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MORE ON FIVE COMMUTATOR IDENTITIES

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(*communicated by Nick Inassaridze*)

Abstract

We prove that five well-known identities universally satisfied by commutators in a group generate all universal commutator identities for commutators of weight 4.

Introduction

For elements x, y of a group we write ${}^x y = xyx^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$. The following commutator identities are universal in the sense that they hold for any elements x, y, z of an arbitrary group:

$$[x, x] = 1,$$

$$[x, yz] = [x, y]\, {}^y[x, z],$$

$$[xy, z] = {}^x[y, z][x, z],$$

$$[[y, x], {}^x z][[x, z], {}^z y][[z, y], {}^y x] = 1,$$

$${}^z[x, y] = [{}^z x, {}^z y].$$

In [**4**] Ellis conjectured that, for any n, these universal relations applied to commutators of weight n generate all universal relations between commutators of weight n. This conjecture is stronger than Miller's result [**10**], who proved that any universal relation among commutators is deduced from four given ones without considering weights. Ellis considers his conjecture as a nonabelian version of the Magnus-Witt theorem (see [**9**] and [**11**]). To make his conjecture precise Ellis introduced the structure of "multiplicative Lie algebra". Then using the methods of homological algebra, he proved his conjecture for $n = 2$ and $n = 3$.

This paper proves Ellis' conjecture for n=4 using essentially the same tools.

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1. Multiplicative Lie algebras

This section is devoted to the formulation of Ellis' conjecture, which he calls a nonabelian version of the Magnus-Witt theorem. We first recall the notion of a multiplicative Lie algebra due to Ellis [**4**].

(1.1) Definition. *A multiplicative Lie algebra consists of a multiplicative (possibly nonabelian) group L together with a binary function $\{\ ,\ \} : L \times L \to L$, which we shall call Lie product, satisfying the following identities for all x, x', y, y', z in L*

$$\{x, x\} = 1\ , \tag{1.2}$$
$$\{x, yy'\} = \{x, y\}\ {}^{y}\{x, y'\}\ , \tag{1.3}$$
$$\{xx', y\} = {}^{x}\{x', y\}\{x, y\}\ , \tag{1.4}$$
$$\{\{y, x\}, {}^{x}z\}\{\{x, z\}, {}^{z}y\}\{\{z, y\}, {}^{y}x\} = 1\ , \tag{1.5}$$
$${}^{z}\{x, y\} = \{{}^{z}x, {}^{z}y\}\ . \tag{1.6}$$

In [**4**] the following identities are deduced from (1.2)-(1.6):

$$\{1, x\} = \{x, 1\} = 1\ , \tag{1.7}$$
$$\{y, x\} = \{x, y\}^{-1}\ , \tag{1.8}$$
$${}^{\{x,y\}}\{x', y'\} = {}^{[x,y]}\{x', y'\}\ , \tag{1.9}$$
$$\{[x, y], x'\} = [\{x, y\}, x']\ , \tag{1.10}$$
$$\{x^{-1}, y\} = {}^{x^{-1}}\{x, y\}^{-1} \quad \text{and} \quad \{x, y^{-1}\} = {}^{y^{-1}}\{x, y\}^{-1} \tag{1.11}$$

for all $x, x', y, y' \in L$. Important examples of multiplicative Lie algebras required for us are

(1.12) Example. *Any group P is a multiplicative Lie algebra with $\{x, y\} = xyx^{-1}y^{-1}$ for all $x, y \in P$.*

(1.13) Example. *For any group P there exists the free multiplicative Lie algebra $\mathcal{L}(P)$ on P which is characterized (up to isomorphism) by the following two properties: P is a subgroup of $\mathcal{L}(P)$; and any group homomorphism $P \to L$ from P to a multiplicative Lie algebra L extends uniquely to a morphism of multiplicative Lie algebras $\mathcal{L}(P) \to L$.*

The free multiplicative Lie algebra functor $\mathcal{L}$ is the left adjoint of the forgetful functor from Multiplicative Lie Algebras to Groups. The construction of $\mathcal{L}$ is given in [**4**] and more precisely in [**1**].

Let P be a group and $\Gamma_n(P)$ be the subgroup of $\mathcal{L}(P)$ generated by the elements $\{\{\ldots\{\{x_1, x_2\}, x_3\}, \ldots\}, x_n\}$ for $x_i \in P$. In particular $\Gamma_1(P) = P$. Then the group identity morphism on P induces a surjective morphism of multiplicative Lie algebras

$$\theta : \mathcal{L}(P) \twoheadrightarrow P$$

in which P has the structure of (1.12), and which restricts to surjective group homomorphisms

$$\theta_n : \Gamma_n(P) \twoheadrightarrow \gamma_n(P)$$

for all $n \geqslant 1$, where $\gamma_1(P) = P,\ \gamma_n(P) = [\gamma_{n-1}(P), P]$ is the lower central series of P. Now we can exactly formulate the Ellis' conjecture.

Conjecture. *If P is a free group, then θ_n are isomorphisms for all $n \geqslant 1$.*

As we had already mentioned, the above conjecture was proved in [**4**] for $n = 2$ and 3. The next section is devoted to the proof for $n = 4$.

2. Ellis conjecture for commutators of weight 4

We begin by recalling the notion of the nonabelian tensor product introduced by Brown and Loday [**3**] for a pair of groups $G,\ H$ which act on themselves by conjugation and each of which acts on the other compatibility, i.e.,

$$^{(^gh)}g' = {}^{ghg^{-1}}g', \quad ^{(^hg)}h' = {}^{hgh^{-1}}h'$$

where $g, g' \in G,\ h, h' \in H$, and $ghg^{-1},\ hgh^{-1}$ are elements of the free product $G * H$. The nonabelian tensor product $G \otimes H$ is the group generated by the symbols $g \otimes h$ subject to the relations

$$gg' \otimes h = ({}^gg' \otimes {}^gh)(g \otimes h)$$

$$g \otimes hh' = (g \otimes h)({}^hg \otimes {}^hh')$$

for all $g, g' \in G$ and $h, h' \in H$.

We will use the additive notations each time $G \otimes H$ is abelian.

In the sequel, unless specified, the tensor product of groups $G \otimes H$ belongs to three kinds for which the compatibility conditions hold:
(1) G is a normal subgroup of H and actions are given by conjugations;
(2) G is an abelian quotient of some normal subgroup of H, the action of H on G is induced by conjugation and the action of G on H is trivial;
(3) $H = P_{ab}$ and G is a quotient of $[P, P]_{ab}$, for some group P, the action of H on G is induced by conjugation and the action of G on H is trivial.

Let P be a group. Define $[P, P]_{ab} \otimes P_{ab}$ according to (3). As $[P, P]_{ab}$ is a P_{ab}-module, [**6**, Proposition 3.2] says that $[P, P]_{ab} \otimes P_{ab}$ is isomorphic to $[P, P]_{ab} \otimes_{P_{ab}} IP_{ab}$, where IP_{ab} denotes the augmentation ideal of P_{ab}. Hence $[P, P]_{ab} \otimes P_{ab}$ is abelian.

(2.1) Lemma. *Let P be a group. Then we have the following equalities in $[P, P]_{ab} \otimes P_{ab}$:*

$$([x, y][x', y']) \otimes z = [x, y] \otimes z + [x', y'] \otimes z\ , \tag{2.2}$$

$$[[a, b], y] \otimes x + [x, [a, b]] \otimes y = 0\ , \tag{2.3}$$

$$[{}^pz, x] \otimes y + [y, {}^pz] \otimes x - [z, x] \otimes y - [y, z] \otimes x = 0\ . \tag{2.4}$$

for any $a, b, x, y, z, p \in P$.

Proof. We only prove the second and third equalities. In fact,

$$
\begin{aligned}
&[[a,b],y]\otimes x+[x,[a,b]]\otimes y=[a,b]\,{}^{y}[b,a]\otimes x+{}^{x}[a,b][b,a]\otimes y\\
&=[a,b]\otimes x+{}^{y}[b,a]\otimes x+{}^{x}[a,b]\otimes y+[b,a]\otimes y\\
&=[a,b]\otimes x+{}^{x}[a,b]\otimes {}^{x}y+[b,a]\otimes y+{}^{y}[b,a]\otimes {}^{y}x\\
&=[a,b]\otimes xy+[b,a]\otimes yx=[a,b]\otimes xy+[b,a]\otimes xy=0\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&[{}^{p}z,x]\otimes y+[y,{}^{p}z]\otimes x-[z,x]\otimes y-[y,z]\otimes x\\
&=[[p,z]z,x]\otimes y+[y,[p,z]z]\otimes x-[z,x]\otimes y-[y,z]\otimes x\\
&=([z,x][[p,z],x])\otimes y+([y,[p,z]][y,z])\otimes x-[z,x]\otimes y-[y,z]\otimes x\\
&=[z,x]\otimes y+[[p,z],x]\otimes y+[y,[p,z]]\otimes x+[y,z]\otimes x-[z,x]\otimes y-[y,z]\otimes x\\
&=[[p,z],x]\otimes y+[y,[p,z]]\otimes x=0\ ,\text{by (2.3).}
\end{aligned}
$$

□

Let P be a group. P and $P\otimes P$ are P-crossed modules and they act on each other via their images in the basis P, i.e.,

$$
{}^{z}(x\otimes y)={}^{z}x\otimes{}^{z}y,\quad {}^{x\otimes y}z={}^{[x,y]}z\ .
$$

Thus, in the next lemma, $(P\otimes P,P)$ is a pair equipped with compatible actions and we can define the nonabelian tensor product $(P\otimes P)\otimes P$. In order to describe $(P\otimes P)\otimes P$ more precisely, assume that F is the free group generated by symbols $x\otimes y$, $x,y\in P$. Then, $(P\otimes P)\otimes P$ will be the group generated by symbols $f\otimes z$, $f\in F$, $z\in P$, subject to the following relations

$$
\begin{gathered}
(ff'\otimes z)(f\otimes z)^{-1}({}^{f}f'\otimes{}^{f}z)^{-1}=1,\\
(f\otimes zz')({}^{z}f\otimes{}^{z}z')^{-1}(f\otimes z)^{-1}=1,\\
\overline{f}\otimes z=1,\overline{f}\in\overline{F},
\end{gathered}
$$

where $f,f'\in F$, $z,z'\in P$, f acts on z via its image in $P\otimes P$ and $\overline{F}$ is the normal subgroup of F generated by the following elements

$$
\begin{gathered}
(xx'\otimes y)(x\otimes y)^{-1}({}^{x}x'\otimes{}^{x}y)^{-1}\ ,\\
(x\otimes yy')({}^{y}x\otimes{}^{y}y')^{-1}(x\otimes y)^{-1}\ .
\end{gathered}
$$

(2.5) Lemma. *Assume that P is a group, F is the aforementioned group, i.e., F is the free group generated by symbols $x\otimes y$, $x,y\in P$ and $[P,P]_{ab}\otimes P_{ab}$ is defined as in Lemma (2.1). Then there is a well-defined homomorphism $\delta:(P\otimes P)\otimes P\longrightarrow [P,P]_{ab}\otimes P_{ab}$, given as follows: if $f=\prod_i(x_i\otimes y_i)^{\epsilon_i}\in F$, $\epsilon_i=\pm1$, and $z\in P$ then*

$$
f\otimes z\mapsto\sum_i\epsilon_i([x_i,y_i]\otimes z+[z,x_i]\otimes y_i+[y_i,z]\otimes x_i)\ ,
$$

where f is identified with its image into $P\otimes P$.

Proof. Taking into account the relations above, we have to check the following equalities:

$$\delta((ff'\otimes z)(f\otimes z)^{-1}({}^{f}f'\otimes {}^{f}z)^{-1})=0, \tag{2.6}$$

$$\delta((f\otimes zz')({}^{z}f\otimes {}^{z}z')^{-1}(f\otimes z)^{-1})=0, \tag{2.7}$$

$$\delta(((xx'\otimes y)(x\otimes y)^{-1}({}^{x}x'\otimes {}^{x}y)^{-1})\otimes z)=0, \tag{2.8}$$

$$\delta(((x\otimes yy')({}^{y}x\otimes {}^{y}y')^{-1}(x\otimes y)^{-1})\otimes z)=0, \tag{2.9}$$

where $f,f'\in F$ and $x,y,x',y',z\in P$.

The proof of (2.6) will be trivial, if we show that $\delta({}^{f}f'\otimes\ {}^{f}z)=\delta(f'\otimes z)$ for all $f,f'\in F$ and $z\in P$. It suffices to take $f=x\otimes y$ and $f'=x'\otimes y'$, where $x,y,x',y'\in P$. Thus, we need to show that $\delta({}^{(x\otimes y)}(x'\otimes y')\otimes^{(x\otimes y)} z)=\delta((x'\otimes y')\otimes z)$, which is equivalent to the equality $\delta((x'\otimes y')\otimes^{[x,y]}z)=\delta((x'\otimes y')\otimes z)$. Clearly $\delta((x'\otimes y')\otimes z)=\delta(({}^{[x,y]}x'\otimes {}^{[x,y]}y')\otimes\ {}^{[x,y]}z)$, hence we have to check that $\delta((x'\otimes y')\otimes^{[x,y]}z)=\delta(({}^{[x,y]}x'\otimes {}^{[x,y]}y')\otimes\ {}^{[x,y]}z)$, which is equivalent to the following:

$$\delta(((x'\otimes y')({}^{[x,y]}x'\otimes {}^{[x,y]}y')^{-1})\otimes z)=0\ .$$

One has:

$$\begin{aligned}
&\delta(((x'\otimes y')({}^{[x,y]}x'\otimes {}^{[x,y]}y')^{-1})\otimes z)\\
&=[x',y']\otimes z+[z,x']\otimes y'+[y',z]\otimes x'-[{}^{[x,y]}x',{}^{[x,y]}y']\otimes z-[z,{}^{[x,y]}x']\otimes {}^{[x,y]}y'\\
&\qquad-[{}^{[x,y]}y',z]\otimes {}^{[x,y]}x'\\
&=[{}^{[x,y]}x',{}^{[x,y]}y']\otimes {}^{[x,y]}z+[{}^{[x,y]}z,{}^{[x,y]}x']\otimes {}^{[x,y]}y'+[{}^{[x,y]}y',{}^{[x,y]}z]\otimes {}^{[x,y]}x'\\
&\qquad-[{}^{[x,y]}x',{}^{[x,y]}y']\otimes z-[z,{}^{[x,y]}x']\otimes {}^{[x,y]}y'-[{}^{[x,y]}y',z]\otimes {}^{[x,y]}x'\\
&=[{}^{[x,y]}z,{}^{[x,y]}x']\otimes {}^{[x,y]}y'+[{}^{[x,y]}y',{}^{[x,y]}z]\otimes {}^{[x,y]}x'-[z,{}^{[x,y]}x']\otimes {}^{[x,y]}y'\\
&\qquad-[{}^{[x,y]}y',z]\otimes {}^{[x,y]}x'\\
&=0\,,\ \text{by previous lemma.}
\end{aligned}$$

We will check only (2.7) and (2.8), because (2.9) is similar to (2.8).

(2.7): One can easily see that it is enough to consider $f=x\otimes y$.

$$\begin{aligned}
&\delta(((x\otimes y)\otimes zz')(({}^{z}x\otimes {}^{z}y)\otimes {}^{z}z')^{-1}((x\otimes y)\otimes z)^{-1})\\
&=[x,y]\otimes zz'+[zz',x]\otimes y+[y,zz']\otimes x-[{}^{z}x,{}^{z}y]\otimes {}^{z}z'-[{}^{z}z',{}^{z}x]\otimes {}^{z}y\\
&\quad-[{}^{z}y,{}^{z}z']\otimes {}^{z}x-\ [x,y]\otimes z-[z,x]\otimes y-[y,z]\otimes x\\
&=[x,y]\otimes z+[{}^{z}x,{}^{z}y]\otimes {}^{z}z'+[{}^{z}z',{}^{z}x]\otimes y+[z,x]\otimes y+\ [y,z]\otimes x+[{}^{z}y,{}^{z}z']\otimes x\\
&\quad-[{}^{z}x,{}^{z}y]\otimes {}^{z}z'-[{}^{z}z',{}^{z}x]\otimes {}^{z}y-[{}^{z}y,{}^{z}z']\otimes {}^{z}x-[x,y]\otimes z-[z,x]\otimes y-[y,z]\otimes x\\
&=0\,,\ \text{since } [a,b]\otimes {}^{z}p=[a,b]\otimes p.
\end{aligned}$$

(2.8):

$$
\begin{aligned}
&\delta(((xx'\otimes y)(x\otimes y)^{-1}({}^{x}x'\otimes {}^{x}y)^{-1})\otimes z)\\
&=[xx',y]\otimes z+[z,xx']\otimes y+[y,z]\otimes xx'-[x,y]\otimes z-[z,x]\otimes y-[y,z]\otimes x\\
&\quad -\ [{}^{x}x',{}^{x}y]\otimes z-[z,{}^{x}x']\otimes {}^{x}y-[{}^{x}y,z]\otimes {}^{x}x'\\
&=[{}^{x}x',{}^{x}y]\otimes z+[x,y]\otimes z+[z,x]\otimes y+\ [{}^{x}z,{}^{x}x']\otimes y+[y,z]\otimes x+[{}^{x}y,{}^{x}z]\otimes {}^{x}x'\\
&\quad -[x,y]\otimes z-[z,x]\otimes y-[y,z]\otimes x-\ [{}^{x}x',{}^{x}y]\otimes z-[z,{}^{x}x']\otimes {}^{x}y-[{}^{x}y,z]\otimes {}^{x}x'\\
&=[{}^{x}z,{}^{x}x']\otimes {}^{x}y+[{}^{x}y,{}^{x}z]\otimes {}^{x}x'-\ [z,{}^{x}x']\otimes {}^{x}y-[{}^{x}y,z]\otimes {}^{x}x'\\
&=0\,,\text{ by the previous lemma.}
\end{aligned}
$$

□

(2.10) Lemma. *Let P be a group and $H_2(P)=0$. Then the homomorphism $\delta:(P\otimes P)\otimes P\longrightarrow [P,P]_{ab}\otimes P_{ab}$ introduced in Lemma (2.5) factors through $(\gamma_2(P)/\gamma_3(P))\otimes_{\mathbb{Z}} P_{ab}$. Thus, $\delta^*:(\gamma_2(P)/\gamma_3(P))\otimes_{\mathbb{Z}} P_{ab}\longrightarrow [P,P]_{ab}\otimes P_{ab}$ given by*

$$[x,y]\otimes z\mapsto [x,y]\otimes z+[z,x]\otimes y+[y,z]\otimes x$$

is well defined.

Proof. By [**3**] we have $[P,P]\cong (P\otimes P)/X_1$, where X_1 is the normal subgroup $P\otimes P$ generated by $x\otimes x$, for all $x\in P$. Therefore,

$$(\gamma_2(P)/\gamma_3(P))\otimes_{\mathbb{Z}} P_{ab}\cong ((P\otimes P)\otimes P)/X_2$$

where X_2 is the normal subgroup of $(P\otimes P)\otimes P$ generated by $(x\otimes x)\otimes z$, $([x,y]\otimes z)\otimes p$, $(x\otimes y)\otimes [p,q]$ for all $x,y,z,p,q,\in P$. By the previous lemma it is enough to check the following:

$$\delta((x\otimes x)\otimes z)=0,$$

$$\delta(([x,y]\otimes z)\otimes p))=0,$$

$$\delta((x\otimes y)\otimes [p,q])=0.$$

We have:

$$\delta((x\otimes x)\otimes z)=[x,x]\otimes z+[z,x]\otimes x+[x,z]\otimes x=0,$$

$$\delta(([x,y]\otimes z)\otimes p))=[[x,y],z]\otimes p+[p,[x,y]]\otimes z+[z,p]\otimes [x,y]=0\,,\text{by Lemma (2.1)},$$

$$\delta((x\otimes y)\otimes [p,q])=[x,y]\otimes [p,q]+[[p,q],x]\otimes y+[y,[p,q]]\otimes x=0\,,\text{by Lemma (2.1)}.$$

□

Given a group P, let $\alpha:[P,P]\otimes P\longrightarrow \gamma_3(P)$ be the commutator map: $\alpha([x,y]\otimes z)=[[x,y],z]$. α induces the following homomorphisms:

$$\alpha_1:[P,P]_{ab}\otimes P_{ab}\longrightarrow \gamma_3(P)/[[P,P],[P,P]],$$

$$\alpha_2:(\gamma_2(P)/\gamma_3(P))\otimes P_{ab}\longrightarrow \gamma_3(P)/\gamma_4(P).$$

Define $i : \operatorname{Ker}\alpha \to \operatorname{Ker}\alpha_1$ and $i' : \operatorname{Ker}\alpha_1 \to \operatorname{Ker}\alpha_2$ as restrictions of the natural projections $[P,P]\otimes P \longrightarrow [P,P]_{ab}\otimes P_{ab}$ and $[P,P]_{ab}\otimes P_{ab} \longrightarrow (\gamma_2(P)/\gamma_3(P))\otimes P_{ab}$, respectively.

(2.11) Lemma. *Let P be a free group and define δ^* as in Lemma (2.10). Then $\operatorname{Ker}\alpha_1 = \operatorname{Im}\delta^*$ and $i' : \operatorname{Ker}\alpha_1 \longrightarrow \operatorname{Ker}\alpha_2$ is an isomorphism.*

Proof. Part one: Using diagram chasing we easily check that $i : \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\alpha_1$ is surjective. [**4**, Theorem 9] says that $\operatorname{Ker}\alpha$ is generated by $([x,y]\otimes {}^{y}z)([y,z]\otimes {}^{z}x)([z,x]\otimes {}^{x}y) \in [P,P]\otimes P$ and $p\otimes p \in [P,P]\otimes P$ for all $x,y,z\in P$ and $p\in[P,P]$. Since i is surjective, the generators of $\operatorname{Ker}\alpha_1$ will be $[x,y]\otimes z+[z,x]\otimes y+[y,z]\otimes x \in [P,P]_{ab}\otimes P_{ab}$ for all $x,y,z\in P$. Hence $\operatorname{Ker}\alpha_1 = \operatorname{Im}\delta^*$.

Part two: Using diagram chasing we easily check that $i' : \operatorname{Ker}\alpha_1 \longrightarrow \operatorname{Ker}\alpha_2$ is surjective. To show the injectivity we need to check the following: if $\omega \in \operatorname{Ker}\alpha_1$, then $\delta^* i'(\omega) = 3\omega$. In fact, by discussion above it suffices to take $w = [x,y]\otimes z + [z,x]\otimes y + [y,z]\otimes x$. We have

$$\begin{aligned}
\delta^* i'(w) &= \delta^*([x,y]\otimes z) + \delta^*([z,x]\otimes y) + \delta^*([y,z]\otimes x)\\
&= [x,y]\otimes z + [z,x]\otimes y + [y,z]\otimes x + [z,x]\otimes y + [y,z]\otimes x\\
&\qquad + [x,y]\otimes x + [y,z]\otimes x + [x,y]\otimes z + [z,x]\otimes y\\
&= 3([x,y]\otimes z + [z,x]\otimes y + [y,z]\otimes x) = 3w.
\end{aligned}$$

Therefore, for injectivity of i', it is sufficient to show that $\operatorname{Ker}\alpha_1$ is torsion free. We have $\operatorname{Ker}\alpha_1 \cong H_3(P_{ab})$ (see [**8**, Theorem 6.7] and [**4**]). Since P is free, $H_3(P_{ab}) \cong P_{ab}\wedge P_{ab}\wedge P_{ab}$ (see [**2, 5, 7**]) which is torsion free. □

Given a group P, we have the short exact sequence of groups

$$1 \longrightarrow \gamma_3(P)/[\gamma_2(P),\gamma_2(P)] \longrightarrow [P,P]_{ab} \longrightarrow \gamma_2(P)/\gamma_3(P) \longrightarrow 1. \tag{2.12}$$

Suppose X be one of the groups in the sequence (2.12). Define $X\otimes P$ according to (2). Assume that the homomorphisms

$$\beta_0 : (\gamma_3(P)/[\gamma_2(P),\gamma_2(P)])\otimes P \longrightarrow \gamma_4(P)/[\gamma_3(P),\gamma_2(P)]$$

$$\beta_1 : [P,P]_{ab}\otimes P \longrightarrow \gamma_3(P)/[\gamma_3(P),\gamma_2(P)],$$

$$\beta_2 : (\gamma_2(P)/\gamma_3(P))\otimes P \longrightarrow \gamma_3(P)/\gamma_4(P),$$

are defined by taking commutators. These homomorphisms are well defined because the restrictions to $\gamma_2(P)$ of the actions of P on $\gamma_3(P)/[\gamma_3(P),\gamma_2(P)]$, $\gamma_4(P)/[\gamma_3(P),\gamma_2(P)]$ and $\gamma_3(P)/\gamma_4(P)$ induced by conjugation are trivial. If P is a free group, then there is a short exact sequence of groups

$$0 \longrightarrow \operatorname{Ker}\beta_0 \longrightarrow \operatorname{Ker}\beta_1 \longrightarrow \operatorname{Ker}\beta_2 \longrightarrow 0. \tag{2.13}$$

In fact, since the groups in Sequence (2.12) are P-modules and act trivially on P, we have $\otimes P \cong \otimes_P IP$, where $IP = \operatorname{Ker}(\mathbb{Z}(P) \to \mathbb{Z})$. Since P is free, IP is a free P-module. Therefore, $(2.12)\otimes P$ is a short exact sequence. This implies that (2.13) is a short exact sequence.

(2.14) Lemma. *Let P be a free group. Assume that*
1) A is the normal subgroup of $(\gamma_3(P)/[\gamma_2(P),\gamma_2(P)])\otimes P$ generated by all $x\otimes y\in(\gamma_3(P)/[\gamma_2(P),\gamma_2(P)])\otimes P$, where $y\in[P,P]$;
2) A' is the natural image of A into $[P,P]_{ab}\otimes P$;
3) B is the normal subgroup of $[P,P]_{ab}\otimes P$ generated by all $z\otimes z\in[P,P]_{ab}\otimes P$ where $z\in[P,P]$ and in $z\otimes z$, the first z is identified with its image in $[P,P]_{ab}$.
Then
(**a**) $\operatorname{Ker}\beta_2\cong\operatorname{Ker}\beta_1/(A'+B)$.
(**b**) $\operatorname{Ker}\beta_0$ *is generated by A and the set of elements* $[y,{}^{y^{-1}}[x,y]]\otimes x+[{}^{y^{-1}}[x,y],x]\otimes{}^{x}y$, $[y,{}^{y^{-1}}[p,q]]\otimes x+[{}^{y^{-1}}[p,q],x]\otimes{}^{x}y+[q,{}^{q^{-1}}[x,y]]\otimes p+[{}^{q^{-1}}[x,y],p]\otimes{}^{p}q$ *for all $x,y,p,q\in P$.*

Proof. (**a**): Denote by j the natural homomorphism $\operatorname{Ker}\beta_1\longrightarrow\operatorname{Ker}\beta_2$. Since $\gamma_2(P)/\gamma_3(P)$ and P act trivially on each other, by [**3**] we have

$$(\gamma_2(P)/\gamma_3(P))\otimes P\cong(\gamma_2(P)/\gamma_3(P))\otimes P_{ab}\ . \tag{2.15}$$

Thanks to this we easily see that j sends $(A'+B)$ to zero and induces a homomorphism $j^*:\operatorname{Ker}\beta_1/(A'+B)\longrightarrow\operatorname{Ker}\beta_2$. Assume that $[P,P]\wedge P$ is defined naturally (i.e. as in [**4**]) and $\alpha^*:[P,P]\wedge P\longrightarrow\gamma_3(P)$ is the commutator map. We have the natural projection

$$[P,P]\wedge P\longrightarrow([P,P]_{ab}\otimes P)/(A'+B),$$

which induces a homomorphism:

$$\operatorname{Ker}\alpha^*\xrightarrow{\tau}\operatorname{Ker}\beta_1/(A'+B).$$

Using diagram chasing we easily check that τ is an epimorphism. Hence the composition $\operatorname{Ker}\alpha^*\xrightarrow{\tau}\operatorname{Ker}\beta_1/(A'+B)\xrightarrow{j^*}\operatorname{Ker}\beta_2$ is an epimorphism. Prove that $j^*\circ\tau$ is an isomorphism. Since P is a direct limit of its finitely generated subgroups and this system is compatible with j^*,τ,α^* and β_2, without lost of generality we can assume that P is a free group with finite basis. Then $H_3(P_{ab})\cong P_{ab}\wedge P_{ab}\wedge P_{ab}\cong\mathbb{Z}^n$ for some $n\in\mathbb{N}$. Moreover, taking into account (2.15) and the previous lemma, we have

$$\operatorname{Ker}\beta_2\ \cong\operatorname{Ker}\alpha_2\cong\operatorname{Ker}\alpha_1\cong H_3(P_{ab})\cong\mathbb{Z}^n,$$

where α_1 and α_2 are defined as above. On the other hand there is an isomorphism $\operatorname{Ker}\alpha^*\cong H_3(P_{ab})$ (see [**4**]). Thus, both of $\operatorname{Ker}\alpha^*$ and $\operatorname{Ker}\beta_2$ are isomorphic to $\mathbb{Z}^n$. Therefore, any epimorphism $\operatorname{Ker}\alpha^*\to\operatorname{Ker}\beta_2$ (in particular $j^*\circ\tau$) is an isomorphism. Hence j^* is an isomorphism and we have (**a**) .

(**b**): Note that B is generated by all $[x,y]\otimes[x,y]\in[P,P]_{ab}\otimes P$ and $([p,q]\otimes[x,y]+[x,y]\otimes[p,q])\in[P,P]_{ab}\otimes P$, where $x,y,p,q\in P$. Therefore, taking into account (2.13) and (**a**), it is sufficient to prove that in $[P,P]_{ab}\otimes P$ the following hold:

$$[x,y]\otimes[x,y]=[y,{}^{y^{-1}}[x,y]]\otimes x+[{}^{y^{-1}}[x,y],x]\otimes{}^{x}y,$$

$$\begin{aligned}&[p,q]\otimes[x,y]+[x,y]\otimes[p,q]\\=\ &[y,{}^{y^{-1}}[p,q]]\otimes x+[{}^{y^{-1}}[p,q],x]\otimes{}^{x}y+\ [q,{}^{q^{-1}}[x,y]]\otimes p+[{}^{q^{-1}}[x,y],p]\otimes{}^{p}q,\end{aligned}$$

for any $x, y, p, q \in P$. Both of these equalities will be clear, if we prove the following:

$$[p,q] \otimes [x,y] = [y, {}^{y^{-1}}[p,q]] \otimes x + [{}^{y^{-1}}[p,q], x] \otimes {}^{x}y.$$

for all $x, y, p, q \in P$. We have:

$$\begin{aligned}
&[p,q] \otimes [x,y] = [p,q] \otimes x\, {}^{y}x^{-1} = [p,q] \otimes x + {}^{x}[p,q] \otimes {}^{xy}x^{-1} \\
&= [p,q] \otimes x + \ ({}^{xyy^{-1}}[p,q] \otimes {}^{xy}x^{-1} + \ {}^{y^{-1}}[p,q] \otimes xy) - \ {}^{y^{-1}}[p,q] \otimes xy \\
&= [p,q] \otimes x + \ {}^{y^{-1}}[p,q] \otimes (xy)x^{-1} - \ {}^{y^{-1}}[p,q] \otimes xy \\
&= [p,q] \otimes x + {}^{y^{-1}}[p,q] \otimes {}^{x}y - \ {}^{y^{-1}}[p,q] \otimes x - \ {}^{xy^{-1}}[p,q] \otimes {}^{x}y \\
&= ([p,q] \otimes x - \ {}^{y^{-1}}[p,q] \otimes x) + \ ({}^{y^{-1}}[p,q] \otimes {}^{x}y - \ {}^{xy^{-1}}[p,q] \otimes {}^{x}y) \\
&= ({}^{yy^{-1}}[p,q] \otimes x + {}^{y^{-1}}[p,q]^{-1} \otimes x) + \ ({}^{y^{-1}}[p,q] \otimes {}^{x}y + \ {}^{xy^{-1}}[p,q]^{-1} \otimes {}^{x}y) \\
&= [y, {}^{y^{-1}}[p,q]] \otimes x + [{}^{y^{-1}}[p,q], x] \otimes {}^{x}y.
\end{aligned}$$

$\square$

(2.16) Lemma. *Let P be a free group and let $\alpha' : \gamma_3(P) \otimes P \longrightarrow \gamma_4(P)/[\gamma_3(P), \gamma_2(P)]$ be the homomorphism defined by taking commutators, i.e., $[[x,y],z] \otimes p \mapsto [[[x,y],z],p]$, for $x,y,z,p \in P$. Then $\operatorname{Ker}\alpha'$ is generated by the subgroups $[\gamma_2(P), \gamma_2(P)] \otimes P$ and $\gamma_3(P) \otimes \gamma_2(P)$ and the set of elements $([y, {}^{y^{-1}}[x,y]] \otimes x)([{}^{y^{-1}}[x,y], x] \otimes {}^{x}y)$ and $([y, {}^{y^{-1}}[p,q]] \otimes x)([{}^{y^{-1}}[p,q], x] \otimes {}^{x}y)([q, {}^{q^{-1}}[x,y]] \otimes p)([{}^{q^{-1}}[x,y], p] \otimes {}^{p}q)$, for all $x, y, p, q \in P$.*

Proof. Assume that A is defined as in Lemma (2.14). Then we have an isomorphism

$$\frac{\gamma_3(P) \otimes P}{\gamma_3(P) \otimes \gamma_2(P) + [\gamma_2(P), \gamma_2(P)] \otimes P} \cong \frac{(\gamma_3(P)/[\gamma_2(P), \gamma_2(P)]) \otimes P}{A}$$

given in a natural way. The rest of the proof is a consequence of Lemma (2.14)(**b**). $\square$

(2.17) Theorem. *If P is a free group, then $\theta_4 : \Gamma_4(P) \longrightarrow \gamma_4(P)$ is an isomorphism.*

Proof. Since surjectivity of θ_4 is obvious, we will prove the injectivity.

Let $\Gamma_4'(P)$ be the subgroup of $\Gamma_4(P)$ generated by $\{\{\{p_1, p_2\}, [p,q]\}, p_3\}$ and $\{\{\{p_1, p_2\}, p_3\}, [p,q]\}$, for all $p_1, p_2, p_3, p, q \in P$. We easily see that $\Gamma_4'(P)$ is a normal subgroup of $\mathcal{L}(P)$ and $\theta_4(\Gamma_4'(P)) = [\gamma_3(P), \gamma_2(P)]$. Define the homomorphisms $\overline{\theta_4}$ and $\widetilde{\theta}_4$:

$$\Gamma_4'(P) \xrightarrow{\overline{\theta_4}} \theta_4(\Gamma_4'(P)) = [\gamma_3(P), \gamma_2(P)] \text{ is the restriction of } \theta_4 \text{ on } \Gamma_4'(P)\ ;$$

$$\Gamma_4(P)/\Gamma_4'(P) \xrightarrow{\widetilde{\theta}_4} \gamma_4(P)/\theta_4(\Gamma_4'(P)) = \gamma_4(P)/[\gamma_3(P), \gamma_2(P)] \text{ is induced by } \theta_4\ .$$

The proof of (2.17) will be done, if we show that $\overline{\theta_4}$ and $\widetilde{\theta_4}$ are injective. Using (1.9) and (1.10) we easily show that $\Gamma_4'(P) \subset \Gamma_3(P)$. Since θ_3 is an isomorphism, $\overline{\theta_4}$ will be injective. In order to show injectivity of $\widetilde{\theta_4}$, we construct the homomorphism

$$(\theta_3^{-1}\widetilde{\otimes}P) : \gamma_3(P) \otimes P \longrightarrow \Gamma_4(P)\ ,$$

$$[[x,y],z] \otimes p \mapsto \{\theta_3^{-1}([[x,y],z]),p\} = \{\{\{x,y\},z\},p\}.$$

Taking into account (1.9), it is trivial to check that $\theta_3^{-1}\widetilde{\otimes}P$ is well defined. Then, the following composition

$$\gamma_3(P) \otimes P \xrightarrow{\theta_3^{-1}\widetilde{\otimes}\, P} \Gamma_4(P)/\Gamma_4'(P) \xrightarrow{\widetilde{\theta}_4} \gamma_4(P)/[\gamma_3(P),\gamma_2(P)]$$

is the map $\alpha' : \gamma_3(P) \otimes P \longrightarrow \gamma_4(P)/[\gamma_3(P),\gamma_2(P)]$ defined in Lemma (2.16). Since $\theta_3^{-1}\widetilde{\otimes}P$ is onto, $\operatorname{Ker}\widetilde{\theta_4} = (\theta_3^{-1}\widetilde{\otimes}P)(\operatorname{Ker}\alpha')$. Hence, the generators of $\operatorname{Ker}\widetilde{\theta_4}$ are the images by $\theta_3^{-1}\widetilde{\otimes}P$ of the set of generators given in Lemma (2.16). Thus, we have to show the following:

$$(\theta_3^{-1}\widetilde{\otimes}P)([\gamma_2(P),\gamma_2(P)] \otimes P) \subset \Gamma_4'(P), \tag{2.18}$$

$$(\theta_3^{-1}\widetilde{\otimes}P)(\gamma_3(P) \otimes \gamma_2(P)) \subset \Gamma_4'(P), \tag{2.19}$$

$$(\theta_3^{-1}\widetilde{\otimes}P)(([y,{}^{y^{-1}}[x,y]] \otimes x)([{}^{y^{-1}}[x,y],x] \otimes {}^x y)) \in \Gamma_4'(P), \tag{2.20}$$

$$(\theta_3^{-1}\widetilde{\otimes}P)(([y,{}^{y^{-1}}[p,q]] \otimes x)([{}^{y^{-1}}[p,q],x] \otimes {}^x y)([q,{}^{q^{-1}}[x,y]] \otimes p)$$
$$([{}^{q^{-1}}[x,y],p] \otimes {}^p q)) \in \Gamma_4'(P). \tag{2.21}$$

(2.18) and (2.19) are trivial inclusions. For (2.20) and (2.21), note that there are the following congruences mod $\Gamma_4'(P)$:

$$\{\{\{z_1,z_2\},{}^z z'\},z_3\} \equiv \{\{\{z_1,z_2\},z'\},z_3\},$$

$$\{\{\{z_1,z_2\},z_3\},{}^z z'\} \equiv \{\{\{z_1,z_2\},z_3\},z'\},$$

$$\{\{\{z_1,z_2\},z_3\},{}^{\{z,z'\}} z_4\} \equiv \{\{\{z_1,z_2\},z_3\},z_4\},$$

for all $z_1,z_2,z_3,z_4,z,z' \in P$. These relations and (1.5) imply that

$$\begin{aligned}&(\theta_3^{-1}\widetilde{\otimes}P)(([y,{}^{y^{-1}}[x,y]] \otimes x)([{}^{y^{-1}}[x,y],x] \otimes {}^x y))\\ &= \{\{y,{}^{y^{-1}}\{x,y\}\},x\}\{\{{}^{y^{-1}}\{x,y\},x\},{}^x y\} \equiv (\{\{x,y\},\{x,y\}\})^{-1} = 1,\end{aligned}$$

$$\begin{aligned}&(\theta_3^{-1}\widetilde{\otimes}P)(([y,{}^{y^{-1}}[p,q]] \otimes x)([{}^{y^{-1}}[p,q],x] \otimes {}^x y)([q,{}^{q^{-1}}[x,y]] \otimes p)([{}^{q^{-1}}[x,y],p] \otimes {}^p q))\\ &= \{\{y,{}^{y^{-1}}\{p,q\}\},x\}\{\{{}^{y^{-1}}\{p,q\},x\},{}^x y\}\{\{q,{}^{q^{-1}}\{x,y\}\},p\}\{\{{}^{q^{-1}}\{x,y\},p\},{}^p q\}\}\\ &\equiv (\{\{x,y\},\{p,q\}\})^{-1}(\{\{p,q\},\{x,y\}\})^{-1} = 1,\end{aligned}$$

where the congruences being still taken mod $\Gamma_4'(P)$. Thus, (2.20) and (2.21) are proved. $\square$

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