))$, $x_{21}\psi_{12}(q(a))$ and $\psi_{21}(q(a))x_{12}$ are compact. Using the fact that $\phi(a) - \psi_{11}(q(a)) \in \phi(J)$, one has

$$(\psi(q(a)) \oplus \phi(a)) \cdot x' - x' \cdot (\psi(p(a)) \oplus \phi(a)) \in \mathcal{K}(E \oplus E' \oplus E), \quad a \in A.$$

To show that $(\psi(q(b)) \oplus \phi(b)) \cdot x'$ and $x' \cdot (\psi(q(b)) \oplus \phi(b))$ are in $\mathcal{K}(E \oplus E' \oplus E)$ when $b \in J$, note that $(\psi(q(b)) \oplus \phi(b)) \cdot x'$ and $x' \cdot (\psi(q(b)) \oplus \phi(b))$ are equal to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi(b)x_{21} & 0 & \phi(b)x_{22} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & x_{12}\phi(b) \\ 0 & 0 & 0 \\ 0 & 0 & x_{22}\phi(b) \end{pmatrix}$$

respectively. They are compact morphisms because each entry of matrices is a compact morphism. □

The category *Rep*(A, B) may be identified with the full C^*-subcategory $D^{(q)}(A, J; B)$ in $D(A, J; B)$, on the objects all pairs of the form $(E, \phi \cdot q)$, considering (E, ϕ) as an object in *Rep*$(A/J; B)$. Let

$$\varepsilon : D^{(q)}(A, J; B) \hookrightarrow D(A, J; B)$$

be the natural inclusion.

Theorem 3.8. *Let $0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0$ be a proper exact sequence of separable C^*-algebras. Then the induced homomorphism*

$$\Gamma_n : \mathbf{K}_n^a(Rep(A/J; B)) \to \mathbf{K}_n^a(D(A, J; B))$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. According to the discussion above, it suffices to show that ε induces an isomorphism

$$\varepsilon_n : \mathbf{K}_n^a(D^{(q)}(A, J; B)) \to \mathbf{K}_n^a(D(A, J; B)),$$

for all $n \in \mathbb{Z}$.

(1). *ε_n is a monomorphism.*

Let $(E, \phi \cdot q)$ be an object in $D^{(q)}(A, J; B)$ and suppose that the class of an element

$$y \in K_n^a(D_{\phi \cdot q}(A, J; E, B))$$

in $\mathbf{K}_n^a(D(A, J; B))$ is zero. This means that there exists an isometry

$$s : (E, \phi \cdot q) \to (E', \psi)$$

in $Rep(A; B)$ such that $\mathrm{Ad}(s)_n(y) = 0$ in $K_n^a(D_\psi(A, J; E', B))$.

According to Lemma 3.6, it is easy to show that

$$s' = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} : (E \oplus H_B, \eta \cdot q) \to (E' \oplus H_B, \eta' \cdot q)$$

is an isometry in $Rep(A/J, B)$, where η and η' are σ-dilations of $\phi \cdot q$ and ψ respectively. By Lemma 3.6 η has form

$$\eta = \begin{pmatrix} \phi & 0 \\ 0 & \chi \end{pmatrix},$$

where χ is a $*$-homomorphism from A to $\mathcal{L}(H_B)$. There is a homomorphism

$$\nu : D_\psi(A, J; E', B) \to D_{\eta' q}(A, J; E' \oplus \mathcal{H}_B, B),$$

defined by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, and an isometry

$$i_1 : (E, \phi) \to (E \oplus H_B, \eta = \begin{pmatrix} \phi & 0 \\ 0 & \chi \end{pmatrix})$$

(inclusion into the first summand). It is clear that $\mathrm{Ad}(s' i_1) = \nu \mathrm{Ad}(s)$. Therefore $\mathrm{Ad}(s' i_1)_n(y) = 0$ in $D_{\eta' q}(A, J; E' \oplus \mathcal{H}_B, B)$ and in $\mathbf{K}_n^a(D^{(q)}(A, J; B))$ too. This means that the class of y in $\mathbf{K}_n^a(D^{(q)}(A, J; B))$ is zero.

(2). *ε_n is an epimorphism.*

Let an element in $\mathbf{K}_n^a(D(A, J; B))$ be represented by an element $x \in K_n^a(D_\phi(A, J; E, B))$. Consider the $*$-homomorphism

$$\theta : D_\phi(A, J; E; B) \to D_{\psi q}(A, J; E \oplus H_B, B)$$

given by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

where ψ is a σ-dilation of ϕ.

Let us show that classes of elements $\theta_n(x)$ and x in $\mathbf{K}_n^a(D(A,J;B))$ coincide. Let ϑ be the composite

$$D_\phi(A,J;E;B) \xrightarrow{\theta} D_{\psi q}(A,J;E \oplus H_B;B) \xrightarrow{\mathrm{Ad}(i_{E\oplus H_B})} D_{\psi q\oplus\phi}(A,J;E\oplus H_B \oplus E;B),$$

where the second arrow is induced by the isometry into the first two summands. It is clear that the $*$-homomorphism ϑ coincides with the $*$-homomorphism defined by

$$x \mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the other hand this homomorphism may be interpreted as the composite

$$D_\phi(A,J;E,B) \xrightarrow{i_1} M_2(D_\phi(A,J;E,B)) \xrightarrow{\rho} D_{\psi\cdot q\oplus\phi}(A,J;E\oplus H_B\oplus E,B)$$

where i_1 is given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and ρ is defined by

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x' = \begin{pmatrix} x_{11} & 0 & x_{12} \\ 0 & 0 & 0 \\ x_{21} & 0 & x_{22} \end{pmatrix},$$

as in Lemma 3.7. Consider another homomorphism η—the composite of the sequence of $*$-homomorphisms

$$D_\phi(A,J;E,B) \xrightarrow{i_2} M_2(D_\phi(A,J;E,B)) \xrightarrow{\rho} D_{\psi\cdot q\oplus\phi}(A,J;E\oplus H_B\oplus E,B)$$

where i_2 is given by $x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$. Since $(i_1)_n = (i_2)_n$, $n \in \mathbb{Z}$, one has $(\vartheta)_n = (\eta)_n$. But $(\eta)_n = (\mathrm{ad}(i_3))_n$, $n \in \mathbb{Z}$, where $i_3 : E \to E \oplus H_B \oplus E$ is an isometry in the third summand. Therefore, classes of the elements $\vartheta_n(x)$, $(\mathrm{ad}(i_3))_n(x)$, x and $\theta_n(x)$ in $\mathbf{K}_n^a(D(A,J;B))$ coincide. Therefore the class of the element $\theta_n(x)$ in $\mathbf{K}_n^a(D^{(q)}(A,J;B))$ is the desired element. □

4. Stability Property of $\mathbb{K}_n^a(\mathrm{Rep}(-;B))$

Everywhere below in this section, $\mathcal{K}$ is the C^*-algebra of compact operators on a countable generated Hilbert space $\mathcal{H}$ considered as an object of S_G via trivial action of the compact group G.

Let $p \in \mathcal{K}$ be a rank one projection and let A be a C^*-algebra in S_G; let $e_A : A \to A \otimes \mathcal{K}$ be the $*$-homomorphism defined by $a \mapsto a \otimes p$, $a \in A$. Then one has the induced functor

$$e_A^* : Rep(A \otimes \mathcal{K}; B) \to Rep(A;B), \tag{4.1}$$

defined by assignments $(E, \varphi) \mapsto (E, \varphi e_A)$ (on objects) and $x \mapsto x$ (on morphisms).

There is a $*$-functor

$$\varepsilon^A : Rep(A; B) \to Rep(A \otimes \mathcal{K}; B)$$

defined by assignments $(E, \phi) \mapsto (E \otimes_k \mathcal{H}, \phi \otimes \mathrm{id}_{\mathcal{K}})$ (on objects) and $f \mapsto f \otimes \mathrm{id}_{\mathcal{H}}$ (on morphisms). Indeed, let (E, ϕ) be an object in $Rep(A; B)$. One has the induced $*$-homomorphism

$$\phi \otimes \mathrm{id}_{\mathcal{K}} : A \otimes \mathcal{K} \to \mathcal{L}(E \otimes_k \mathcal{H}).$$

Let $f : (E, \phi) \to (E', \phi')$ be a morphism in $Rep(A; B)$, i. e. $f\phi(a) - \phi'(a)f \in \mathcal{K}(E, E')$, $a \in A$. Then

$$(f \otimes \mathrm{id}_{\mathcal{H}})((\phi \otimes \mathrm{id}_{\mathcal{K}})(a \otimes \kappa)) - ((\phi \otimes \mathrm{id}_{\mathcal{K}})(a \otimes \kappa))(f \otimes \mathrm{id}_{\mathcal{H}}) = \\ = (f\phi(a) - \phi(a)f) \otimes \kappa \in \mathcal{K}(E \otimes_k \mathcal{H}, E' \otimes_k \mathcal{H}) \quad (4.2)$$

for all $a \in A$, $\kappa \in \mathcal{K}$.

Now, in view of Remark 3.1 stability property of $\mathbb{K}_n^a(\mathrm{Rep}(-; B))$ may be formulated as follows.

Theorem 4.1. *For any rank one projection $p \in \mathcal{K}$ and any C^*-algebra A in S_G the homomorphism*

$$e_n^A = \mathbf{K}_n^a(e_A^*) : \mathbf{K}_n^a(Rep(A \otimes \mathcal{K}; B)) \to \mathbf{K}_n^a(Rep(A; B)),$$

induced by the functor 4.1 is an isomorphism.

Proof. Let $\varepsilon_n^A : \mathbf{K}_n^a(Rep(A; B)) \to \mathbf{K}_n^a(Rep(A \otimes \mathcal{K}; B))$ be the homomorphism induced by the functor ε^A. It is easy to verify that the family $\{\varepsilon_n^A\}$ is a natural transformation from the functor $\mathbf{K}_n^a(Rep(-; B))$ to $\mathbf{K}_n^a(Rep(- \otimes \mathcal{K}; B))$. Therefore, the following diagram

$$\begin{tikzcd} \mathbf{K}_n^a(Rep(A \otimes \mathcal{K}; B)) \arrow[r, "e_n^A"] \arrow[d, "\varepsilon_n^{A \otimes \mathcal{K}}"'] & \mathbf{K}_n^a(Rep(A; B)) \arrow[d, "\varepsilon_n^A"] \\ \mathbf{K}_n^a(Rep(A \otimes \mathcal{K} \otimes \mathcal{K}; B)) \arrow[r, "e_n^{A \otimes \mathcal{K}}"'] & \mathbf{K}_n^a(Rep(A \otimes \mathcal{K}; B)) \end{tikzcd}$$

commutes, and it shows that for our purposes it suffices to verify that

$$e_n^A \varepsilon_n^A = \mathrm{id}_{\mathbf{K}_n^a(Rep(A;B))}, \quad (4.3)$$

for all $A \in S_G$. Indeed, first note that

1. The equality 4.3 shows that e_n^A is an epimorphism.

2. Since $\mathbf{K}_n^a(Rep(- \otimes \mathcal{K}; B))$ is a stable functor, according to the identity 4.3, one easily shows that $\varepsilon_n^{A \otimes \mathcal{K}}$ and $e_n^{A \otimes \mathcal{K}}$ are isomorphisms (cf. Proposition 10.6 in [**33**]). Therefore e_n^A is a monomorphism.

Checking the equality 4.3. We construct a useful isometry $\sigma_E : E \to E \otimes_k \mathcal{H}$, for any countably generated B-module E, which is a morphism from (E, ϕ) to $(E \otimes_k \mathcal{H}, (\phi \otimes \mathrm{id}_{\mathcal{K}})e_A)$.

Choose $y \in \mathcal{H}$ so that $p(y) = y$ and $||y|| = 1$ and consider a B-homomorphism σ_E given by $x \mapsto x \otimes y$. For any $z \in \mathcal{H}$ there exists $\lambda_z \in k$ determined uniquely by the equation $p(z) = \lambda_z y$. Define σ_E^* by the map $x \otimes z \mapsto \lambda_z x$. The B-homomorphism σ_E^* is adjoint to σ_E. Since $\sigma_E^* \sigma_E(x) = \sigma_E^*(x \otimes y) = x$, one concludes that σ_E is an isometry. Since $\sigma_E \phi(a) = ((\phi \otimes \mathrm{id}_{\mathcal{K}}) e_A(a))\sigma_E$, the isometry σ_E is a morphism from (E, ϕ) into $(E \otimes_k \mathcal{H}, (\phi \otimes \mathrm{id}_{\mathcal{K}}) e_A)$.

Consider restriction of $e_A^* \varepsilon$ to $D_\phi(A; E; B)$. We have a $*$-homomorphism

$$(e_A^* \varepsilon)_E : D_\phi(A; E; B) \to D_{(\phi \otimes \mathrm{id}) e_A}(A; E \otimes_k \mathcal{H}; B) \tag{4.4}$$

mapping $x \in D_\phi(A; E; B)$ to $x \otimes \mathrm{id}_{\mathcal{H}} \in D_{(\phi \otimes \mathrm{id}) e_A}(A; E \otimes_k \mathcal{H}; B)$. But

$$(\sigma_E x)(z) = (\sigma_E)(x(z)) = x(z) \otimes y = ((x \otimes p)\sigma_E)(z) \text{ for any } x \in D_\phi(A; E; B), z \in E.$$

Since $\sigma_E \sigma_E^* = \mathrm{id}_E \otimes p$, one concludes that $\sigma_E x \sigma_E^* = x \otimes p$. Therefore

$$(e_A^* \varepsilon)_E(x) = \sigma_E x \sigma_E^* + x \otimes (1 - p). \tag{4.5}$$

Let $\psi(x) = (e_A^* \varepsilon)_E(x)$, $\psi_0(x) = x \otimes p$ and $\psi_1(x) = x \otimes (1 - p)$. Then ψ_0 and ψ_1 are $*$-homomorphisms, $\psi = \psi_0 + \psi_1$ and

$$\psi_0(x)\psi_1(x) = \psi_1(x)\psi_0(x) = 0. \tag{4.6}$$

Here we show that $e_A^* \varepsilon$ induces the identity homomorphism of the group $\mathbf{K}_n^a(Rep(A, B))$ onto itself. Indeed, choose an element $r \in \mathbf{K}_n^a(Rep(A, B))$. By definition of $\mathbf{K}_n^a$-groups the element r is represented by an element $r_\phi \in K_n^a(D_\phi(A; E; B))$. Then the element $\mathbf{K}_n^a(e_A^* \varepsilon)(r)$ is represented by the element

$$K_n^a((e_A^* \varepsilon)_E)(r_\phi) = K_n^a(\psi_0 + \psi_1)(r_\phi). \tag{4.7}$$

Since K_n^a is an additive functor, according to 4.6 and Lemma 2.1.18 in ([**11**]), it follows that

$$K_n^a(\psi_0 + \psi_1)(r_\phi) = K_n^a(\psi_0)(r_\phi) + K_n^a(\psi_1)(r_\phi).$$

Since $\psi_0 = ad(s_E)$, the class of $K_n^a(\psi_0)(r_\phi)$ is equal to the class of r_ϕ. Thus the proof will be completed if we show that the class of $K_n^a(\psi_1)(r_\phi)$ in $\mathbf{K}_n^a(Rep(A; B))$ is zero. Indeed, let

$$s : E \otimes \mathcal{H} \to (E \otimes \mathcal{H}) \oplus \mathcal{H}_B)$$

be the isometry defined by $e \otimes h \mapsto e \otimes h \oplus 0$. Then the $*$-homomorphism $ad(s)\psi_1$ may be factored through the $*$-homomorphism

$$D_\phi(A; E; B) \to D_0(A; E \otimes (1 - p)\mathcal{H} \oplus \mathcal{H}_B; B).$$

But, according to Kasparov's stabilization theorem, one has

$$D_0(A; E \otimes (1 - p)\mathcal{H} \oplus \mathcal{H}_B; B) \approx \mathcal{L}(\mathcal{H}_B).$$

Since $K_n^a(\mathcal{L}(\mathcal{H}_B)) = 0$, one concludes that classes of $K_n^a(\psi_1)(z)$ and $K_n^a(ad(s)(\psi_1))(z)$ are equal to zero in $\mathbf{K}_n^a(Rep(A; B))$. Thus the homomorphism $\mathbf{K}_n^a(e_A \varepsilon)$ is the identity. $\square$

5. On the Isomorphism $\mathbf{K}_0^a(\mathrm{Rep}(-;B)) \simeq KK_{-1}(-;B)$

First we recall the definition of Kasparov's group $K^1(A,B)$, which will be denoted by $KK_{-1}(A,B)$, where A and B are trivially graded C^*-algebras with actions of a second countable compact group.

Consider a triple $(\varphi, E; p)$, where E is a trivially graded countably generated right B-module, $\varphi : A \to \mathcal{L}_B(E)$ is a $*$-homomorphism and $p \in \mathcal{L}_B(E)$ is an invariant element so that

$$\begin{gathered} p\varphi(a) - \varphi(a)p \in \mathcal{K}_B(E), \\ (p^* - p)\varphi(a) \in \mathcal{K}_B(E), \quad (p^2 - p)\varphi(a) \in \mathcal{K}_B(E) \end{gathered} \tag{5.1}$$

for all $a \in A$. Such a triple will be called a *Kasparov-Fredholm A, B- module*. If all left parts in 5.1 are zero, then such a triple is said to be *degenerate*.

Define the sum of Kasparov-Fredholm A, B-modules by the formula

$$(\varphi, E; p) \oplus (\varphi', E'; p') = (\varphi \oplus \varphi', E \oplus E'; p \oplus p').$$

Consider the equivalence relations:

- (*Unitary isomorphism*) A, B-modules $(\varphi, E; p)$ and $(\varphi', E'; p')$ will be said to be unitarily isomorphic if there exists a unitary isomorphism $u : E \to E'$ such that
$$u\varphi(a)u^* = \varphi'(a), \quad upu^* = p'$$
for all $a \in A$.
- (*Homology*) A, B-modules $(\varphi, E; p)$ and $(\varphi', E; p')$ will be said to be homologous if
$$p'\varphi'(a) - p\varphi(a) \in \mathcal{K}_B(E)$$
for all $a \in A$.

Simple checking shows that the equivalence relations defined above are well behaved with respect to sum.

Let $\mathcal{E}^1(A,B)$ be the abelian monoid of classes of A, B-modules with respect to the equivalence relation generated by the unitary isomorphism and homology. Denote by $\mathcal{D}^1(A,B)$ the submonoid of $\mathcal{E}^1(A,B)$ consisting of only those classes which are classes of all degenerate triples. By definition

$$E^1(A,B) = \mathcal{E}^1(A,B)/\mathcal{D}^1(A,B).$$

Using the Kasparov stabilization theorem, one easily shows that the definition of $E^1(A,B)$ coincides with Kasparov's original definition of $E^1(A,B)$ which is isomorphic to $KK_{-1}(A,B)$ by lemma 2 of section 7 of [**24**].

Recall that objects of $\mathrm{Rep}(A,B)$, by definition, have form $(\varphi, E; p)$, where $p : (\varphi, E) \to (\varphi, E)$ is a projection in the category $Rep(A;B)$. More precisely,

$$\varphi(a)p - p\varphi(a) \in \mathcal{K}_B(E), \quad p^* = p, \quad p^2 = p.$$

A unitary isomorphism $s : (\varphi, E; p) \to (\psi, E', q)$ in $\mathrm{Rep}(A,B)$ is a usual partial isometry $s : E \to E'$ such that

$$s\varphi(a) - \psi(a)s \in \mathcal{K}_B(E, E'), \quad s^*s = p, \quad ss^* = q.$$

Let $\tilde{\mathcal{E}}^1(A,B)$ be the abelian monoid of unitary isomorphism classes of objects in Rep(A; B). According to Lemma 1.1, one easily checks that the Grothendieck group of $\tilde{\mathcal{E}}^1(A,B)$ may be identified with $K_0(\mathrm{Rep}(A,B))$ (cf. [**16**]).

There is a natural homomorphism

$$\lambda_1 : K_0(\mathrm{Rep}(A,B)) \to E^1(A,B),$$

defined by the map $(\varphi, E; p) \mapsto (\varphi, E; p)$. Indeed, let $s : (\varphi, E; p) \to (\varphi', E'; p')$ be a unitary isomorphism in Rep(A; B). Consider the isomorphism in $Rep(A;B)$

$$\bar{s} : (\varphi \oplus \psi, E \oplus E') \to (\psi \oplus \varphi, E' \oplus E),$$

where

$$\bar{s} = \begin{pmatrix} s & 1 - ss^* \\ 1 - s^*s & s \end{pmatrix}.$$

It is clear that $(\varphi \oplus \psi, E \oplus E', \bar{p})$ is isomorphic to $(s(\varphi \oplus \psi)s^*, E' \oplus E, \bar{q})$, which is homologous to $(\psi \oplus \varphi, E' \oplus E, \bar{q})$, with

$$\bar{p} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{p}' = \begin{pmatrix} 1 & 0 \\ 0 & p' \end{pmatrix}.$$

This means that classes of $(\varphi, E; p)$ and $(\varphi', E'; p')$ coincide in $E^1(A,B)$.

Let $\mathrm{Rep}_J(A,B) = \mathrm{Rep}(A,B)/\mathrm{D}(A,J;B)$ be the idempotent-complete C^*-category universally obtained from the category $Rep(A,B)/D(A,J;B))$ (see section 1). Let $(\varphi; E; p)$ be a Kasparov-Fredholm A,B-module. Then p defines a projector $\dot{p}$ in the category $Rep_A(A,B)$. Thus the triple $(\varphi; E; \dot{p})$ is an object in $\mathrm{Rep}_A(A,B)$.

There is a well-defined homomorphism

$$\mu : E^1(A,B) \to K_0(\mathrm{Rep}_A(A,B))$$

defined by $(\varphi; E; p) \mapsto (\varphi; E; \dot{p})$. This is checked below.

We recall definition of operatorial homotopy:

- (*Operatorial homotopy*) An A,B-module $(\varphi, E; p)$ is operatorially homotopic to $(\varphi, E; p')$ if there exists a continuous map $p_t : [0;1] \to \mathcal{L}_B(E)$ such that $(\varphi, E; p_t)$ is an A,B-module for any $t \in [0;1]$.

If $(\varphi, E; p)$ is homologous to $(\psi, E; q)$, then $(\varphi, E; p) \oplus (\psi, E; 0)$ is operatorially homotopic to $(\varphi, E; 0) \oplus (\psi, E; q)$. Indeed, the desired homotopy is defined by the formula

$$\left(\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}, E \oplus E, \frac{1}{1+t^2} \begin{pmatrix} p & tpq \\ tqp & t^2 q \end{pmatrix} \right), \quad t \in [0;\infty]$$

(cf. section 7 in [**24**]). Thus the projections $p \dot{\oplus} 0$ and $0 \dot{\oplus} q$ are homotopic. Then, using Lemma 4 from section 6 in [**24**], one concludes that the objects $(\varphi, E; \dot{p}) \oplus (\psi, E; \dot{0})$ and $(\varphi, E; \dot{0}) \oplus (\psi, E; \dot{q})$ are unitarily isomorphic objects in $\mathrm{Rep}_A(A,B)$. Let $(\varphi, E; p)$ be unitarily isomorphic to $(\psi, E; q)$. Then $(\varphi, E; \dot{p})$ is isomorphic to $(\psi, E; \dot{q})$ in the category $\mathrm{Rep}_A(A,B)$. Therefore μ is well-defined.

We are now ready to prove the following theorem.

Theorem 5.1. *The natural homomorphism*

$$\lambda_1 : K_0(\mathrm{Rep}(A, B)) \to E^1(A, B) \simeq KK_{-1}(A, B)$$

is an isomorphism.

Proof. The homomorphism λ_1 is an epimorphism. Indeed, let $(\varphi, E; p)$ be a Kasparov-Fredholm A, B-module. Applying techniques of the Lemmata 17.4.2-17.4.3 in [**2**], one can suppose that $p^* = p$ and $||p|| \leqslant 1$. Then it is equivalent to $(\varphi \oplus 0, E \oplus E; p')$, where

$$p' = \begin{pmatrix} p & \sqrt{p-p^2} \\ \sqrt{p-p^2} & 1-p \end{pmatrix}.$$

Simple checking shows that p' is a projection and $(\varphi \oplus 0, E \oplus E; p')$ is an object in $\mathrm{Rep}(A, B)$. To show that λ_1 is a monomorphism, consider the commutative diagram

$$\begin{tikzcd} K_0(\mathrm{Rep}(A,B)) \arrow[r, "\lambda_1"] \arrow[d, "\|"'] & E^1(A,B) \arrow[d, "\mu"] \\ K_0(\mathrm{Rep}(A,B)) \arrow[r, "\xi"] & K_0(\mathrm{Rep}_A(A,B)). \end{tikzcd}$$

By Theorem 3.5, ξ is an isomorphism. Therefore λ_1 is a monomorphism. ☐

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