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THE ASSOCIATIVE OPERAD AND THE WEAK ORDER ON THE SYMMETRIC GROUPS

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(*communicated by James Stasheff*)

Abstract

The associative operad is a certain algebraic structure on the sequence of group algebras of the symmetric groups. The weak order is a partial order on the symmetric group. There is a natural linear basis of each symmetric group algebra, related to the group basis by Möbius inversion for the weak order. We describe the operad structure on this second basis: the surprising result is that each operadic composition is a sum over an interval of the weak order. We deduce that the coradical filtration is an operad filtration. The Lie operad, a suboperad of the associative operad, sits in the first component of the filtration. As a corollary to our results, we derive a simple explicit expression for Dynkin's idempotent in terms of the second basis.

There are combinatorial procedures for constructing a planar binary tree from a permutation, and a composition from a planar binary tree. These define set-theoretic quotients of each symmetric group algebra. We show that they are non-symmetric operad quotients of the associative operad. Moreover, the Hopf kernels of these quotient maps are non-symmetric suboperads of the associative operad.

Introduction

One of the simplest symmetric operads is the associative operad $\mathcal{A}s$. This is an algebraic structure carried by the sequence of vector spaces $\mathcal{A}s_n = \Bbbk S_n$, $n \geqslant 1$, where S_n is the symmetric group on n letters. In particular this entails structure maps, for each $n, m \geqslant 1$ and $1 \leqslant i \leqslant n$,

$$\mathcal{A}s_n \otimes \mathcal{A}s_m \xrightarrow{\circ_i} \mathcal{A}s_{n+m-1}$$

satisfying certain axioms (Section 1). The Lie operad $\mathcal{L}ie$ is a symmetric suboperad of $\mathcal{A}s$.

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The space $\mathcal{A}s_* := \bigoplus_{n\geqslant 1} \Bbbk S_n$ carries the structure of a (non-unital) graded Hopf algebra, first defined by Malvenuto and Reutenauer [**17**], and studied recently in a number of works, including [**1, 7, 15**]. It is known that $\mathcal{L}ie_* := \bigoplus_{n\geqslant 1} \mathcal{L}ie_n$ sits inside the subspace of $\mathcal{A}s_*$ consisting of primitive elements for this Hopf algebra. This led us to consider whether this subspace is itself a suboperad of $\mathcal{A}s$. In the process to answering this question, we found a number of interesting results linking the non-symmetric operad structure of $\mathcal{A}s$ to the combinatorics of the symmetric groups, and in particular to a partial order on S_n known as the left weak Bruhat order (or weak order, for simplicity).

Let F_σ denote the standard basis element of $\Bbbk S_n$ corresponding to $\sigma \in S_n$. Define a new basis of $\Bbbk S_n$ by means of the formula

$$M_\sigma = \sum_{\sigma\leqslant\tau} \mu(\sigma,\tau) F_\tau \,,$$

where μ is the Möbius function of the weak order (Section 1.3). This basis was used in [**1**] to provide a simple explicit description of the primitive elements and the coradical filtration of the Hopf algebra of Malvenuto and Reutenauer (as well as the rest of the Hopf algebra structure).

One of our main results, Theorem 1.1, provides an explicit description for the non-symmetric operad structure of $\mathcal{A}s$ (the maps $\circ_i$) in this basis. We find that each $M_\sigma \circ_i M_\tau$ is a sum over the elements of an interval in the weak order, which we describe explicitly. This result, and the combinatorics needed for its proof, are given in Sections 1.3, 1.4, and 1.5.

Together with the results of [**1**], Theorem 1.1 allows us to conclude that the space of primitive elements is a non-symmetric suboperad of $\mathcal{A}s$, and moreover that the coradical filtration is an operadic filtration. These notions are reviewed in Section 2 and the result is obtained in Theorem 2.1. An alternative proof suggested by the referee is given in Remark 2.3.

There are combinatorial procedures for constructing a planar binary tree with n internal vertices from a permutation in S_n, and a subset of $[n-1]$ from such a tree. The composite procedure associates to a permutation the set of its descents. The behavior of these constructions with respect to the Hopf algebra structure is well-understood [**10, 15, 23**]. In Sections 3 and 4 we show that they lead to non-symmetric operad quotients of $\mathcal{A}s$. These quotients possess partial orders and linear bases analogous to those of $\Bbbk S_n$, and the non-symmetric operad structure maps on the basis M are again given by sums over intervals in the weak order (which degenerate to a point in the case of subsets). These results are obtained in Propositions 3.3 and 4.5. Special attention is granted to the quotient of $\mathcal{A}s$ defined by passing to descents. The Hopf kernel of this map is shown to be a non-symmetric suboperad of $\mathcal{A}s$ in Proposition 3.7.

In Section 5 we turn to the symmetric operad $\mathcal{L}ie$. The subspace $\mathcal{L}ie_n$ of $\mathcal{A}s_n$ is generated as right S_n-module by a special element θ_n called Dynkin's idempotent. Our main result here is a surprisingly simple expression for this element in the basis

M (Theorem 5.3):

$$\theta_n = \sum_{\sigma \in S_n,\, \sigma(1)=1} M_\sigma\,.$$

We obtain this result by noting that the classical definition of θ_n can be recast as the iteration of a certain operation $\{\ ,\ \} : \mathcal{A}s \times \mathcal{A}s \to \mathcal{A}s$ that preserves the suboperad $\mathcal{L}ie$ and the non-symmetric suboperad of primitive elements, and by calculating an explicit expression for $\{M_\sigma, M_\tau\}$ (Proposition 5.1).

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Notation

The set $\{1, 2, \ldots, n\}$ is denoted $[n]$. We work over a commutative ring $\Bbbk$ of arbitrary characteristic. We refer to $\Bbbk$-modules as "spaces".

The symmetric group S_n is the group of bijections $\sigma : [n] \to [n]$. We specify permutations by the list of their values, that is $\sigma = (\sigma_1, \ldots, \sigma_n)$ is the permutation such that $\sigma(i) = \sigma_i$.

1. The associative operad

1.1. Symmetric and non-symmetric operads

A *non-symmetric unital operad* is a sequence of spaces $\mathcal{P} = \{\mathcal{P}_n\}_{n \geqslant 1}$ together with linear maps

$$\mathcal{P}_n \otimes \mathcal{P}_m \xrightarrow{\circ_i} \mathcal{P}_{n+m-1}\,,$$

one for each $n, m \geqslant 1$ and $1 \leqslant i \leqslant n$, and a distinguished element $1 \in \mathcal{P}_1$, such that

$$\begin{aligned} (x \circ_i y) \circ_{j+m-1} z &= (x \circ_j z) \circ_i y \quad \text{if } 1 \leqslant i < j \leqslant n, \\ (x \circ_i y) \circ_{i+j-1} z &= x \circ_i (y \circ_j z) \quad \text{if } 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m, \end{aligned} \tag{1}$$

$$\begin{aligned} x \circ_i 1 &= x \quad \text{if } 1 \leqslant i \leqslant n, \\ 1 \circ_1 x &= x \end{aligned} \tag{2}$$

for every $x \in \mathcal{P}_n$, $y \in \mathcal{P}_m$, and $z \in \mathcal{P}_l$. For various equivalent and related definitions, see [**12**] or [**18**, II.1].

Given permutations $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n$ and $\tau = (\tau_1, \ldots, \tau_m) \in S_m$, define a permutation $B_i(\sigma, \tau) \in S_{n+m-1}$ by

$$B_i(\sigma, \tau) := (a_1, \ldots, a_{i-1}, b_1, \ldots, b_m, a_{i+1}, \ldots, a_n) \tag{3}$$

where

$$a_j := \begin{cases} \sigma_j & \text{if } \sigma_j < \sigma_i \\ \sigma_j + m - 1 & \text{if } \sigma_j > \sigma_i \end{cases} \quad \text{and} \quad b_k := \tau_k + \sigma_i - 1\,. \tag{4}$$

Note that $a_j \in [1, \sigma_i - 1] \cup [\sigma_i + m, n + m - 1]$ and $b_k \in [\sigma_i, \sigma_i + m - 1]$, so $B_i(\sigma, \tau)$ is indeed a permutation. For instance, if $\sigma = (2, 3, 1, 4)$, $\tau = (\mathbf{2}, \mathbf{3}, \mathbf{1})$, and $i = 2$ then $B_2(\sigma, \tau) = (2, \mathbf{4}, \mathbf{5}, \mathbf{3}, 1, 6)$. One may understand this construction in terms of permutation matrices. Associate to $\sigma \in S_n$ the $n \times n$-matrix whose (i, j)-entry is

Kronecker's δ_{i,σ_j}. Then the matrix of $B_i(\sigma,\tau)$ is obtained by inserting the matrix of τ in the (σ_i, i) entry of the matrix of σ. In the above example,

$$\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}, \text{ and } B_2(\sigma,\tau) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A *symmetric unital operad* is a sequence of spaces $\mathcal{P}$ with the same structure as above, plus a right linear action of the symmetric group S_n on $\mathcal{P}_n$ such that

$$(x \cdot \sigma) \circ_i (y \cdot \tau) = (x \circ_{\sigma(i)} y) \cdot B_i(\sigma, \tau) \tag{5}$$

for every $x \in \mathcal{P}_n$, $y \in \mathcal{P}_m$, $\sigma \in S_n$, $\tau \in S_m$, and $1 \leqslant i \leqslant n$.

Associated to any (symmetric or non-symmetric) operad $\mathcal{P}$, there is the graded vector space

$$\mathcal{P}_* := \bigoplus_{n \geqslant 1} \mathcal{P}_n.$$

There is a pair of adjoint functors

$$\{\text{symmetric operads}\} \underset{\textit{symmetrization}}{\overset{\textit{forgetful}}{\rightleftarrows}} \{\text{non-symmetric operads}\},$$

the symmetrization functor $\mathcal{S}$ being left adjoint to the forgetful functor $\mathcal{F}$. Given a symmetric operad $\mathcal{P}$, the non-symmetric operad $\mathcal{F}\mathcal{P}$ is obtained by forgetting the symmetric group actions. Conversely, every non-symmetric operad $\mathcal{P}$ gives rise to a symmetric operad $\mathcal{S}\mathcal{P}$ with spaces $\mathcal{S}\mathcal{P}_n := \mathcal{P}_n \otimes \Bbbk S_n$. These spaces are equipped with the action of S_n by right multiplication on the second tensor factor. There is then a unique way to extend the structure maps $\circ_i$ from $\mathcal{P}$ to $\mathcal{S}\mathcal{P}$ in a way that is compatible with the action.

An *algebra* over a non-symmetric operad $\mathcal{P}$ is a space A together with structure maps

$$\mathcal{P}_n \otimes A^{\otimes n} \to A$$

subject to certain associativity and unitality conditions. An algebra over a symmetric operad $\mathcal{P}$ is a space A as above for which the structure maps factor through the quotient $\mathcal{P}_n \otimes_{\Bbbk S_n} A^{\otimes n}$.

An algebra over a non-symmetric operad $\mathcal{P}$ is the same thing as an algebra over the symmetrization $\mathcal{S}\mathcal{P}$. On the other hand, if $\mathcal{P}$ is a symmetric operad, algebras over $\mathcal{P}$ and algebras over $\mathcal{F}\mathcal{P}$ differ.

1.2. The associative operad

Let $\mathcal{A}s_n := \Bbbk S_n$ be the group algebra of the symmetric group. The basis element corresponding to a permutation $\sigma \in S_n$ is denoted F_σ. This enables us to distinguish between various linear bases of $\mathcal{A}s_n$, all of which are indexed by permutations. In particular, a basis M_σ is introduced in Section 1.3 below.

The collection $\mathcal{A}s := \{\mathcal{A}s_n\}_{n\geqslant 1}$ carries a structure of symmetric operad, uniquely determined by the requirements

$$F_{1_n} \circ_i F_{1_m} = F_{1_{n+m-1}} \quad \text{for any } n, m \geqslant 1,\ 1 \leqslant i \leqslant n,$$
$$F_\sigma = F_{1_n} \cdot \sigma \quad \text{for any } n \geqslant 1,\ \sigma \in S_n,$$

where $1_n = (1, 2, \ldots, n)$ denotes the identity permutation in S_n. In view of (5), the structure maps $\mathcal{A}s_n \otimes \mathcal{A}s_m \xrightarrow{\circ_i} \mathcal{A}s_{n+m-1}$ are given by

$$F_\sigma \circ_i F_\tau := F_{B_i(\sigma,\tau)}\,. \tag{6}$$

This is the *associative* operad $\mathcal{A}s$. It is a symmetric operad. The non-symmetric operad $\mathcal{F}\mathcal{A}s$ is denoted $\mathcal{A}$. We refer to $\mathcal{A}$ as the non-symmetric associative operad. Algebras over $\mathcal{A}s$ are associative algebras; we do not have a simple description for the algebras over $\mathcal{A}$.

The *commutative* operad $\mathcal{C}$ is the sequence of 1-dimensional spaces $\Bbbk\{x_n\}$ with structure maps

$$x_n \circ_i x_m = x_{n+m-1}\,.$$

It is a symmetric operad with the trivial symmetric group actions. The symmetric operad $\mathcal{S}\mathcal{F}\mathcal{C}$ is the associative operad $\mathcal{A}s$.

The Lie operad $\mathcal{L}ie$ is defined in Section 5.3. It is a symmetric suboperad of $\mathcal{A}s$ and algebras over $\mathcal{L}ie$ are Lie algebras. The non-symmetric operad $\mathcal{F}\mathcal{L}ie$ is denoted $\mathcal{L}$. It is a (non-symmetric) suboperad of $\mathcal{A}$.

1.3. The weak order and the monomial basis

The set of inversions of a permutation $\sigma \in S_n$ is

$$\mathrm{Inv}(\sigma) \ := \ \{(i,j) \in [n] \times [n] \mid i < j \text{ and } \sigma_i > \sigma_j\}\,.$$

The set of inversions determines the permutation.

Let $\sigma, \tau \in S_n$. The *left weak Bruhat order* on S_n is defined by

$$\sigma \leqslant \tau \iff \mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\tau)\,.$$

This is a partial order on S_n. We refer to it as the *weak order* for simplicity. For the Hasse diagram of the weak order on S_4, see [**1**, Figure 1] or Figure 1.

Let $\sigma \leqslant \tau$ in S_n. The *Möbius function* $\mu(\sigma, \tau)$ is defined by the recursion

$$\sum_{\sigma\leqslant\rho\leqslant\tau} \mu(\sigma,\rho) = \begin{cases} 1 & \text{if } \sigma = \tau\,, \\ 0 & \text{if } \sigma < \tau\,. \end{cases}$$

The Möbius function of the weak order takes values in $\{-1, 0, 1\}$. Explicit descriptions can be found in [**4**, Corollary 3] or [**8**, Theorem 1.2]. We will not need these descriptions.

The *monomial* basis $\{M_\sigma\}$ of $\mathcal{A}_n$ is defined as follows [**1**, Section 1.3]. For each $n \geqslant 1$ and $\sigma \in S_n$, let

$$M_\sigma := \sum_{\sigma\leqslant\tau} \mu(\sigma,\tau) F_\tau\,. \tag{7}$$

For instance,

$$M_{(4,1,2,3)} = F_{(4,1,2,3)} - F_{(4,1,3,2)} - F_{(4,2,1,3)} + F_{(4,3,2,1)} .$$

By Möbius inversion,

$$F_\sigma = \sum_{\sigma \leqslant \tau} M_\tau . \tag{8}$$

Before giving the full description of the operad structure of $\mathcal{A}$ on the basis $\{M_\sigma\}$, consider one particular example. Using (6) and (7), one finds by direct calculation that

$$M_{(1,2,3)} \circ_2 M_{(2,1)} = M_{(1,3,2,4)} + M_{(1,4,2,3)} + M_{(2,3,1,4)} + M_{(2,4,1,3)} + M_{(3,4,1,2)} . \tag{9}$$

The five permutations appearing on the right hand side form an interval in the weak order on S_4; the bottom element is $(1,3,2,4)$ and the top element is $(3,4,1,2)$. This is a general fact. Fix $n, m \geqslant 1$ and $i \in [n]$. Below we show the existence of a map $T_i : S_n \times S_m \to S_{n+m-1}$ such that the permutations appearing in the expansion of $M_\sigma \circ_i M_\tau$ form an interval with bottom element $B_i(\sigma,\tau)$ and top element $T_i(\sigma,\tau)$.

Theorem 1.1. *For any $\sigma \in S_n$, $\tau \in S_m$, and $1 \leqslant i \leqslant n$,*

$$M_\sigma \circ_i M_\tau = \sum_{B_i(\sigma,\tau) \leqslant \rho \leqslant T_i(\sigma,\tau)} M_\rho . \tag{10}$$

A main ingredient in the proof of Theorem 1.1 is the construction of a map $P_i : S_{n+m-1} \to S_n \times S_m$ which is right adjoint to B_i, so that the pair (B_i, P_i) forms a Galois connection in the sense of [**16**, Section IV.5]. The map P_i is related to B_i and T_i through the following result. We put on $S_n \times S_m$ the partial order obtained by taking the Cartesian product of the weak orders on S_n and S_m.

Proposition 1.2. *The maps*

$$P_i : S_{n+m-1} \to S_n \times S_m , \quad B_i : S_n \times S_m \to S_{n+m-1} , \text{ and } T_i : S_n \times S_m \to S_{n+m-1}$$

satisfy the following properties:

(i) B_i is order-preserving and

$$B_i(\sigma,\tau) \leqslant B_i(\sigma',\tau') \Leftrightarrow (\sigma,\tau) \leqslant (\sigma',\tau') . \tag{11}$$

(ii) P_i is order-preserving and is a right adjoint for B_i, that is

$$B_i(\sigma,\tau) \leqslant \rho \Longleftrightarrow (\sigma,\tau) \leqslant P_i(\rho).$$

(iii) The fiber of the map P_i over each pair (σ,τ) is an interval in the poset S_{n+m-1}, with bottom element $B_i(\sigma,\tau)$ and top element $T_i(\sigma,\tau)$:

$$\{\rho \in S_{n+m-1} | P_i(\rho) = (\sigma,\tau)\} = [B_i(\sigma,\tau); T_i(\sigma,\tau)].$$

Note that this data does *not* define a *lattice congruence* in the sense of [**22**], since the map T_i is not order-preserving. The fact that B_i is order-preserving has the following additional consequence: if $x \leqslant y$ in S_{n+m-1}, then the bottom element in the fiber of P_i which contains x is less than or equal to the bottom element in the

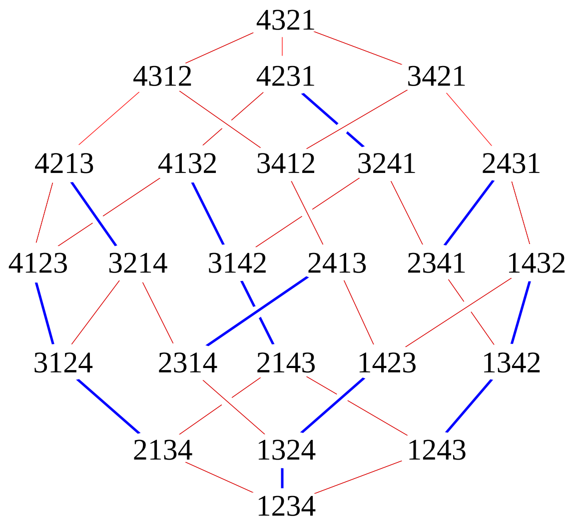


Figure 1: The fibers of $P_1 : S_4 \to S_3 \times S_2$

fiber of P_i which contains y. The fibers of P_i are shown in Figure 1, for $n = 3$, $m = 2$, $i = 1$ (elements joined by a thick edge belong to the same fiber).

The proof of (i) in Proposition 1.2 is given at the end of this section. The construction of P_i and the proof of (ii) in Proposition 1.2 is given in Section 1.4. The construction of T_i and the proof of (iii) in Proposition 1.2 is given in Section 1.5. Below we deduce Theorem 1.1 from these results.

Proof of Theorem 1.1. By Proposition 1.2, (10) may be restated as follows:

$$M_\sigma \circ_i M_\tau = \sum_{P_i(\rho)=(\sigma,\tau)} M_\rho \,. \tag{12}$$

Define a map $\tilde{\circ}_i$ by

$$M_\sigma \tilde{\circ}_i M_\tau := \sum_{P_i(\rho)=(\sigma,\tau)} M_\rho \,.$$

Then, by (8),

$$F_\sigma \tilde{\circ}_i F_\tau = \sum_{\substack{\sigma \leqslant \sigma' \\ \tau \leqslant \tau'}} M_{\sigma'} \tilde{\circ}_i M_{\tau'} = \sum_{\substack{(\sigma,\tau)\leqslant(\sigma',\tau') \\ P_i(\rho)=(\sigma',\tau')}} M_\rho = \sum_{(\sigma,\tau)\leqslant P_i(\rho)} M_\rho \,.$$

Since P_i is right adjoint for B_i,

$$F_\sigma \tilde{\circ}_i F_\tau = \sum_{B_i(\sigma,\tau)\leqslant\rho} M_\rho = F_{B_i(\sigma,\tau)} \,.$$

Thus $F_\sigma \tilde{\circ}_i F_\tau = F_\sigma \circ_i F_\tau$ and hence also, by linearity, $M_\sigma \tilde{\circ}_i M_\tau = M_\sigma \circ_i M_\tau$, which proves (12) and (10). $\square$

Proof of (i) in Proposition 1.2. Let $I_1 := [1, i-1]$, $I_2 := [i, i+m-1]$, $I_3 := [i+m, n+m-1]$. For $\alpha \in I_1$, $\beta \in I_2$, and $\gamma \in I_3$, let $\tilde{\alpha} := \alpha$, $\tilde{\beta} := \beta - i + 1$, and $\tilde{\gamma} := \gamma - m + 1$. Choose $\alpha_j, \beta_j \in I_j$ for each $j = 1, 2, 3$.

The following assertions are easy to check: for $k, l \neq 2$ and $k \leqslant l$,

$$\begin{aligned}
(\alpha_k, \beta_l) \in \mathrm{Inv}(B_i(\sigma, \tau)) &\iff (\tilde{\alpha}_k, \tilde{\beta}_l) \in \mathrm{Inv}(\sigma)\,,\\
(\alpha_2, \beta_2) \in \mathrm{Inv}(B_i(\sigma, \tau)) &\iff (\tilde{\alpha}_2, \tilde{\beta}_2) \in \mathrm{Inv}(\tau)\,,\\
(\alpha_1, \beta_2) \in \mathrm{Inv}(B_i(\sigma, \tau)) &\iff (\alpha_1, i) \in \mathrm{Inv}(\sigma)\,,\\
(\alpha_2, \beta_3) \in \mathrm{Inv}(B_i(\sigma, \tau)) &\iff (i, \tilde{\beta}_3) \in \mathrm{Inv}(\sigma)\,.
\end{aligned}$$

The result follows. □

1.4. Construction of the map P_i

First we set some notation. Given a sequence of distinct integers $a = (a_1, \ldots, a_n)$, we use $\{a\}$ to denote the underlying set $\{a_1, \ldots, a_n\}$. The *standardization* of a is the unique permutation $\mathrm{st}(a)$ of $[n]$ such that $\mathrm{st}(a)_i < \mathrm{st}(a)_j$ if and only if $a_i < a_j$ for every $i, j \in [n]$. The sequence a is determined by the set $\{a\}$ and the permutation $\mathrm{st}(a)$.

Given two sets of integers A and B and an integer m, we write $A < m$ to indicate that $x < m$ for every $x \in A$, and $A < B$ if $A < y$ for every $y \in B$.

Fix $n, m \geqslant 1$ and $i \in [n]$. All constructions below depend on n, m, and i, even when not explicitly mentioned.

Given $\rho \in S_{n+m-1}$, define three sequences $\mathfrak{L}_j(\rho)$, $j = 1, 2, 3$, by

$$\mathfrak{L}_1(\rho) := (\rho_1, \ldots, \rho_{i-1})\,,\ \mathfrak{L}_2(\rho) := (\rho_i, \ldots, \rho_{i+m-1}),$$
$$\text{and}\ \ \mathfrak{L}_3(\rho) := (\rho_{i+m}, \ldots, \rho_{n+m-1})\,.$$

Next we define a map $P_i : S_{n+m-1} \to S_n \times S_m$ (the map depends on n and m but this is omitted from the notation). Let $\rho \in S_{n+m-1}$. If $m = 1$ we set $P_i(\rho) := (\rho, 1)$.

Assume $m \geqslant 2$. Define

$$u^\rho := \min\{\mathfrak{L}_2(\rho)\} \ \text{ and } \ v^\rho := \max\{\mathfrak{L}_2(\rho)\}\,.$$

Write

$$\{\mathfrak{L}_1(\rho)\} = C^\rho_{1,b} \cup C^\rho_{1,m} \cup C^\rho_{1,t} \ \text{ and } \ \{\mathfrak{L}_3(\rho)\} = C^\rho_{3,b} \cup C^\rho_{3,m} \cup C^\rho_{3,t} \tag{13}$$

with

$$C^\rho_{j,b} < u^\rho < C^\rho_{j,m} < v^\rho < C^\rho_{j,t}$$

for $j = 1, 3$.

Thus $[u^\rho, v^\rho] = C^\rho_{1,m} \cup C^\rho_{3,m} \cup \{\mathfrak{L}_2(\rho)\}$. If $n^\rho_{1,m}$ (resp. $n^\rho_{3,m}$) denotes the number of elements in $C^\rho_{1,m}$ (resp. $C^\rho_{3,m}$), then $v^\rho - u^\rho + 1 = m + n^\rho_{1,m} + n^\rho_{3,m}$.

Define a permutation $B\rho \in S_{n+m-1}$ by $\mathrm{st}(\mathfrak{L}_j(B\rho)) = \mathrm{st}(\mathfrak{L}_j(\rho))$ for $j = 1, 2, 3$ and

$$\begin{aligned}
\{\mathfrak{L}_1(B\rho)\} :=& C^\rho_{1,b} \cup [u^\rho, u^\rho + n^\rho_{1,m} - 1] \cup C^\rho_{1,t}\,,\\
\{\mathfrak{L}_2(B\rho)\} :=& [u^\rho + n^\rho_{1,m}, v^\rho - n^\rho_{3,m}]\,,\\
\{\mathfrak{L}_3(B\rho)\} :=& C^\rho_{3,b} \cup [v^\rho - n^\rho_{3,m} + 1, v^\rho] \cup C^\rho_{3,t}\,.
\end{aligned} \tag{14}$$

Finally, define

$$P_i(\rho) := (\mathrm{st}(\mathcal{L}_1(B\rho), u^\rho + n^\rho_{1,m}, \mathcal{L}_3(B\rho)), \mathrm{st}(\mathcal{L}_2(\rho)))\,. \tag{15}$$

By construction,

$$B_i(P_i(\rho)) = B\rho\,. \tag{16}$$

Lemma 1.3. *The map P_i is order-preserving. In addition,*

$$B_i \circ P_i \leqslant Id \quad \textit{and} \quad P_i \circ B_i = Id\,.$$

Proof. Given ρ in S_{n+m-1} and $h \in \{1, 3\}$, let

$$I^h_\rho := \{k_h \in I_h \mid \exists\, i_2, j_2 \in I_2, \rho(i_2) < \rho(k_h) < \rho(j_2)\} = \{k_h \in I_h \mid \rho(k_h) \in C^\rho_{h,m}\}$$

and

$$J_\rho := \mathrm{Inv}(\rho) \cap (I^1_\rho \times I_2 \cup I_2 \times I^3_\rho \cup I^1_\rho \times I^3_\rho).$$

Note that if $\rho = B_i(\sigma, \tau)$ then $J_\rho = \emptyset$.

By (13) and (14) we have

$$\mathrm{Inv}(B_i P_i(\rho)) = \mathrm{Inv}(B\rho) = \mathrm{Inv}(\rho) - J_\rho\,, \tag{17}$$

which implies

$$B_i P_i(\rho) \leqslant \rho.$$

Let $\rho' \in S_{n+m-1}$ be such that $\rho \leqslant \rho'$. Let (k, l) be in $J_{\rho'} \cap \mathrm{Inv}(\rho)$. If $k \in I^1_{\rho'}$, there exist $i_2, j_2 \in I_2$ such that $\rho'(i_2) < \rho'(k) < \rho'(j_2)$. Hence, $(k, j_2) \notin \mathrm{Inv}(\rho')$, so $(k, j_2) \notin \mathrm{Inv}(\rho)$. But $(k, l) \in \mathrm{Inv}(\rho)$ implies $\rho(l) < \rho(k) < \rho(j_2)$ hence $k \in I^1_\rho$ if $l \in I_2$. If $l \in I^3_{\rho'}$, there exists $j \in I_2$ such that $\rho'(j) < \rho'(l)$, hence $\rho(j) < \rho(l) < \rho(k) < \rho(j_2)$ and $k \in I^1_\rho$. The same argument applies to l: if $l \in I^3_{\rho'}$ then $l \in I^3_\rho$ when $k \in I^1_\rho \cup I_2$. As a consequence, $J_{\rho'} \cap \mathrm{Inv}(\rho) \subseteq J_\rho$ and $\mathrm{Inv}(B_i P_i(\rho)) \subseteq \mathrm{Inv}(B_i P_i(\rho'))$. By (11) we get that P_i is order-preserving.

If $\rho = B_i(\sigma, \tau)$ then $I^1_\rho = I^3_\rho = \emptyset$ and $\mathrm{Inv}(B_i P_i(\rho)) = \mathrm{Inv}(\rho)$. It follows that $B_i P_i(\rho) = B_i(\sigma, \tau)$, and the injectivity of B_i implies

$$P_i \circ B_i = Id\,.$$

□

Proof of (ii) in Proposition 1.2. Assume $B_i(\sigma, \tau) \leqslant \rho$. Since P_i is order-preserving and $P_i B_i = Id$ one obtains $(\sigma, \tau) \leqslant P_i(\rho)$. Conversely, if the latter inequality is satisfied, then using that B_i is order preserving and that $B_i P_i \leqslant Id$ one obtains the former inequality. □

1.5. Construction of the map T_i

We proceed to define a map $T_i : S_n \times S_m \to S_{n+m-1}$. Let $\sigma \in S_n$ and $\tau \in S_m$. If $m = 1$ we set $T_i(\sigma, 1) := \sigma$.

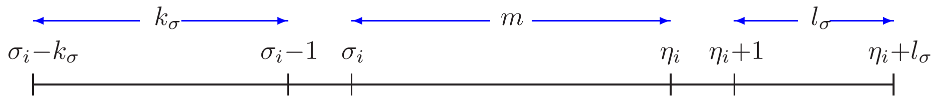
Assume $m > 1$. Let $L_j := \mathcal{L}_j(B_i(\sigma, \tau))$ for $j = 1, 2, 3$. We will define $T_i(\sigma, \tau)$ by specifying the three sequences $\mathcal{L}_j(T_i(\sigma, \tau))$, $j = 1, 2, 3$.

Let $\eta_i := \sigma_i + m - 1$. Define

$$k_\sigma := \max\bigl\{j \in [0, i-1] \mid [\sigma_i - j, \sigma_i] \subseteq \{L_1\} \cup \{\sigma_i\}\bigr\},$$
$$l_\sigma := \max\bigl\{j \in [0, n-i] \mid [\eta_i, \eta_i + j] \subseteq \{\eta_i\} \cup \{L_3\}\bigr\}.$$

The numbers k_σ and l_σ depend only on σ (and m and i), but not on τ. The interval $[\sigma_i - k_\sigma, \sigma_i - 1]$ consists of the elements of $\{L_1\}$ that immediately precede σ_i, and $[\eta_i + 1, \eta_i + l_\sigma]$ consists of the elements of $\{L_3\}$ that immediately follow η_i. The remaining elements of $\{L_1\}$ fall into two classes: $A^\sigma_{1,b}$, which consists of elements smaller than $\sigma_i - k_\sigma$, and $A^\sigma_{1,t}$, which consists of elements bigger than σ_i. Similarly, the remaining elements of $\{L_3\}$ fall into two classes: $A^\sigma_{3,b}$, which consists of elements smaller than η_i, and $A^\sigma_{3,t}$, which consists of elements bigger than $\eta_i + l_\sigma$.

The interval $[\sigma_i - k_\sigma, \eta_i + l_\sigma]$ is thus partitioned as follows:



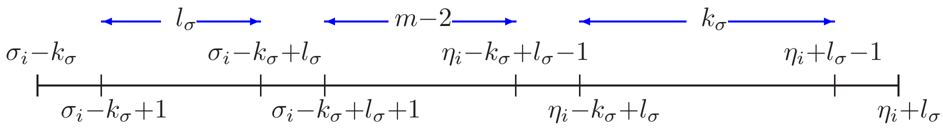
The numbers on top indicate the cardinality of each interval.

It follows from (4) that $A^\sigma_{3,b}$ must consist of elements smaller than $\sigma_i - k_\sigma$, and $A^\sigma_{1,t}$ must consist of elements bigger than $\eta_i + l_\sigma$. Thus, we have

$$\begin{aligned}\{L_1\} &= A^\sigma_{1,b} \cup [\sigma_i - k_\sigma, \sigma_i - 1] \cup A^\sigma_{1,t}\,,\\ \{L_2\} &= [\sigma_i, \eta_i]\,,\\ \{L_3\} &= A^\sigma_{3,b} \cup [\eta_i + 1, \eta_i + l_\sigma] \cup A^\sigma_{3,t}\,,\end{aligned} \tag{18}$$

with $A^\sigma_{1,b}$, $A^\sigma_{3,b} < \sigma_i - k_\sigma$ and $\eta_i + l_\sigma < A^\sigma_{1,t}$, $A^\sigma_{3,t}$.

Consider now the following partition of the interval $[\sigma_i - k_\sigma, \eta_i + l_\sigma]$:



We define $T_i(\sigma, \tau)$ as the unique permutation in S_{n+m-1} with sequences $L'_j := \mathcal{L}_j(T_i(\sigma, \tau))$ determined by

$$\mathrm{st}(L'_j) = \mathrm{st}(L_j)$$

for $j \in \{1, 2, 3\}$, and

$$\begin{aligned}\{L'_1\} &= A^\sigma_{1,b} \cup [\eta_i + l_\sigma - k_\sigma, \eta_i + l_\sigma - 1] \cup A^\sigma_{1,t}\,,\\ \{L'_2\} &= \{\sigma_i - k_\sigma\} \cup [\sigma_i - k_\sigma + l_\sigma + 1, \eta_i + l_\sigma - k_\sigma - 1] \cup \{\eta_i + l_\sigma\}\,,\\ \{L'_3\} &= A^\sigma_{3,b} \cup [\sigma_i - k_\sigma + 1, \sigma_i - k_\sigma + l_\sigma] \cup A^\sigma_{3,t}\,.\end{aligned} \tag{19}$$

Here is an example. Let $\sigma = (5, 8, 2, 4, 6, 1, 7, 3)$, $\tau = (2, 4, 3, 1)$, and $i = 5$. We have

$$B_5(\sigma, \tau) = (5, 11, 2, 4, 7, 9, 8, 6, 1, 10, 3)\,,$$

and

$$L_1 = (5, 11, 2, 4)\,, \quad L_2 = (7, 9, 8, 6)\,, \quad \text{and} \quad L_3 = (1, 10, 3)\,.$$

Since $\sigma_5 = 6$ and $\eta_5 = 9$, we have

$$\{L_1\} = \{2\} \cup [4, 5] \cup \{11\}\,, \quad \{L_2\} = [6, 9]\,, \quad \text{and} \quad \{L_3\} = \{1, 3\} \cup [10, 10] \cup \emptyset\,.$$

Hence $k_\sigma = 2$, $l_\sigma = 1$, and

$$\{L'_1\} = \{2\} \cup [8, 9] \cup \{11\}\,, \quad \{L'_2\} = \{4\} \cup [6, 7] \cup \{10\}\,, \quad \text{and} \quad \{L'_3\} = \{1, 3\} \cup [5, 5] \cup \emptyset\,.$$

Finally,

$$L'_1 = (8, 11, 2, 9)\,, \quad L'_2 = (6, 10, 7, 4)\,, \quad \text{and} \quad L'_3 = (1, 5, 3)\,,$$

so

$$T_5(\sigma, \tau) = (8, 11, 2, 9, 6, 10, 7, 4, 1, 5, 3)\,.$$

Lemma 1.4. *We have*

$$B_i \circ T_i = Id \quad \textit{and} \quad Id \leqslant T_i \circ B_i\,.$$

Proof. We compare $B_i \circ P_i \circ T_i$ to B_i. Given $(\sigma, \tau) \in S_n \times S_m$, it suffices to compare $B\rho$ to $B_i(\sigma, \tau)$ with $\rho := T_i(\sigma, \tau)$, in view of (16). Since the standardization of the lists $\mathfrak{L}_j$ for these two permutations are the same, it suffices to compare the sets $\{\mathfrak{L}_j(B\rho)\}$ and $\{\mathfrak{L}_j(B_i(\sigma, \tau)\}$. These sets coincide in view of (18), (19), (13), and (14). Since B_i is injective, it follows that

$$P_i \circ T_i = Id\,.$$

With the notation of (18), let

$$\begin{aligned} I_1^\sigma &= \{\alpha \in I_1 \mid \sigma_i - k_\sigma \leqslant B_i(\sigma, \tau)(\alpha) \leqslant \sigma_i - 1\}\,, \\ I_3^\sigma &= \{\beta \in I_3 \mid \eta_i + 1 \leqslant B_i(\sigma, \tau)(\beta) \leqslant \eta_i + l_\sigma\}\,, \\ K_{B_i(\sigma,\tau)} &= \{(\alpha, \beta) \in I_1^\sigma \times I_2 \mid B_i(\sigma, \tau)(\beta) < \eta_i\} \\ &\quad \cup \{(\alpha, \beta) \in I_2 \times I_3^\sigma \mid \sigma_i < B_i(\sigma, \tau)(\alpha)\} \cup I_1^\sigma \times I_3^\sigma\,. \end{aligned}$$

The inversion set of $T_i(\sigma, \tau)$ is

$$\operatorname{Inv}(T_i(\sigma, \tau)) = \operatorname{Inv}(B_i(\sigma, \tau)) \cup K_{B_i(\sigma,\tau)}\,. \tag{20}$$

In view of (17) and (20), to show that $T_i \circ P_i \geqslant Id$ it suffices to prove that $J_\rho \subseteq K_{B_iP_i(\rho)}$. Recall from (16) that $B_iP_i(\rho) = B\rho$. Let $(\sigma, \tau) := P_i(\rho)$. Then $k_\sigma \geqslant n^\rho_{1,m}$ and $l_\sigma \geqslant n^\rho_{3,m}$. These imply that if $j \in I^\alpha_\rho$ then $j \in I^\sigma_\alpha$. Assume $(j, l) \in I^1_\rho \times I_2$ is an inversion for ρ. Then $\rho(l) < \rho(j) < v^\rho$ and $B\rho(l) < v^\rho - n^\rho_{3,m} = \sigma_i + m - 1 = \eta_i$. It follows that (j, l) lies in $K_{B\rho}$. The same argument holds if $(j, l) \in I_2 \times I^3_\rho$. Hence $J_\rho \subseteq K_{B_iP_i(\rho)}$ and

$$T_i \circ P_i \geqslant Id.$$

□

Proof of (iii) in Proposition 1.2. If $B_i(\sigma, \tau) \leqslant \rho \leqslant T_i(\sigma, \tau)$, then applying the order-preserving map P_i and using $P_iB_i = P_iT_i = Id$ one gets $P_i(\rho) = (\sigma, \tau)$. Conversely, if $P_i(\rho) = (\sigma, \tau)$ then $B_iP_i(\rho) \leqslant \rho \leqslant T_iP_i(\rho)$. □

2. A filtration of the non-symmetric associative operad

The space $H := \bigoplus_{n \geqslant 1} \Bbbk S_n$ carries a graded Hopf algebra structure, first introduced by Malvenuto and Reutenauer [**17**]. The component of degree n is $\Bbbk S_n$. We are interested in the graded coalgebra structure, which is defined for $\sigma \in S_n$ by

$$\Delta(F_\sigma) = \sum_{i=1}^{n-1} F_{\mathrm{st}(\sigma_1, \dots, \sigma_i)} \otimes F_{\mathrm{st}(\sigma_{i+1}, \dots, \sigma_n)} \,. \tag{21}$$

(See 1.4 for the notion of standardization.) This structure is studied in various recent works, including [**1, 7, 15, 17**] (these references deal with the counital version of this coalgebra, which is obtained by adding a copy of the base ring in degree 0 and adding the terms $1 \otimes F_\sigma$ and $F_\sigma \otimes 1$ to (21)).

We use $\sigma^i_{(1)}$ and $\sigma^i_{(2)}$ to denote the permutations $\mathrm{st}(\sigma_1, \dots, \sigma_i)$ and $\mathrm{st}(\sigma_{i+1}, \dots, \sigma_n)$, respectively. In particular, $\sigma^0_{(2)} = \sigma$ and $\sigma^n_{(1)} = \sigma$.

The iterated coproducts are defined by

$$\Delta^{(1)} := \Delta \quad \text{and} \quad \Delta^{(k+1)} := (\Delta \otimes id^{\otimes k}) \circ \Delta^{(k)} \,.$$

Let $k \geqslant 1$. The k-th component of the *coradical filtration* is

$$H^{(k)} := \ker(\Delta^{(k)} : H \to H^{\otimes(k+1)}) \,.$$

The first component $H^{(1)}$ is the space of primitive elements of H. Note that $H^{(1)} \subseteq H^{(2)} \subseteq H^{(3)} \subseteq \cdots$.

Since Δ is degree-preserving, each space $H^{(k)}$ is graded by setting $H^{(k)}_n := H^{(k)} \cap \Bbbk S_n$. For $k \geqslant 0$, let $\mathcal{A}^{(k)}$ denote the sequence of spaces $\{H^{(k+1)}_n\}_{n \geqslant 1}$. It makes sense to wonder if $\mathcal{A}^{(0)}$ is a suboperad of the associative operad $\mathcal{A}$. This question is motivated by the well-known fact that $\mathcal{A}^{(0)}$ contains the Lie operad $\mathcal{L}$, a symmetric suboperad of $\mathcal{A}$ (see Section 5.3 for more on this). One readily sees that $\mathcal{A}^{(0)}$ is not stable under the right action of the symmetric groups; hence, $\mathcal{A}^{(0)}$ is not a symmetric suboperad of $\mathcal{A}$. However, we show below that it is a non-symmetric suboperad. More generally, we show that the sequence $\{\mathcal{A}^{(k)}\}_{k \geqslant 0}$ is a filtration of the non-symmetric operad $\mathcal{A}$.

Theorem 2.1. *The sequence $\{\mathcal{A}^{(k)}\}_{k \geqslant 0}$ is a filtration of the non-symmetric associative operad $\mathcal{A}$, i.e.,*

$$\mathcal{A}^{(k)}_n \circ_i \mathcal{A}^{(h)}_m \subseteq \mathcal{A}^{(k+h)}_{n+m-1} \quad \textit{for any } n, m \geqslant 1,\ k, h \geqslant 0,\ \textit{and } i = 1, \dots, n. \tag{22}$$

In particular, the space of primitive elements $\mathcal{A}^{(0)}$ is a non-symmetric suboperad of $\mathcal{A}$.

We do not have a simple description of the algebras over the non-symmetric operad $\mathcal{A}^{(0)}$.

We will deduce Theorem 2.1 from Theorem 1.1. For an alternative proof, see Remark 2.3.

Our proof of Theorem 2.1 makes use of an explicit description of the coradical filtration in terms of the M-basis found in [**1**]. A permutation $\sigma \in S_n$ has a global descent at a position $p \in [n{-}1]$ if

$$\forall\ i \leqslant p \text{ and } j \geqslant p{+}1\,,\ \ \sigma_i > \sigma_j\,.$$

Let $\mathrm{GDes}(\sigma) \subseteq [n{-}1]$ be the set of global descents of σ. Corollary 6.3 in [**1**] states that the set

$$\{M_\sigma \mid \sigma \text{ has at most } k \text{ global descents}\} \tag{23}$$

is a linear basis of $H^{(k+1)}$.

Given a set of integers S and an integer m, $\mathsf{S}+m$ is the set with elements $s+m$ for $s \in \mathsf{S}$.

Lemma 2.2. *For any $\sigma \in S_n$, $\tau \in S_m$, and $i \in [n]$,*

$$\begin{multline}\mathrm{GDes}\big(T_i(\sigma,\tau)\big) \cap \big([1,i-1] \cup [i+m-1,n+m-2]\big) = \\ (\mathrm{GDes}(\sigma) \cap [1,i-1]) \cup \big((\mathrm{GDes}(\sigma) \cap [i,n-1]) + m-1\big), \end{multline} \tag{24}$$

$$\mathrm{GDes}\big(T_i(\sigma,\tau)\big) \cap [i,i+m-2] = \begin{cases} \mathrm{GDes}(\tau)+i-1 & \textit{if } A^\sigma_{1,b}=\emptyset \textit{ and } A^\sigma_{3,t}=\emptyset\,,\\ \emptyset & \textit{if not.}\end{cases} \tag{25}$$

Proof. Equation (24) follows easily from the definition of T_i. Let $\rho := T_i(\sigma,\tau)$. Let $j \in [i,i{+}m{-}2]$ be a global descent of ρ. There exist $\alpha \in [i,j]$ and $\beta \in [j{+}1,i{+}m{-}1]$ such that $\rho(\alpha) = \eta_i + l_\sigma$ and $\rho(\beta) = \sigma_i - k_\sigma$. Since j is a global descent, one has

$$\rho(u) > \sigma_i - k_\sigma \quad\text{and}\quad \rho(v) < \eta_i + l_\sigma$$

for all $u \in [1,i-1]$ and $v \in [i{+}m,n{+}m{-}1]$. Hence $A^\sigma_{1,b}=\emptyset$, $A^\sigma_{3,t}=\emptyset$ and $j-i+1$ is a global descent of τ. Conversely, assume that $A^\sigma_{1,b}=\emptyset$ and $A^\sigma_{3,t}=\emptyset$, and let j be a global descent of τ. There exist $\alpha \in [1,j]$ and $\beta \in [j+1,m]$ such that $\tau(\alpha)=m$ and $\tau(\beta)=1$, so $\rho(\alpha{+}i{-}1) = \eta_i + l_\sigma$ and $\rho(\beta{+}i{-}1) = \sigma_i - k_\sigma$. Then, using (19),

$$\forall\, s \leqslant j+i-1,\ \rho(s) > \rho(\beta+i-1) \implies \rho(s) > \sigma_i - k_\sigma + l_\sigma \text{ and}$$

$$\forall\, t > j+i-1,\ \rho(t) < \rho(\alpha+i-1) \implies \rho(t) < \eta_i - k_\sigma + l_\sigma$$

imply

$$\forall\, s \leqslant j+i-1,\ \rho(s) > \{\mathcal{L}_3(\rho)\} \text{ and } \forall t > j+i-1, \{\mathcal{L}_1(\rho)\} > \rho(t).$$

Since $\{\mathcal{L}_1(\rho)\} > \{\mathcal{L}_3(\rho)\}$, the identity (25) follows. □

Proof of Theorem 2.1. It follows from (25) that the number of global descents of $T_i(\sigma,\tau)$ is at most the sum of the numbers of global descents of σ and τ. Thus, if σ has at most k global descents and τ has at most h global descents then $T_i(\sigma,\tau)$ has at most $k{+}h$ global descents. Now, the map GDes is an order-preserving application between the weak order on S_n and the poset of subsets of $[n-1]$ under inclusion [**1**, Proposition 2.13]. Therefore, the number of global descents of ρ is at most $k+h$ for any permutation $\rho \leqslant T_i(\sigma,\tau)$. Hence, all permutations appearing in the right hand side of (10) have at most $k+h$ global descents, which proves (22), in view of (23). □

As already mentioned, the set $\{M_\sigma \mid \sigma$ has at most k global descents$\}$ is a linear basis of $\mathcal{A}_*^{(k)}$. In this sense, the operad $\mathcal{A}$ is filtered by the number of global descents. Note that this is *not* an operad grading though. Referring to the calculation of $M_{(1,2,3)} \circ_2 M_{(2,1)}$, we see that $(1,2,3)$ has 0 global descents, $(2,1)$ has 1, and the first four permutations appearing in the right hand side of (9) have 0 global descents, while the last one has 1.

Remark 2.3. The referee pointed out that a different and direct proof of Theorem 2.1 is possible, along the following lines.

View $\sigma \in S_n$ as the list $(\sigma(1), \dots, \sigma(n))$. According to (21), each summand in the coproduct $\Delta(F_\sigma)$ is obtained by cutting the list between two entries and standardizing the labels of the two resulting lists. One such cut is illustrated below, for $\sigma = (2,1,6,5,7,2,4)$:

$$(2,1,6\,|\,5,7,3,8,4) \xrightarrow{\text{cut}} \mathrm{st}(2,1,6) = (2,1,3)\,, \quad \mathrm{st}(5,7,3,8,4) = (3,4,1,5,2)\,.$$

The iterated coproduct $\Delta^{(k)}(F_\sigma)$ is obtained similarly, by making k distinct cuts in the list of entries of σ in all possible ways, as illustrated below for $k = 3$:

$$(2,1,6\,|\,5,7\,|\,3\,|\,8,4) \xrightarrow{\text{cut}} \mathrm{st}(2,1,6) = (2,1,3)\,,\ \mathrm{st}(5,7) = (1,2)\,,$$
$$\mathrm{st}(3) = (1)\,,\ \mathrm{st}(8,4) = (2,1)\,.$$

Let $\tau \in S_m$, and $1 \leqslant i \leqslant n$. According to (6), the operadic composition $F_\sigma \circ_i F_\tau$ is equal to $F_{B_i(\sigma,\tau)}$, where $B_i(\sigma,\tau)$ is obtained by inserting τ in place of the i-th entry of σ, and then relabeling the entries according to (4). It is not hard to see that first making cuts among entries of σ and then inserting τ yields the same result as first inserting τ and then making cuts among entries of σ (the relabeling produced by inserting is compatible with standardization). This is illustrated below in the running example, with $i = 4$ and $\tau = (2,1)$.

$$(2,1,6,5,7,3,8,4) \xrightarrow{\text{insert}(2,1)} (2,1,7\,|6,5,8\,|\,3|\,9,4) \xrightarrow{\text{cut}} (2,1,3), (2,1,3), (1), (2,1);$$

$$(2,1,6|\,5,7|3|\,8,4) \xrightarrow{\text{cut}} (2,1,3), (1,2), (1), (2,1) \xrightarrow{\text{insert}(2,1)} (2,1,3), (2,1,3), (1), (2,1).$$

Similarly, if the cuts are to be made among the entries of τ, doing it before or after inserting τ in σ leads to the same result.

Now suppose that $x \in \mathcal{A}_n^{(k)}$ and $y \in \mathcal{A}_m^{(h)}$, so that

$$\Delta^{(k+1)}(x) = 0 \quad \text{and} \quad \Delta^{(h+1)}(y) = 0\,.$$

Then $\Delta^{(k+h+1)}(x \circ_i y)$ is a sum of terms of the form $B_i(\sigma,\tau)$ in which $h+k+1$ distinct cuts are made. In any such term, at least $k+1$ cuts are made between entries of x, or (exclusively) at least $h+1$ cuts are made between entries of y. By the argument in the preceding paragraph, the sum of the terms with at least $k+1$ cuts between entries of x admits $\Delta^{(k+1)}(x)$ as a factor, and is therefore 0, and the sum of the terms with at least $h+1$ cuts between entries of y admits $\Delta^{(h+1)}(y)$ as a factor, and so is also 0. Since this accounts for all terms in $\Delta^{(k+h+1)}(x \circ_i y)$, we have that $\Delta^{(k+h+1)}(x \circ_i y) = 0$. Thus $x \circ_i y \in \mathcal{A}_{n+m-1}^{(k+h)}$ and Theorem 2.1 is proven.

In [**11**, Theorem 3.2.2], it is shown that one can associate a coalgebra to any *Hopf operad*; when applied to the associative operad $\mathcal{A}$, this construction produces the coalgebra H. The referee pointed out that the preceding proof can be extended to this context, resulting in the fact that the coradical filtration is a non-symmetric operadic filtration.

3. Quotient operad: descent sets

There is a combinatorial procedure for constructing a subset of $[n-1]$ from a permutation of $[n]$. In this section we show that it leads to an operad quotient of the non-symmetric associative operad $\mathcal{A}$.

A permutation $\sigma \in S_n$ has a descent at a position $p \in [n-1]$ if

$$\sigma_p > \sigma_{p+1}\,.$$

Let $\mathrm{Des}(\sigma) \subseteq [n-1]$ be the set of descents of σ. Note that $\mathrm{GDes}(\sigma) \subseteq \mathrm{Des}(\sigma)$.

For each $n \geqslant 1$, let Q_n denote the poset of subsets of $[n-1]$ under inclusion (the Boolean poset). The map $\mathrm{Des} : S_n \to Q_n$ is an order-preserving application from the weak order on S_n to the Boolean poset Q_n. Let $\mathfrak{Q}$ denote the sequence of spaces $\{\Bbbk Q_n\}_{n\geqslant 1}$. The basis element of $\Bbbk Q_n$ corresponding to a subset $\mathsf{S} \subseteq [n-1]$ is denoted F_{S}.

Given a set of integers S and an integer p, let

$$\mathsf{S} + p := \{s + p \mid s \in \mathsf{S}\}\,.$$

Given $\mathsf{S} \subseteq [n-1]$, $\mathsf{T} \subseteq [m-1]$, and $i \in [n]$, let

$$B_i(\mathsf{S},\mathsf{T}) := (\mathsf{S}\cap\{1,\ldots,i-1\})\cup(\mathsf{T}+i-1)\cup(\mathsf{S}\cap\{i,\ldots,n-1\}+m-1) \subseteq [n+m-2]\,. \tag{26}$$

Lemma 3.1. *For any $\sigma \in S_n$, $\tau \in S_m$, and $i \in [n]$,*

$$\mathrm{Des}\big(B_i(\sigma,\tau)\big) = B_i\big(\mathrm{Des}(\sigma),\mathrm{Des}(\tau)\big)\,. \tag{27}$$

Proof. Write $B_i(\sigma,\tau) = (a_1,\ldots,a_{i-1},b_1,\ldots,b_m,a_{i+1},\ldots,a_n)$, as in (3). Since

$$\mathrm{st}(a_1,\ldots,a_{i-1}) = \mathrm{st}(\sigma_1,\ldots,\sigma_{i-1}), \qquad \mathrm{st}(a_{i+1},\ldots,a_n) = \mathrm{st}(\sigma_{i+1},\ldots,\sigma_n)$$

$$\text{and} \quad \mathrm{st}(b_1,\ldots,b_m) = \mathrm{st}(\tau_1,\ldots,\tau_m),$$

the two sets appearing in (27) contain the same elements from

$$[1,i-2] \cup [i,i+m-2] \cup [i+m,n+m-2]\,.$$

If $i-1 \in \mathrm{Des}\big(B_i(\sigma,\tau)\big)$ then $a_{i-1} > b_1$, that is, by (4), $a_{i-1} > \tau_1 + \sigma_i - 1 \geqslant \sigma_i$. This can only occur if $\sigma_{i-1} > \sigma_i$, i.e., $i-1 \in \mathrm{Des}(\sigma)$. Also,

$$\begin{aligned} i+m-1 \in \mathrm{Des}(B_i(\sigma,\tau)) &\iff b_m = \tau_m + \sigma_i - 1 > a_{i+1} \iff \sigma_i > \sigma_{i+1} \\ &\iff i \in \mathrm{Des}(\sigma) \iff i+m-1 \in B_i\big(\mathrm{Des}(\sigma),\mathrm{Des}(\tau)\big)\,. \end{aligned}$$

This completes the proof of (27). □

Proposition 3.2. *The sequence $\mathcal{Q}$ is a non-symmetric operad under the structure maps*

$$F_{\mathsf{S}} \circ_i F_{\mathsf{T}} := F_{B_i(\mathsf{S},\mathsf{T})} \,. \tag{28}$$

Moreover, the map $\mathsf{D} : \mathcal{A} \to \mathcal{Q}$, $\mathsf{D}(F_\sigma) := F_{\mathrm{Des}(\sigma)}$, is a surjective morphism of non-symmetric operads.

Proof. According to (6) and Lemma 3.1, $\mathcal{Q}$ inherits the structure of $\mathcal{A}$. Hence $\mathcal{Q}$ is a non-symmetric operad and D a morphism. □

In analogy with (7), the monomial basis of $\Bbbk Q_n$ is defined by

$$M_{\mathsf{S}} := \sum_{\mathsf{S}\subseteq\mathsf{T}} (-1)^{\#(T\setminus\mathsf{S})} F_{\mathsf{T}} \,. \tag{29}$$

For instance, with $n = 4$,

$$M_{\{1\}} = F_{\{1\}} - F_{\{1,2\}} - F_{\{1,3\}} + F_{\{1,2,3\}} \,.$$

By Möbius inversion,

$$F_{\mathsf{S}} = \sum_{\mathsf{S}\subseteq\mathsf{T}} M_{\mathsf{T}} \,. \tag{30}$$

The operad structure of $\mathcal{Q}$ takes the same form in the M-basis as in the F-basis.

Proposition 3.3. *For any $\mathsf{S} \subseteq [n-1]$, $\mathsf{T} \subseteq [m-1]$, and $i \in [n]$,*

$$M_{\mathsf{S}} \circ_i M_{\mathsf{T}} = M_{B_i(\mathsf{S},\mathsf{T})} \,. \tag{31}$$

Proof. Define a map $\tilde{\circ}_i$ by

$$M_{\mathsf{S}} \tilde{\circ}_i M_{\mathsf{T}} := M_{B_i(\mathsf{S},\mathsf{T})} \,.$$

Then, by (30),

$$F_{\mathsf{S}} \tilde{\circ}_i F_{\mathsf{T}} = \sum_{\substack{\mathsf{S}\subseteq\mathsf{S}'\\ \mathsf{T}\subseteq\mathsf{T}'}} M_{\mathsf{S}'} \tilde{\circ}_i M_{\mathsf{T}'} = \sum_{\substack{\mathsf{S}\subseteq\mathsf{S}'\\ \mathsf{T}\subseteq\mathsf{T}'}} M_{B_i(\mathsf{S}',\mathsf{T}')} \,.$$

Suppose one is given $B_i(\mathsf{S},\mathsf{T})$, n, m, and i. An inspection of (26) reveals that one can then determine S and T. Fix S and T. It follows that the map B_i is a bijection between the set

$$\{(\mathsf{S}',\mathsf{T}') \mid \mathsf{S} \subseteq \mathsf{S}' \subseteq [n-1] \text{ and } \mathsf{T} \subseteq \mathsf{T}' \subseteq [m-1]\}$$

and the set

$$\{\mathsf{R} \mid B_i(\mathsf{S},\mathsf{T}) \subseteq \mathsf{R} \subseteq [n+m-2]\} \,.$$

Therefore,

$$F_{\mathsf{S}} \tilde{\circ}_i F_{\mathsf{T}} = \sum_{B_i(\mathsf{S},\mathsf{T})\subseteq\mathsf{R}} M_{\mathsf{R}} = F_{B_i(\mathsf{S},\mathsf{T})} \,.$$

Thus $F_{\mathsf{S}} \tilde{\circ}_i F_{\mathsf{T}} = F_{\mathsf{S}} \circ_i F_{\mathsf{T}}$ and hence also, by linearity, $M_{\mathsf{S}} \tilde{\circ}_i M_{\mathsf{T}} = M_{\mathsf{S}} \circ_i M_{\mathsf{T}}$, which proves (31). □

Corollary 3.4. *The map $F_{\mathsf{S}} \mapsto M_{\mathsf{S}}$ defines an automorphism of the non-symmetric operad $\mathcal{Q}$.*

Proof. This is an immediate consequence of (28) and (31). □

The analogous formula to (31) at the level of the operad $\mathcal{A}$ is (10). The latter involves a sum over an interval, while in the former the interval has degenerated to a single point. For instance, the formula that corresponds to (9) is simply ($n = 3$, $m = 2$)

$$M_{\emptyset} \circ_2 M_{\{1\}} = M_{\{2\}} \,. \tag{32}$$

Note that $F_\sigma \mapsto M_\sigma$ does *not* define an automorphism of the operad $\mathcal{A}$, in contrast to Corollary 3.4. A further comparison between the two formulas leads to the following observations.

First, let us recall the expression of the map D on the M-bases of $\mathcal{A}$ and $\mathcal{Q}$. We say that a permutation $\sigma \in S_n$ is closed if $\mathrm{Des}(\sigma) = \mathrm{GDes}(\sigma)$, as in [**1**, Definition 7.1]. The map Des (and also the map GDes) defines an isomorphism between the subposet of S_n consisting of closed permutations and the Boolean poset Q_n [**1**, Proposition 2.11].

The map D is given as follows [**1**, Theorem 7.3]

$$\mathsf{D}(M_\sigma) \;=\; \begin{cases} M_{\mathrm{Des}(\sigma)} & \text{if } \sigma \text{ is closed,} \\ 0 & \text{if not.} \end{cases} \tag{33}$$

Corollary 3.5. *Let $\sigma \in S_n$, $\tau \in S_m$, and $1 \leqslant i \leqslant n$. If σ and τ are both closed, then there is exactly one closed permutation ρ in the interval $[B_i(\sigma,\tau), T_i(\sigma,\tau)]$. Otherwise, this interval contains no closed permutations.*

Proof. This follows from (10), (31), and (33). □

For instance, the permutations $\sigma = (1,2)$ and $\tau = (2,3,1)$ are closed. We have $B_2(\sigma,\tau) = (1,3,4,2)$, $T_2(\sigma,\tau) = (3,2,4,1)$, and the only closed permutation between these two is $(2,3,4,1)$.

Next, we discuss the analog of the filtration $\mathcal{A}^{(k)}$ of the operad $\mathcal{A}$ (Section 2). In this case there is a stronger result: there is not only a filtration of the operad $\mathcal{Q}$ but a grading.

The space $\bigoplus_{n\geqslant 1} \Bbbk Q_n$ carries a Hopf algebra structure, for which the map $\mathsf{D} : \bigoplus_{n\geqslant 1} \Bbbk S_n \to \bigoplus_{n\geqslant 1} \Bbbk Q_n$ becomes a Hopf algebra surjection [**17**, Theorem 3.3]. This is the Hopf algebra of quasi-symmetric functions. Let $k \geqslant 0$. The $(k+1)$-th component of the coradical filtration of this Hopf algebra has the following linear basis:

$$\{M_{\mathsf{S}} \mid \#\mathsf{S} \leqslant k\} \,.$$

Let $\mathcal{Q}^{(k)}$ denote the corresponding sequence of spaces, i.e., the spaces with linear bases

$$\{M_{\mathsf{S}} \mid \#\mathsf{S} \leqslant k \,,\ \mathsf{S} \subseteq [n-1]\}_{n\geqslant 1}$$

and $\mathcal{Q}^k$ the sequence of spaces with linear bases

$$\{M_{\mathsf{S}} \mid \#\mathsf{S} = k \,,\ \mathsf{S} \subseteq [n-1]\}_{n\geqslant 1} \,.$$

Corollary 3.6. *The sequence $\{\mathcal{Q}^k\}_{k\geqslant 0}$ is a grading of the non-symmetric operad $\mathcal{Q}$.*

Proof. This follows from the fact that $\#B_i(\mathsf{S},\mathsf{T}) = \#\mathsf{S} + \#\mathsf{T}$, which is immediate from (26). ☐

In this sense, the non-symmetric operad $\mathcal{Q}$ is graded by the cardinality of subsets. The associated filtration is $\{\mathcal{Q}^{(k)}\}_{k\geqslant 0}$. The map $\mathsf{D} : \mathcal{A} \to \mathcal{Q}$ sends the filtration $\{\mathcal{A}^{(k)}\}_{k\geqslant 0}$ to the filtration $\{\mathcal{Q}^{(k)}\}_{k\geqslant 0}$ (since a map of connected Hopf algebras preserves the coradical filtrations, or in view of (33)).

Note that the space of primitive elements $\mathcal{Q}^{(0)}$ is a suboperad of $\mathcal{Q}$. Each homogeneous component is one-dimensional, being spanned by $M_{\emptyset_n}$, where $\emptyset_n$ denotes the empty set viewed as a subset of $[n-1]$. The operad structure is simply

$$M_{\emptyset_n} \circ_i M_{\emptyset_m} = M_{\emptyset_{n+m-1}}\,.$$

We now turn to a finer analysis of the map $\mathsf{D} : \mathcal{A} \to \mathcal{Q}$. Since it is a morphism of operads (Proposition 3.2), its kernel is an operadic ideal of $\mathcal{A}$. On the other hand, recalling the coalgebra structure of $\bigoplus_{n\geqslant 1} \Bbbk S_n$, it makes sense to consider the *Hopf kernel* of D, which is the space

$$\mathcal{K}_* = \ker(\mathsf{D}) \cap \ker\big((id \otimes \mathsf{D}) \circ \Delta\big)\,.$$

(This coincides with the standard definition of Hopf kernel [**19**, Chapter 7] in the graded connected case.) This is particularly relevant in view of the fact that there is a vector space decomposition

$$\bigoplus_{n\geqslant 1} \Bbbk S_n \cong \mathcal{K}_* \otimes \Big(\bigoplus_{n\geqslant 1} \Bbbk Q_n\Big)\,.$$

(This follows from the fact that $\mathsf{D} : \bigoplus_{n\geqslant 1} \Bbbk S_n \to \bigoplus_{n\geqslant 1} \Bbbk Q_n$ is a morphism of Hopf algebras which admits a coalgebra splitting, see [**1**, Theorem 8.1].)

Since D is degree-preserving, $\mathcal{K}_*$ is a graded space. Let $\mathcal{K}_n$ denote the homogeneous component of degree n and let $\mathcal{K} := \{\mathcal{K}_n\}_{n\geqslant 1}$ denote the corresponding sequence of spaces. We have the following surprising result.

Proposition 3.7. *The sequence of spaces $\mathcal{K}$ forms a non-symmetric suboperad of $\mathcal{A}$.*

Proof. Let us say that a permutation σ is an *eventual identity* if there exists $k \geqslant 1$ such that

$$\sigma = (*, \dots, *, 1, 2, \dots k)$$

where $*$ stands for an arbitrary value. According to [**1**, Theorem 8.2], the set

$$\{M_\sigma \mid \sigma \text{ is } \textit{not} \text{ an eventual identity}\}$$

is a linear basis of $\mathcal{K}_*$. Let $\rho \in S_{n+m-1}$ and write $P_i(\rho) = (\sigma, \tau)$, where $P_i : S_{n+m-1} \to S_n \times S_m$ is the map of Section 1.3. In view of (12), it suffices to show that if ρ is an eventual identity then at least one of σ and τ are eventual identities. Assume that $\rho = (x_1, \dots, x_{i-1}, y_1, \dots, y_m, x_{i+1}, \dots, x_n)$ is an eventual identity: $\rho = (*, \dots, *, 1, 2, \dots k)$. If $i + m - 1 \leqslant m + n - 1 - k$ then, with the notation of (15),

$\{1,\ldots,k\} \in C^{\rho}_{3,b}$. This implies that $\sigma = (*,\ldots,*,1,2,\ldots k)$. If $i+m-1 > m+n-1-k$ then $(y_1,\ldots,y_m) = (*,\ldots,*,1,2,\ldots k)$ or $(y_1,\ldots,y_m) = (l,l+1,\ldots,k)$. But $\tau = \mathrm{st}(y_1,\ldots,y_m)$ so in either case it is an eventual identity. $\square$

Remark 3.8. We thus have a graded vector space decomposition

$$\mathcal{A}_* \cong \mathcal{K}_* \otimes \mathcal{Q}_* ,$$

in which $\mathcal{K}$ is a suboperad and $\mathcal{Q}$ is a quotient of the non-symmetric operad $\mathcal{A}$. It would be interesting to fully describe the operad structure of $\mathcal{A}$ in terms of $\mathcal{K}$ and $\mathcal{Q}$.

We close this section by discussing descriptions of the non-symmetric operad $\mathcal{Q}$ in terms of other combinatorial objects.

A composition of n is a sequence of positive integers $\alpha = (a_1,\ldots,a_k)$ such that $a_1+\cdots+a_k = n$. There is a standard bijection between compositions of n and subsets of $[n-1]$, that associates the set $\mathsf{S} = \{A_1, A_2, \ldots, A_{k-1}\}$ to $\alpha = (a_1,\ldots,a_k)$, where

$$A_p := a_1 + \cdots + a_p \,. \tag{34}$$

In this situation, the basis element F_{S} may also be denoted F_α. Equation (28) may be reformulated as follows: given compositions $\alpha = (a_1,\ldots,a_h)$ of n and $\beta = (b_1,\ldots,b_k)$ of m, we have

$$F_\alpha \circ_i F_\beta = F_{B_i(\alpha,\beta)}$$

where
$B_i(\alpha,\beta) :=$

$$\begin{cases} (a_1,\ldots,a_\ell, b_1+i-1+A_\ell, b_2,\ldots,b_{k-1}, b_k+A_{\ell+1}-i, a_{\ell+2},\ldots,a_h) & \text{if } k>1\,,\\ (a_1,\ldots,a_\ell, a_{\ell+1}+m-1, a_{\ell+2},\ldots,a_h) & \text{if } k=1\,, \end{cases}$$

where A_p is as in (34) and ℓ is defined by $A_\ell < i \leqslant A_{\ell+1}$.

A simpler description may be obtained by indexing basis elements of $\mathcal{Q}$ with binary strings. Given $\mathsf{S} \subseteq [n-1]$, let

$$\epsilon_i := \begin{cases} + & \text{if } i \in \mathsf{S},\\ - & \text{if } i \notin \mathsf{S}. \end{cases}$$

The map that associates the sequence $(\epsilon_1,\ldots,\epsilon_{n-1})$ to S is a bijection between subsets of $[n-1]$ and binary strings. With this indexing, the operad structure of $\mathcal{Q}$ takes the following form:

$$F_{(\epsilon_1,\ldots,\epsilon_{n-1})} \circ_i F_{(\delta_1,\ldots,\delta_{m-1})} = F_{(\epsilon_1,\ldots,\epsilon_{i-1},\delta_1,\ldots,\delta_{m-1},\epsilon_i,\ldots,\epsilon_{n-1})} \,.$$

Remark 3.9. The associated symmetric operad $\mathcal{SQ}$ is the symmetric operad MN^0 considered by Chapoton in [**6**, Section 2]. It can be seen that an algebra over the operad $\mathcal{Q}$ is a vector space equipped with two associative operations $\cdot$ and $*$ satisfying

$$(a\cdot b) * c = a \cdot (b * c) \quad (a * b)\cdot c = a * (b\cdot c)\,.$$

These objects have been considered by Richter under the name “Doppelalgebren” [**24**] and by Pirashvili [**21**].

4. The quotient operad of planar binary trees

There are combinatorial procedures for constructing a rooted planar binary tree with n internal nodes from a permutation of $[n]$, and a subset of $[n-1]$ from such a tree, which factor the construction of the descent set of a permutation of Section 3. In this section we show that they lead to successive operad quotients of the non-symmetric associative operad.

Let Y_n be the set of rooted, planar binary trees with n internal nodes (and thus $n+1$ leaves), henceforth called simply "trees". If $t \in Y_n$, we write $n := |t|$ and refer to this number as the degree of t. Let 1_0 denote the unique element of Y_0 (the unique tree with one leaf and no root).

Given trees $s \in Y_n$ and $t \in Y_m$, let $s \backslash t \in Y_{n+m}$ be the tree obtained by gluing the root of t to the rightmost branch of s, and $s/t \in Y_{n+m}$ the tree obtained by gluing the root of s to the leftmost branch of t. We also set $1_0 \backslash t = t = t/1_0$. The *grafting* of s and t is the tree $s \vee t \in Y_{n+m+1} := (s/\mathrm{Y})\backslash t = s/(\mathrm{Y}\backslash t)$, where $\mathrm{Y} \in Y_1$ denotes the unique tree with one internal node. Equivalently, $s \vee t$ is obtained by drawing a new root and joining it to the roots of s and t. For illustrations of all these operations, see [**2**, Section 1].

For $n \geqslant 1$, every tree $t \in Y_n$ has a unique decomposition $t = t_l \vee t_r$ with $t_l \in Y_p$, $t_r \in Y_q$, $p, q \geqslant 0$ and $n = p + q + 1$.

We are interested in the map $\lambda : S_n \to Y_n$, as considered in [**14**, Section 2.4] (equivalent constructions are described in [**25**, pp. 23-24] and [**5**, Def. 9.9]). The map λ is defined recursively as follows. We set $\lambda(id_0) = 1_0$. For $n \geqslant 1$, let $\sigma \in S_n$ and $j := \sigma^{-1}(n)$. Let $\sigma_l := \mathrm{st}(\sigma_1, \ldots, \sigma_{j-1})$, $\sigma_r := \mathrm{st}(\sigma_{j+1}, \ldots, \sigma_n)$, and define

$$\lambda(\sigma) := \lambda(\sigma_l) \vee \lambda(\sigma_r)\,. \tag{35}$$

Now, given $s \in Y_n$, $t \in Y_n$, and $1 \leqslant i \leqslant n$, we define a tree $B_i(s,t) \in Y_{n+m-1}$ by the following recursion. Write $s = s_l \vee s_r$ and $t = t_l \vee t_r$. Let $j := |s_l| + 1$. Define

$$B_i(s,t) := \begin{cases} s_l \vee (s_r \circ_{i-j} t) & \text{if } j < i, \\ (s_l/t_l) \vee (t_r \backslash s_r) & \text{if } j = i, \\ (s_l \circ_i t) \vee s_r & \text{if } j > i. \end{cases} \tag{36}$$

Lemma 4.1. *For any $\sigma \in S_n$, $\tau \in S_m$, and $1 \leqslant i \leqslant n$,*

$$\lambda\big(B_i(\sigma,\tau)\big) = B_i\big(\lambda(\sigma), \lambda(\tau)\big)\,. \tag{37}$$

Proof. This may be done by induction. We omit details. □

Let $\mathcal{Y}$ denote the sequence of spaces $\{\Bbbk Y_n\}_{n\geqslant 1}$. The basis element of $\Bbbk Y_n$ corresponding to a tree $t \in Y_n$ is denoted F_t.

Proposition 4.2. *The sequence $\mathcal{Y}$ is a non-symmetric operad under the structure maps*

$$F_s \circ_i F_t := F_{B_i(s,t)}\,. \tag{38}$$

Moreover, the map $\Lambda : \mathcal{A} \to \mathcal{Y}$, $\Lambda(F_\sigma) := F_{\lambda(\sigma)}$, is a surjective morphism of non-symmetric operads.

Proof. This follows from (6) and Lemma 4.1. □

Let $L : Y_n \to Q_n$ be the following map. A tree $t \in Y_n$ has $n+1$ leaves, which we number from 1 to $n-1$ left-to-right, excluding the two outermost leaves. Let $L(t)$ be the set of labels of those leaves that point left. Then $\mathrm{Des} = L \circ \lambda$ [**14**, Section 4.4].

Corollary 4.3. *The map $\mathsf{L} : \mathcal{Y} \to \mathcal{Q}$, $\mathsf{L}(F_t) := F_{L(t)}$, is a surjective morphism of non-symmetric operads. There is a commutative diagram of surjective morphisms of non-symmetric operads*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Lambda"] \arrow[rr, bend right, "\mathsf{D}"'] & \mathcal{Y} \arrow[r, "\mathsf{L}"] & \mathcal{Q} \end{tikzcd}$$

Proof. This follows from Propositions 3.2 and 4.2. □

We refer to [**5**, Sec. 9] for the definition of the *Tamari* partial order on Y_n. The monomial basis of $\Bbbk Y_n$ is defined by

$$M_s := \sum_{s \leqslant t} \mu(s,t) F_t\,, \tag{39}$$

where μ denotes the Möbius function of the Tamari order.

To describe the map Λ on the monomial bases, we need the notion of 132-avoiding permutation. A permutation $\sigma \in S_n$ meets the pattern 132 if there is a triple of indices $1 \leqslant i < j < k \leqslant n$ such that $\sigma(i) < \sigma(k) < \sigma(j)$; i.e., if there is a 3-letter substring $(\sigma_i, \sigma_j, \sigma_k)$ of σ that standardizes to $(1,3,2)$. Otherwise, it is said that σ is 132-avoiding.

The map Λ is given as follows [**2**, Theorem 3.1]

$$\Lambda(M_\sigma) = \begin{cases} M_{\lambda(\sigma)} & \text{if } \sigma \text{ is 132-avoiding,} \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

Let $\gamma : Y_n \to S_n$ be the map denoted Max in [**15**, Definition 2.4]. The image of γ consists of the set of 132-avoiding permutations [**2**, Section 1.2]. In addition, $\lambda \circ \gamma = id$.

Proposition 4.4. *Let $n, m \geqslant 1$ and $1 \leqslant i \leqslant n$ be fixed. There is a unique map $P_i : Y_{n+m-1} \to Y_n \times Y_m$ such that*

$$\begin{tikzcd} Y_{n+m-1} \arrow[r, "\gamma"] \arrow[d, dashed, "P_i"'] & S_{n+m-1} \arrow[d, "P_i"] \\ Y_n \times Y_m \arrow[r, "\gamma\times\gamma"'] & S_n \times S_m \end{tikzcd}$$

commutes.

Proof. Since γ is injective, the claim is equivalent to the following statement: if $P_i(\rho) = (\sigma, \tau)$ and ρ is 132-avoiding, then so are σ and τ. This can be seen from (15). □

The operad structure of $\mathcal{Y}$ takes the following form on the monomial basis.

Proposition 4.5. *For any trees $s \in Y_n$, $t \in Y_m$, and $1 \leqslant i \leqslant n$,*

$$M_s \circ_i M_t = \sum_{P_i(r)=(s,t)} M_r\,. \tag{41}$$

Proof. This follows by applying Λ to (12) for $\sigma = \gamma(s)$ and $\tau = \gamma(t)$, in view of (40) and Proposition 4.4. $\square$

Next we show that each fiber of P_i is an interval for the Tamari order on Y_n, in analogy with the situation for S_n (10) and for Q_n (31). To this end we need to recall one more map relating permutations to trees, the map $\rho : S_n \to Y_n$ defined in [**2**, Section 2]. The key property that relates the maps λ, γ, and ρ is [**2**, Theorem 2.1]

$$\sigma \leqslant \gamma(r) \leqslant \tau \iff \lambda(\sigma) \leqslant r \leqslant \rho(\tau)\,. \tag{42}$$

Let $n, m \geqslant 1$ and $1 \leqslant i \leqslant n$. Define a map $T_i : Y_n \times Y_m \to Y_{n+m-1}$ by means of the second diagram below

$$\begin{tikzcd} Y_n \times Y_m \arrow[r, "\gamma\times\gamma"] \arrow[d, "B_i"'] & S_n \times S_m \arrow[d, "B_i"] \\ Y_{n+m-1} & S_{n+m-1} \arrow[l, "\lambda"] \end{tikzcd} \qquad \begin{tikzcd} Y_n \times Y_m \arrow[r, "\gamma\times\gamma"] \arrow[d, dashed, "T_i"'] & S_n \times S_m \arrow[d, "T_i"] \\ Y_{n+m-1} & S_{n+m-1} \arrow[l, "\rho"] \end{tikzcd} \tag{43}$$

The first diagram commutes by Lemma 4.1 and the fact that $\lambda \circ \gamma = id$.

Corollary 4.6. *For any trees $s \in Y_n$, $t \in Y_m$, and $1 \leqslant i \leqslant n$,*

$$M_s \circ_i M_t = \sum_{B_i(s,t) \leqslant r \leqslant T_i(s,t)} M_r\,. \tag{44}$$

Proof. According to the definition of P_i,

$$P_i(r) = (s,t) \iff \big(\gamma(s), \gamma(t)\big) = P_i\big(\gamma(r)\big)\,.$$

By Proposition 1.2 this is equivalent to

$$B_i\big(\gamma(s), \gamma(t)\big) \leqslant \gamma(r) \leqslant T_i\big(\gamma(s), \gamma(t)\big)\,.$$

In view of (42) this is in turn equivalent to

$$\lambda B_i\big(\gamma(s), \gamma(t)\big) \leqslant r \leqslant \rho T_i\big(\gamma(s), \gamma(t)\big)\,,$$

and by (43), also to

$$B_i(s,t) \leqslant r \leqslant T_i(s,t)\,.$$

Formula (44) is thus equivalent to (41). $\square$

Unlike the case of the operad $\mathcal{Q}$, the interval $[B_i(s,t), T_i(s,t)]$ does not degenerate to a point, in general. For instance,

M $\circ_2$ M $=$ M $+$ M .

Compare with (9) and (32).

Remark 4.7. It can be seen that an algebra over the non-symmetric operad $\mathcal{Y}$ is a vector space equipped with two associative operations $\cdot$ and $*$ satisfying the identity

$$(a \cdot b) * c = a \cdot (b * c)\,.$$

This follows from results of Pirashvili [**21**]. There is a different non-symmetric operad structure on the sequence $\mathcal{Y}$ which was introduced by Loday [**13**] (Loday works directly with the symmetrization of this operad). Algebras over this operad are called *dendriform algebras* and they are studied in [**14, 15**]. The Tamari order is relevant to the structure of free dendriform algebras [**15**] .

5. The Lie operad and Dynkin's idempotent

5.1. The associative operad as a twisted Lie algebra

As in [**15**, Def. 1.9], given permutations $\sigma \in S_n$ and $\tau \in S_m$, let σ/τ and $\sigma\backslash\tau$ be the following permutations in S_{n+m}:

$$\sigma/\tau := (\sigma_1,\, \sigma_2,\, \ldots,\, \sigma_n,\;\; \tau_1 + n,\, \tau_2 + n,\, \ldots,\, \tau_m + n)\,,$$
$$\sigma\backslash\tau := (\sigma_1 + m,\, \sigma_2 + m,\, \ldots,\, \sigma_n + m,\;\; \tau_1,\, \tau_2,\, \ldots,\, \tau_m)\,.$$

The operation $F_\sigma * F_\tau := F_{\sigma/\tau}$ endows $\mathcal{A}_*$ with the structure of a *free twisted associative algebra* in the sense of Barratt [**3**, Definitions 2 and 3]. Define a new operation on $\mathcal{A}_*$ by

$$\{F_\sigma, F_\tau\} := F_{\sigma/\tau} - F_{\sigma\backslash\tau}\,. \tag{45}$$

The operation $\{F_\sigma, F_\tau\}$ endows $\mathcal{A}_*$ with the structure of a *twisted Lie algebra* [**3**, Definitions 4 and 5]. Explicitly, this means that for any $x \in \mathcal{A}_n$, $y \in \mathcal{A}_m$, and $z \in \mathcal{A}_p$, one has

$$\{y, x\} = -\{x, y\} \cdot Z_{m,n}\,,$$
$$\{x, \{y, z\}\} + \{y, \{z, x\}\} \cdot Z_{n,m+p} + \{z, \{x, y\}\} \cdot Z_{n+m,p} = 0\,.$$

Here, $Z_{n,m} := 1_n\backslash 1_m \in S_{m+m}$, and $F_\sigma \cdot \tau := F_{\sigma\cdot\tau}$, where $\sigma \cdot \tau$ denotes the ordinary product of permutations in S_n.

It follows from (6) and the fact that $M_{(1,2)} = F_{(1,2)} - F_{(2,1)}$, that

$$\{x, y\} := (M_{(1,2)} \circ_2 y) \circ_1 x \tag{46}$$

for any $x, y \in \mathcal{A}$. In view of Theorem 1.1, this implies that $\{M_\sigma, M_\tau\}$ is a linear combination of basis elements M_ρ with non-negative integer coefficients. In fact, these structure constants are 0 or 1, and they admit the following description.

Recall that an (n, m)-shuffle is a permutation $\zeta \in S_{n+m}$ such that

$$\zeta_1 < \cdots < \zeta_n \;\text{ and }\; \zeta_{n+1} < \cdots < \zeta_{n+m}\,.$$

Let $\mathrm{Sh}(n, m)$ denote the set of all (n, m)-shuffles. This set forms an interval for the weak order of S_{n+m}. The smallest element is the identity $1_{n+m} := (1, 2, \ldots, n+m)$ and the biggest element is $Z_{n,m}$.

Proposition 5.1. *For any $\sigma \in S_n$ and $\tau \in S_m$,*

$$\{M_\sigma, M_\tau\} = \sum_{\substack{\zeta\in \mathrm{Sh}(n,m)\\ \zeta\neq Z_{n,m}}} M_{\zeta\cdot(\sigma/\tau)}\,. \tag{47}$$

Proof. Define an operation $\{\ ,\ \}'$ by means of (47). Using (8), we calculate

$$\{F_\sigma, F_\tau\}' = \sum_{\sigma\leqslant\sigma',\tau\leqslant\tau'} \{M_\sigma, M_\tau\}' = \sum_{\substack{\sigma\leqslant\sigma',\tau\leqslant\tau'\\ \zeta\in\mathrm{Sh}(n,m),\zeta\neq Z_{n,m}}} M_{\zeta\cdot(\sigma'/\tau')}\,.$$

On the other hand, according to [**1**, Prop. 2.5], the permutations in S_{n+m} that are bigger than σ/τ are precisely those of the form

$$\zeta\cdot(\sigma'/\tau') \text{ for some } \zeta\in\mathrm{Sh}(n,m),\ \sigma\leqslant\sigma'\in S_n\,, \text{ and } \tau\leqslant\tau'\in S_m\,.$$

Note that $\sigma\backslash\tau = Z_{n,m}\cdot(\sigma/\tau)$. Hence, again by [**1**, Prop. 2.5], the permutations that are bigger than $\sigma\backslash\tau$ are those of the form

$$Z_{n,m}\cdot(\sigma'/\tau') \text{ for some } \sigma\leqslant\sigma'\in S_n \text{ and } \tau\leqslant\tau'\in S_m\,.$$

Therefore,

$$\begin{aligned}\{F_\sigma, F_\tau\} = F_{\sigma/\tau} - F_{\sigma\backslash\tau} &= \sum_{\substack{\sigma\leqslant\sigma',\tau\leqslant\tau'\\ \zeta\in\mathrm{Sh}(n,m)}} M_{\zeta\cdot(\sigma'/\tau')} - \sum_{\sigma\leqslant\sigma',\tau\leqslant\tau'} M_{Z_{n,m}\cdot(\sigma'/\tau')}\\ &= \sum_{\substack{\sigma\leqslant\sigma',\tau\leqslant\tau'\\ \zeta\in\mathrm{Sh}(n,m),\zeta\neq Z_{n,m}}} M_{\zeta\cdot(\sigma'/\tau')} \ = \ \{F_\sigma, F_\tau\}'\,.\end{aligned}$$

□

5.2. Primitive elements as a twisted Lie algebra

Let $\mathcal{A}^{(0)}$ be the (non-symmetric) suboperad of the non-symmetric associative operad $\mathcal{A}$ discussed in Section 2. As a vector space, $\mathcal{A}^{(0)}$ consists of the primitive elements of the Hopf algebra $H = \bigoplus_{n\geqslant 1} \Bbbk S_n$. Therefore, $\mathcal{A}^{(0)}_*$ is closed under the commutator bracket

$$[x, y] := xy - yx\,. \tag{48}$$

Here xy denotes the product in the Hopf algebra H of two elements $x, y \in H$.

This operation differs from the operation $\{x, y\}$ of Section 5.1. For instance,

$$[F_1, F_1] = 0 \text{ while } \{F_1, F_1\} = F_{(1,2)} - F_{(2,1)}\,.$$

In view of (46) and the facts that $M_{12}\in\mathcal{A}^{(0)}_*$ and that $\mathcal{A}^{(0)}$ is a suboperad of $\mathcal{A}$, it follows that $\mathcal{A}^{(0)}_*$ is also closed under the operation $\{x, y\}$. Thus $\mathcal{A}^{(0)}_*$ is a twisted Lie subalgebra of $\mathcal{A}_*$. More generally, if $x\in\mathcal{A}^{(k)}_*$ and $y\in\mathcal{A}^{(h)}_*$, then $\{x, y\}\in\mathcal{A}^{(k+h)}_*$, by Theorem 2.1.

5.3. The Lie operad as a twisted Lie algebra

Let $\mathcal{L}ie$ be the symmetric suboperad of $\mathcal{A}s$ generated by the element

$$M_{(1,2)} = F_{(1,2)} - F_{(2,1)}\,.$$

Thus $\mathcal{L}ie_*$ is the smallest subspace of $\mathcal{A}s_*$ containing $M_{(1,2)}$, closed under the operations $\circ_i$ for every i, and closed under the action of S_n by right multiplication. This is the Lie operad. Let $\mathcal{L} := \mathcal{F}\mathcal{L}ie$ be the non-symmetric operad obtained by forgetting the symmetric group actions (Section 1.1).

Since the suboperad $\mathcal{A}^{(0)}$ of $\mathcal{A}$ is not symmetric, we cannot immediately conclude that $\mathcal{L} \subseteq \mathcal{A}^{(0)}$. Nevertheless, Patras and Reutenauer have constructed a Hopf subalgebra of H for which $\mathcal{L}_*$ consists precisely of the primitive elements [**20**]. It follows that $\mathcal{L}$ is a suboperad of $\mathcal{A}^{(0)}$. The inclusions

$$\mathcal{L}_n \subset \mathcal{A}_n^{(0)} \subset \mathcal{A}_n$$

are strict for $n \geqslant 3$. It is known that the dimension of $\mathcal{L}_n$ is $(n-1)!$ [**23**, 5.6.2]. From the description of $\mathcal{A}^{(0)}$ given in Section 2 we see that the dimension of $\mathcal{A}_n^{(0)}$ is the number of permutations with no global descents; it lies between $(n-1)!$ and $n!$. More information on this number is given in [**1**, Cor. 6.4].

In view of (46), $\mathcal{L}_*$ is closed under the operation $\{\ ,\ \}$. Thus $\mathcal{L}_*$ is a twisted Lie subalgebra of $\mathcal{A}_*^{(0)}$ and of $\mathcal{A}_*$. Since the primitive elements of any Hopf algebra are closed under the commutator bracket, $\mathcal{L}_*$ is also closed under the commutator bracket (48).

5.4. Dynkin's idempotent

The classical definition of this particular element of $\mathcal{L}_n$ goes as follows [**23**, Thm. 8.16]. Let V be a vector space. Consider the natural left action of S_n on $V^{\otimes n}$ given by

$$F_\sigma \cdot v_1 \dots v_n := v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(n)}$$

for any $v_1, \dots, v_n \in V$. Let $T(V)$ be the tensor algebra of V, and let $[\mu, \eta] := \mu\eta - \eta\mu$ be the usual commutator bracket on this algebra (not to be confused with the commutator bracket on H mentioned in Section 5.2).

Let θ_n be the unique element in $\Bbbk S_n$ such that for every V and $v_1, \dots, v_n \in V$ we have

$$\theta_n \cdot (v_1 \dots v_n) = \Big[\cdots \big[[v_1, v_2], v_3\big], \cdots, v_n\Big]$$

($n-1$ left nested commutator brackets). Dynkin's idempotent is $\frac{1}{n}\theta_n$ (it is defined when n is invertible in $\Bbbk$). We may reformulate this definition as follows.

Lemma 5.2. *For any $n \geqslant 1$,*

$$\theta_n = \Big\{\cdots \big\{\{F_1, F_1\}, F_1\big\}, \cdots, F_1\Big\} \tag{49}$$

($n-1$ left nested brackets).

Proof. It suffices to show that for any $\sigma \in S_n$, $\tau \in S_m$, and $v_1, \dots, v_{n+m} \in V$,

$$\{F_\sigma, F_\tau\} \cdot (v_1 \dots v_{n+m}) = \left[F_\sigma \cdot (v_1 \dots v_n),\, F_\tau \cdot (v_{n+1} \dots v_{n+m})\right].$$

This follows from the facts that

$$F_{\sigma/\tau} \cdot (v_1 \dots v_{n+m}) = v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(n)} v_{n+\tau^{-1}(1)} \dots v_{n+\tau^{-1}(m)}$$

and

$$F_{\sigma\backslash\tau}\cdot(v_1\dots v_{n+m}) = v_{n+\tau^{-1}(1)}\dots v_{n+\tau^{-1}(m)}v_{\sigma^{-1}(1)}\dots v_{\sigma^{-1}(n)}\,.$$

□

We can now derive a surprisingly simple explicit expression for Dynkin's idempotent in terms of the basis M.

Theorem 5.3. *For any $n \geqslant 1$,*

$$\theta_n = \sum_{\sigma\in S_n,\,\sigma(1)=1} M_\sigma\,. \tag{50}$$

Proof. We argue by induction on n. For $n = 1$, $\theta_1 = F_1 = M_1$.

Assume $n \geqslant 2$. By Lemma 5.2, $\theta_n = \{\theta_{n-1}, M_1\}$. By induction hypothesis and Proposition 5.1, we can conclude

$$\theta_n = \sum_{\substack{\tau\in S_{n-1},\,\tau(1)=1\\ \zeta\in\mathrm{Sh}(n-1,1),\,\zeta\neq Z_{n-1,1}}} M_{\zeta\cdot(\tau/1)}\,.$$

Now, $Z_{n-1,1}$ is the only $(n-1,1)$-shuffle Z such that $Z(n) = 1$. All other $(n-1,1)$-shuffles ζ satisfy $\zeta(1) = 1$. Thus all permutations $\sigma = \zeta\cdot(\tau/1)$ appearing in the above sum satisfy $\sigma(1) = 1$. By uniqueness of the parabolic decomposition, these $(n-1)\cdot(n-2)!$ permutations are all distinct. Hence, they are all the permutations $\sigma \in S_n$ such that $\sigma(1) = 1$. □

Note that any permutation $\sigma \in S_n$ with $\sigma(1) = 1$ has no global descents. This confirms the fact that $\theta_n \in \mathcal{A}_n^{(0)}$.

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