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EQUALIZERS IN THE CATEGORY OF COCOMPLETE COCATEGORIES

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Abstract

We prove existence of equalizers in certain categories of cocomplete cocategories. This allows us to complete the proof of the fact that A_∞-functor categories arise as internal Hom-objects in the category of differential graded cocomplete augmented cocategories.

1. Introduction

We refer to [**11**] and to Lyubashenko-Ovsienko [**20**] for an introduction to A_∞-structures and their links to homological algebra.

The notion of an A_∞-category appeared in Fukaya's work on Floer homology [**3**]. Its relation to mirror symmetry became apparent after Kontsevich's talk at ICM '94 [**14**]. Following Kontsevich, one should consider A_∞-categories as models for noncommutative varieties. This approach is being developed by Kontsevich and Soibelman in [**17**].

For a pair of A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, there is an A_∞-category $A_\infty(\mathcal{A},\mathcal{B})$ whose objects are A_∞-functors and whose morphisms are A_∞-transformations. These A_∞-functor categories have been considered for example by Kontsevich [**15**], Fukaya [**4**], Lefèvre-Hasegawa [**18**], Lyubashenko [**19**]. They provide models for the internal Hom-functor of the homotopy category of differential graded categories (Drinfeld [**2**], Toën [**22**], cf. [**13**] for a survey), where the internal Hom-functor is not a derived functor. Furthermore, as detailed in [**12**], A_∞-functor categories yield a natural construction of the B_∞-structure on the Hochschild complex of an associative algebra (Getzler-Jones [**5**], Kadeishvili [**9**], Voronov-Gerstenhaber [**23**]), which is important for proving Deligne's conjecture and Tamarkin's version of Kontsevich's formality theorem (cf. for example Kontsevich-Soibelman [**16**], Tamarkin [**21**], Hinich [**8**]).

In order to interpret A_∞-functor categories as internal Hom-objects, one passes to a suitable category of cocategories following an idea of Lyubashenko [**19**]. For this suitable category, one can either take the monoidal subcategory generated by the images of graded quivers under the bar construction, as in [**19**], or the category of all (cocomplete augmented etc.) cocategories. The former approach is developed

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further in the forthcoming book by Bespalov, Lyubashenko, and Manzyuk [**1**] using the technique of closed multicategories. The latter approach has been taken by the first author in [**12**]. He proved in Theorem 5.3 of [**12**] that the monoidal category of cocomplete augmented cocategories was closed. However, the proof of the theorem was incomplete: it relied on the assumption that the category of cocomplete augmented cocategories has equalizers. In this paper, we close this gap. Note that it suffices to prove existence of equalizers in the category of cocomplete cocategories since it is equivalent to the category of cocomplete augmented cocategories, see Remark 2.

Theorem 1. *Suppose k is a field. Then the category of cocomplete k-cocategories admits equalizers. The analogous assertions hold in the graded and in the differential graded settings.*

Since the proofs in the three cases are quite similar, we have chosen to present the proof for the ungraded version providing remarks concerning modifications necessary in the other cases. The proof occupies Section 4.

In the case of coalgebras, there is a different proof based on the duality between coalgebras and algebras, successfully applied in works of Kontsevich-Soibelman [**17**] and Hamilton-Lazarev [**7**]. We would like to briefly outline it.[1]

The category of finite dimensional coalgebras is anti-equivalent to the category of finite dimensional algebras. Furthermore, finite dimensional coalgebras are objects of finite presentation, in the terminology of [**10**, Definition 6.3.3], in the category of coalgebras. Since an arbitrary coalgebra is a union of finite dimensional subcoalgebras, see [**6**] or [**17**, Proposition 2.1.2], it follows by [**10**, Proposition 6.3.4] that the category of coalgebras is equivalent to the category of ind-objects in the category of finite dimensional coalgebras. Moreover, a finite dimensional subcoalgebra of a cocomplete coalgebra is conilpotent, therefore the category of cocomplete coalgebras is equivalent to the category of ind-objects in the category of finite dimensional conilpotent coalgebras, which is in turn anti-equivalent to the category of pro-objects in the category of finite dimensional nilpotent algebras (the category of formal algebras in the terminology of [**7**]). It suffices to establish the existence of equalizers in the latter category. However, by the dual of [**10**, Proposition 6.1.16], this follows from the existence of equalizers in the category of finite dimensional nilpotent algebras.

Apparently, with some work the above argument can be generalized to cocategories, although we did not check the details. The only drawback of this approach, in our opinion, is that it is indirect. Our proof relies on a direct verification and yields an explicit description of equalizers, which is necessary in order to compute internal Hom-objects in the category of cocomplete cocategories and to relate these to A_∞-functor categories.

[1]We thank the anonymous referee for drawing our attention to this approach.

2. Preliminaries

Let k be a commutative ring. A *k-quiver* $\mathcal{A}$ consists of a set of objects $\operatorname{Ob}\mathcal{A}$ and of k-modules $\mathcal{A}(X,Y)$, for each pair of objects $X,Y \in \operatorname{Ob}\mathcal{A}$. A *morphism of k-quivers* $f : \mathcal{A} \to \mathcal{B}$ consists of a map $\operatorname{Ob} f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto f(X)$, and of k-linear maps

$$f = f_{X,Y} : \mathcal{A}(X,Y) \to \mathcal{B}(f(X), f(Y)),$$

for each pair of objects $X,Y \in \operatorname{Ob}\mathcal{A}$. Let $\mathscr{Q}$ denote the category of k-quivers. For a set S, denote by $\mathscr{Q}/S$ the subcategory of $\mathscr{Q}$ whose objects are k-quivers $\mathcal{A}$ such that $\operatorname{Ob}\mathcal{A} = S$, and whose morphisms are morphisms of k-quivers $f\colon \mathcal{A} \to \mathcal{B}$ such that $\operatorname{Ob} f = \mathrm{id}_S$. The category $\mathscr{Q}/S$ is monoidal. The tensor product of quivers $\mathcal{A}$ and $\mathcal{B}$ is given by

$$(\mathcal{A} \otimes \mathcal{B})(X,Z) = \bigoplus_{Y \in S} \mathcal{A}(X,Y) \otimes \mathcal{B}(Y,Z), \quad X, Z \in S.$$

The unit object is the *discrete quiver* kS given by $\operatorname{Ob} kS = S$, $kS(X,X) = k$ and $kS(X,Y) = 0$ if $X \neq Y$, $X,Y \in S$. Recall that a *cocategory* $(\mathcal{C}, \Delta)$ is a coassociative coalgebra in the monoidal category $\mathscr{Q}/\operatorname{Ob}\mathcal{C}$. Thus, a cocategory consists of a k-quiver $\mathcal{C}$ and of a morphism $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ in $\mathscr{Q}/\operatorname{Ob}\mathcal{C}$, the comultiplication, satisfying the usual coassociativity condition. For $X,Y,Z \in \operatorname{Ob}\mathcal{C}$, denote by

$$\Delta_{X,Y,Z} = \mathrm{pr}_{X,Y,Z} \circ \Delta : \mathcal{C}(X,Z) \to \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)$$

the components of Δ. Since the k-linear map

$$(\Delta_{X,Y,Z})_{Y \in \operatorname{Ob}\mathcal{C}} : \mathcal{C}(X,Z) \to \prod_{Y \in \operatorname{Ob}\mathcal{C}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)$$

factors as

$$\mathcal{C}(X,Z) \xrightarrow{\Delta} \bigoplus_{Y \in \operatorname{Ob}\mathcal{C}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \hookrightarrow \prod_{Y \in \operatorname{Ob}\mathcal{C}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z),$$

it follows that, for each $t \in \mathcal{C}(X,Z)$, the element $\Delta_{X,Y,Z}(t)$ vanishes for all but finitely many $Y \in \operatorname{Ob}\mathcal{C}$. The coassociativity is expressed by the following equation:

$$\begin{aligned} &\big[\mathcal{C}(W,Z) \xrightarrow{\Delta_{W,X,Z}} \mathcal{C}(W,X) \otimes \mathcal{C}(X,Z) \xrightarrow{1 \otimes \Delta_{X,Y,Z}} \mathcal{C}(W,X) \otimes \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)\big] \\ = &\big[\mathcal{C}(W,Z) \xrightarrow{\Delta_{W,Y,Z}} \mathcal{C}(W,Y) \otimes \mathcal{C}(Y,Z) \xrightarrow{\Delta_{W,X,Y} \otimes 1} \mathcal{C}(W,X) \otimes \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)\big]. \end{aligned}$$

A *cocategory homomorphism* $f\colon (\mathcal{C}, \Delta) \to (\mathcal{D}, \Delta)$ is a morphism of k-quivers $f\colon \mathcal{C} \to \mathcal{D}$ compatible with the comultiplication in the sense of the equation

$$\big[\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\Delta} \mathcal{D} \otimes \mathcal{D}\big] = \big[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{f \otimes f} \mathcal{D} \otimes \mathcal{D}\big],$$

where the morphism $f \otimes f\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{D} \otimes \mathcal{D}$ is given by $\operatorname{Ob} f \otimes f = \operatorname{Ob} f$ and

$$\begin{aligned} (f \otimes f)_{X,Z} = \big[&\bigoplus_{Y \in \operatorname{Ob}\mathcal{C}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \xrightarrow{\bigoplus_{Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y} \otimes f_{Y,Z}} \\ &\bigoplus_{Y \in \operatorname{Ob}\mathcal{C}} \mathcal{D}(f(X), f(Y)) \otimes \mathcal{D}(f(Y), f(Z)) \hookrightarrow \bigoplus_{U \in \operatorname{Ob}\mathcal{D}} \mathcal{D}(f(X), U) \otimes \mathcal{D}(U, f(Z))\big], \end{aligned}$$

for each pair of objects $X, Z \in \operatorname{Ob}\mathcal{C}$. Explicitly, for $X, Z \in \operatorname{Ob}\mathcal{C}$, $U \in \operatorname{Ob}\mathcal{D}$, the following equation holds true:

$$\begin{aligned}&\bigl[\mathcal{C}(X,Z) \xrightarrow{f_{X,Z}} \mathcal{D}(f(X),f(Z)) \xrightarrow{\Delta_{f(X),U,f(Z)}} \mathcal{D}(f(X),U)\otimes\mathcal{D}(U,f(Z))\bigr] \\ &= \sum_{f(Y)=U}^{Y\in\operatorname{Ob}\mathcal{C}} \bigl[\mathcal{C}(X,Z) \xrightarrow{\Delta_{X,Y,Z}} \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z) \\ &\qquad\qquad \xrightarrow{f_{X,Y}\otimes f_{Y,Z}} \mathcal{D}(f(X),U)\otimes\mathcal{D}(U,f(Z))\bigr]. \qquad (1)\end{aligned}$$

In particular, the right hand side vanishes if U is not in the image of f.

Let $\mathcal{C}$ be a cocategory. Let $\Delta^{(n)} : \mathcal{C} \to \mathcal{C}^{\otimes n}$ denote the comultiplication iterated $n-1$ times, so that $\Delta^{(1)} = \mathrm{id}_{\mathcal{C}}$, $\Delta^{(2)} = \Delta$, $\Delta^{(3)} = (\Delta\otimes 1)\circ\Delta = (1\otimes\Delta)\circ\Delta$, and so on. Denote by

$$\begin{aligned}&\Delta^{(n)}_{X_0,\dots,X_n} = \operatorname{pr}_{X_0,\dots,X_n}\circ\Delta^{(n)} : \\ &\qquad\qquad \mathcal{C}(X_0,X_n) \to \mathcal{C}(X_0,X_1)\otimes\mathcal{C}(X_1,X_2)\otimes\dots\otimes\mathcal{C}(X_{n-1},X_n)\end{aligned}$$

the components of $\Delta^{(n)}$, for $X_0,\dots,X_n \in \operatorname{Ob}\mathcal{C}$. Suppose $f : \mathcal{C} \to \mathcal{D}$ is a cocategory homomorphism. By induction on n, it follows that

$$\begin{aligned}&\Delta^{(n)}_{f(X),U_1,\dots,U_{n-1},f(Y)} \circ f_{X,Y} \\ &\quad = \sum_{f(Z_i)=U_i,\ i=1,\dots,n-1}^{Z_1,\dots,Z_{n-1}\in\operatorname{Ob}\mathcal{C}} (f_{X,Z_1}\otimes f_{Z_1,Z_2}\otimes\dots\otimes f_{Z_{n-1},Y})\circ\Delta^{(n)}_{X,Z_1,\dots,Z_{n-1},Y} : \\ &\qquad\qquad \mathcal{C}(X,Y) \to \mathcal{D}(f(X),U_1)\otimes\mathcal{D}(U_1,U_2)\otimes\dots\otimes\mathcal{D}(U_{n-1},f(Y)), \qquad (2)\end{aligned}$$

for an arbitrary collection of objects $X, Y \in \operatorname{Ob}\mathcal{C}$, $U_1,\dots,U_{n-1} \in \operatorname{Ob}\mathcal{D}$.

A cocategory $\mathcal{C}$ is *cocomplete* if, for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{C}$,

$$\mathcal{C}(X,Y) = \bigcup_{n\geqslant 1} \operatorname{Ker}(\Delta^{(n)} : \mathcal{C}(X,Y) \to \mathcal{C}^{\otimes n}(X,Y)).$$

Equivalently, $\mathcal{C}$ is cocomplete if for each $t \in \mathcal{C}(X,Y)$ there is $n \geqslant 1$ such that

$$\Delta^{(n)}_{X,Z_1,\dots,Z_{n-1},Y}(t) = 0,$$

for all $Z_1,\dots,Z_{n-1} \in \operatorname{Ob}\mathcal{C}$.

Example 1. An arbitrary $\Bbbk$-quiver $\mathcal{A}$ gives rise to a cocategory $(T^{\geqslant 1}\mathcal{A}, \Delta)$, where $T^{\geqslant 1}\mathcal{A} = \bigoplus_{n=1}^{\infty} T^n\mathcal{A}$, $T^n\mathcal{A} = \mathcal{A}^{\otimes n}$ is the n-fold tensor product in $\mathscr{Q}/\operatorname{Ob}\mathcal{A}$, and Δ is the cut comultiplication. Thus,

$$T^{\geqslant 1}\mathcal{A}(X,Y) = \bigoplus_{Z_1,\dots,Z_{n-1}\in\operatorname{Ob}\mathcal{A}}^{n\geqslant 1} \mathcal{A}(X,Z_1)\otimes\mathcal{A}(Z_1,Z_2)\otimes\dots\otimes\mathcal{A}(Z_{n-1},Y),$$

for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{A}$, and Δ is given by

$$\Delta(f_1\otimes f_2\otimes\dots\otimes f_n) = \sum_{i=1}^{n-1} f_1\otimes\dots\otimes f_i \bigotimes f_{i+1}\otimes\dots\otimes f_n.$$

Since $\Delta(T^n\mathcal{A}) \subset \bigoplus_{p+q=n}^{p,q>0} T^p\mathcal{A} \otimes T^q\mathcal{A}$, it follows that $T^{\geqslant 1}\mathcal{A}$ is a cocomplete cocategory.

Remark 1. The correspondence $\mathcal{A} \mapsto T^{\geqslant 1}\mathcal{A}$ extends to a functor $T^{\geqslant 1} : \mathscr{Q} \to \mathscr{Q}$. It is proven in [**1**, Chapter 8] that the functor $T^{\geqslant 1}$ admits the structure of a comonad, and that the category of $T^{\geqslant 1}$-coalgebras is isomorphic to the category of cocomplete cocategories.

Remark 2. A cocategory $\mathcal{C}$ is *counital* if it is equipped with a morphism $\varepsilon : \mathcal{C} \to k\operatorname{Ob}\mathcal{C}$ in $\mathscr{Q}/\operatorname{Ob}\mathcal{C}$ such that the two counit equations hold. Note that, for an arbitrary set S, the k-quiver kS admits the natural structure of a counital cocategory, namely the comultiplication is the canonical isomorphism $kS \xrightarrow{\sim} kS \otimes kS$ in $\mathscr{Q}/S$ and the counit is the identity map $kS \to kS$. An *augmented* cocategory is a counital cocategory endowed with a morphism of counital cocategories $\eta : k\operatorname{Ob}\mathcal{C} \to \mathcal{C}$ such that $\operatorname{Ob}\eta = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$ and $\varepsilon \circ \eta = \mathrm{id}_{k\operatorname{Ob}\mathcal{C}}$. A morphism of augmented cocategories is a cocategory homomorphism compatible with the counit and the augmentation. The category of augmented cocategories is equivalent to the category of cocategories [**1**, Lemma 8.12]: given a cocategory $(\mathcal{A}, \Delta)$, there is the natural structure of an augmented cocategory on the k-quiver $T^{\leqslant 1}\mathcal{A} = k\operatorname{Ob}\mathcal{A} \oplus \mathcal{A}$, where the counit and the augmentation are the projection $\varepsilon = \mathrm{pr}_0 : T^{\leqslant 1}\mathcal{A} \to k\operatorname{Ob}\mathcal{A}$ and the inclusion $\eta = \mathrm{in}_0 : k\operatorname{Ob}\mathcal{A} \to T^{\leqslant 1}\mathcal{A}$ respectively, and the comultiplication is given by the formulas

$$\begin{aligned}
\Delta|_{k\operatorname{Ob}\mathcal{A}} &= \big[k\operatorname{Ob}\mathcal{A} \xrightarrow{\sim} k\operatorname{Ob}\mathcal{A} \otimes k\operatorname{Ob}\mathcal{A} \xrightarrow{\mathrm{in}_0 \otimes \mathrm{in}_0} T^{\leqslant 1}\mathcal{A} \otimes T^{\leqslant 1}\mathcal{A}\big], \\
\Delta|_{\mathcal{A}} &= \big[\mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes k\operatorname{Ob}\mathcal{A} \xrightarrow{\mathrm{in}_1 \otimes \mathrm{in}_0} T^{\leqslant 1}\mathcal{A} \otimes T^{\leqslant 1}\mathcal{A}\big] \\
&\quad + \big[\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{in}_1 \otimes \mathrm{in}_1} T^{\leqslant 1}\mathcal{A} \otimes T^{\leqslant 1}\mathcal{A}\big] \\
&\quad + \big[\mathcal{A} \xrightarrow{\sim} k\operatorname{Ob}\mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{in}_0 \otimes \mathrm{in}_1} T^{\leqslant 1}\mathcal{A} \otimes T^{\leqslant 1}\mathcal{A}\big].
\end{aligned}$$

Conversely, given an augmented cocategory $(\mathcal{C}, \Delta, \varepsilon, \eta)$, the reduced k-quiver $\overline{\mathcal{C}} = \operatorname{Ker}\varepsilon$ becomes a cocategory. The functors $\mathcal{A} \mapsto T^{\leqslant 1}\mathcal{A}$ and $\mathcal{C} \mapsto \overline{\mathcal{C}}$ are quasi-inverse equivalences. By definition, an augmented cocategory $\mathcal{C}$ is *cocomplete* if its reduction $\overline{\mathcal{C}}$ is cocomplete. Thus the category of cocomplete cocategories is equivalent to the category of cocomplete augmented cocategories.

The above definitions admit obvious graded and differential graded variants. For instance, a *graded* (resp. *differential graded*) *k-quiver* $\mathcal{A}$ consists of a set of objects $\operatorname{Ob}\mathcal{A}$ and of graded k-modules (resp. cochain complexes of k-modules) $\mathcal{A}(X, Y)$, for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{A}$. A *morphism of graded* (resp. *differential graded*) *k-quivers* $f : \mathcal{A} \to \mathcal{B}$ consists of a map $\operatorname{Ob} f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto f(X)$, and of morphisms of graded k-modules of degree 0 (resp. cochain maps)

$$f = f_{X,Y} : \mathcal{A}(X, Y) \to \mathcal{B}(f(X), f(Y)),$$

for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{A}$. The tensor product of quivers with the same set of objects is defined analogously to the considered case, using tensor product of graded k-modules (resp. of cochain complexes). The definitions of cocategory, cocategory homomorphism etc. are modified accordingly.

3. Subcocategory generated by a set of objects

Let $\mathcal{C}$ be a cocomplete cocategory, $\mathcal{B} \subset \mathcal{C}$ a full subquiver, i.e., $\operatorname{Ob}\mathcal{B} \subset \operatorname{Ob}\mathcal{C}$ and $\mathcal{B}(X,Y) = \mathcal{C}(X,Y)$, for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{B}$. Then, in general, $\mathcal{B}$ is not a cocategory since the comultiplication

$$\Delta : \mathcal{C}(X,Z) \to \bigoplus_{Y\in\operatorname{Ob}\mathcal{C}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)$$

does not take values in $\bigoplus_{Y\in\operatorname{Ob}\mathcal{B}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)$ if $X, Z \in \operatorname{Ob}\mathcal{B}$. However, at least if k is a field, for each subset $S \subset \operatorname{Ob}\mathcal{C}$ there exists a maximal cocomplete subcocategory $\mathcal{C}_S \subset \mathcal{C}$ such that $\operatorname{Ob}\mathcal{C}_S = S$. It is constructed as follows. For a pair of objects $X, Y \in S$, denote

$$\mathcal{N}(X,Y) = \bigoplus_{\exists i:\, Z_i \notin S}^{Z_1,\dots,Z_{n-1}\in\operatorname{Ob}\mathcal{C},\; n\geqslant 1} \mathcal{C}(X,Z_1)\otimes\mathcal{C}(Z_1,Z_2)\otimes\cdots\otimes\mathcal{C}(Z_{n-1},Y),$$

and define a k-linear map $N = N_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{N}(X,Y)$ by

$$N = \sum_{\exists i:\, Z_i \notin S}^{Z_1,\dots,Z_{n-1}\in\operatorname{Ob}\mathcal{C},\; n\geqslant 1} \Delta^{(n)}_{X,Z_1,\dots,Z_{n-1},Y}.$$

For each $t \in \mathcal{C}(X,Y)$, the sum in the right hand side is finite since $\mathcal{C}$ is cocomplete. Let $\mathcal{C}_S(X,Y) = \operatorname{Ker} N$, so that we have an exact sequence of k-vector spaces:

$$0 \to \mathcal{C}_S(X,Y) \xrightarrow{\iota} \mathcal{C}(X,Y) \xrightarrow{N} \mathcal{N}(X,Y).$$

Choose a splitting $\pi : \mathcal{C}(X,Y) \to \mathcal{C}_S(X,Y)$ such that $\pi \circ \iota = \mathrm{id}_{\mathcal{C}_S(X,Y)}$. Suppose that $X, Y, Z \in S$. Then the composite

$$\bigl[\mathcal{C}_S(X,Z) \xrightarrow{\iota} \mathcal{C}(X,Z) \xrightarrow{\Delta_{X,Y,Z}} \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z) \xrightarrow{N\otimes 1} \mathcal{N}(X,Y)\otimes\mathcal{C}(Y,Z)\bigr]$$

vanishes. Indeed, by coassociativity, $(N\otimes 1)\circ\Delta_{X,Y,Z}$ equals

$$\sum_{\exists i:\, Z_i \notin S}^{Z_1,\dots,Z_{n-1}\in\operatorname{Ob}\mathcal{C},\; n\geqslant 1} (\Delta^{(n)}_{X,Z_1,\dots,Z_{n-1},Y}\otimes 1)\circ\Delta_{X,Y,Z}$$
$$= \sum_{\exists i:\, Z_i \notin S}^{Z_1,\dots,Z_{n-1}\in\operatorname{Ob}\mathcal{C},\; n\geqslant 1} \Delta^{(n+1)}_{X,Z_1,\dots,Z_{n-1},Y,Z} :$$
$$\mathcal{C}(X,Z) \to \bigoplus_{\exists i:\, Z_i \notin S}^{Z_1,\dots,Z_{n-1}\in\operatorname{Ob}\mathcal{C},\; n\geqslant 1} \mathcal{C}(X,Z_1)\otimes\mathcal{C}(Z_1,Z_2)\otimes\cdots\otimes\mathcal{C}(Z_{n-1},Y)\otimes\mathcal{C}(Y,Z).$$

It follows from the definition of ι that $\Delta^{(k)}_{X,U_1,\dots,U_{k-1},Y}\circ\iota = 0$, for an arbitrary sequence of objects $U_1,\dots,U_{k-1} \in \operatorname{Ob}\mathcal{C}$ such that $U_i \notin S$ for some i. Therefore $(N\otimes 1)\circ\Delta_{X,Y,Z}\circ\iota = 0$. Since the sequence

$$0 \to \mathcal{C}_S(X,Y)\otimes\mathcal{C}(Y,Z) \xrightarrow{\iota\otimes 1} \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z) \xrightarrow{N\otimes 1} \mathcal{N}(X,Y)\otimes\mathcal{C}(Y,Z)$$

is exact, the composite $\Delta_{X,Y,Z} \circ \iota$ factors through $\mathcal{C}_S(X,Y) \otimes \mathcal{C}(Y,Z)$. In other words, there exists a unique k-linear map $\phi\colon \mathcal{C}_S(X,Z) \to \mathcal{C}_S(X,Y) \otimes \mathcal{C}(Y,Z)$ such that $(\iota \otimes 1) \circ \phi = \Delta_{X,Y,Z} \circ \iota$. Since $\iota \otimes 1$ is an embedding split by $\pi \otimes 1$, the map ϕ is necessarily given by the composite

$$\big[\mathcal{C}_S(X,Z) \xrightarrow{\iota} \mathcal{C}(X,Z) \xrightarrow{\Delta_{X,Y,Z}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \xrightarrow{\pi\otimes 1} \mathcal{C}_S(X,Y) \otimes \mathcal{C}(Y,Z)\big].$$

In particular, the equation $(\iota \otimes 1) \circ \phi = \Delta_{X,Y,Z} \circ \iota$ takes the form

$$(\iota \circ \pi \otimes 1) \circ \Delta_{X,Y,Z} \circ \iota = \Delta_{X,Y,Z} \circ \iota\colon \mathcal{C}_S(X,Z) \to \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z). \tag{3}$$

Similarly, the following equation holds true:

$$(1 \otimes \iota \circ \pi) \circ \Delta_{X,Y,Z} \circ \iota = \Delta_{X,Y,Z} \circ \iota\colon \mathcal{C}_S(X,Z) \to \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z). \tag{4}$$

Combining these equations yields

$$(\iota \otimes \iota) \circ (\pi \otimes \pi) \circ \Delta_{X,Y,Z} \circ \iota = \Delta_{X,Y,Z} \circ \iota\colon \mathcal{C}_S(X,Z) \to \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z).$$

Define

$$\Delta'_{X,Y,Z} = \big[\mathcal{C}_S(X,Z) \xrightarrow{\iota} \mathcal{C}(X,Z) \xrightarrow{\Delta_{X,Y,Z}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \xrightarrow{\pi\otimes\pi} \mathcal{C}_S(X,Y) \otimes \mathcal{C}_S(Y,Z)\big]. \tag{5}$$

Then the above equation is equivalent to $(\iota\otimes\iota)\circ\Delta'_{X,Y,Z} = \Delta_{X,Y,Z}\circ\iota$. Coassociativity of Δ' follows from coassociativity of Δ since

$$\begin{aligned}(\iota \otimes \iota \otimes \iota) \circ (\Delta' \otimes 1) \circ \Delta' &= (\Delta \otimes 1) \circ (\iota \otimes \iota) \circ \Delta' = (\Delta \otimes 1) \circ \Delta \circ \iota \\ &= (1 \otimes \Delta) \circ \Delta \circ \iota = (1 \otimes \Delta) \circ (\iota \otimes \iota) \circ \Delta' = (\iota \otimes \iota \otimes \iota) \circ (1 \otimes \Delta') \circ \Delta',\end{aligned}$$

and $\iota \otimes \iota \otimes \iota$ is an embedding split by $\pi \otimes \pi \otimes \pi$. Thus, $\mathcal{C}_S$ becomes a cocategory. The quiver map $\iota\colon \mathcal{C}_S \to \mathcal{C}$ with $\operatorname{Ob}\iota\colon S \hookrightarrow \operatorname{Ob}\mathcal{C}$ is a cocategory homomorphism. Indeed, it was shown above that equation (1) holds true for $X, Y, Z \in S$. If $X, Z \in S$, $Y \in \operatorname{Ob}\mathcal{C} \smallsetminus S$, then $\Delta_{X,Y,Z} \circ \iota = 0$ by the definition of $\mathcal{C}_S$, therefore equation (1) is satisfied in this case as well. By (2), the equation $\iota^{\otimes n} \circ \Delta'^{(n)}_{X,Z_1,\dots,Z_{n-1},Y} = \Delta^{(n)}_{X,Z_1,\dots,Z_{n-1},Y} \circ \iota$ holds true, for all $X, Z_1, \dots, Z_{n-1}, Y \in S$. This implies that the cocategory $\mathcal{C}_S$ is cocomplete: given an element $t \in \mathcal{C}_S(X,Y)$, there is $n \geqslant 1$ such that $\Delta^{(n)}_{X,Z_1,\dots,Z_{n-1},Y} \circ \iota(t) = 0$, for an arbitrary collection of objects $Z_1, \dots, Z_{n-1} \in \operatorname{Ob}\mathcal{C}$. Since $\iota^{\otimes n}$ is an embedding split by $\pi^{\otimes n}$, it follows that $\Delta'^{(n)}_{X,Z_1,\dots,Z_{n-1},Y}(t) = 0$ for all $Z_1, \dots, Z_{n-1} \in S$.

Proposition 1 (Universal property of $\mathcal{C}_S$). *An arbitrary cocategory homomorphism $h\colon \mathcal{B} \to \mathcal{C}$ with $h(\operatorname{Ob}\mathcal{B}) \subset S$ factors uniquely through $\mathcal{C}_S$.*

Proof. It follows from equation (2) that $\Delta^{(n)}_{h(X),U_1,\dots,U_{n-1},h(Y)} \circ h = 0$ if $U_i \notin S$ for some i, thus $N \circ h = 0\colon \mathcal{B}(X,Y) \to \mathcal{N}(h(X),h(Y))$, for each pair $X, Y \in \operatorname{Ob}\mathcal{B}$. Therefore, $h\colon \mathcal{B}(X,Y) \to \mathcal{C}(h(X),h(Y))$ factors through $\mathcal{C}_S(h(X),h(Y))$, i.e., there exists a unique linear map $\overline{h} = \overline{h}_{X,Y}\colon \mathcal{B}(X,Y) \to \mathcal{C}_S(h(X),h(Y))$ such that $\iota \circ \overline{h} = h$. The quiver map $\overline{h}\colon \mathcal{B} \to \mathcal{C}_S$ with $\operatorname{Ob}\overline{h} = \operatorname{Ob} h\colon \operatorname{Ob}\mathcal{B} \to S$ is a cocategory homomorphism since

$$(\iota \otimes \iota) \circ \Delta' \circ \overline{h} = \Delta \circ \iota \circ \overline{h} = \Delta \circ h = (h \otimes h) \circ \Delta = (\iota \otimes \iota) \circ (\overline{h} \otimes \overline{h}) \circ \Delta,$$

and $\iota \otimes \iota$ is an embedding split by $\pi \otimes \pi$. Uniqueness of $\overline{h}$ is obvious. □

Remark 3. The same construction makes sense in graded and differential graded contexts. The proofs transport literally except the following subtlety in the case of differential graded cocategories: in general, the embedding $\iota : \mathcal{C}_S(X,Y) \to \mathcal{C}(X,Y)$ does not admit a splitting which is a cochain map. Nevertheless, the argument can be modified as follows. Choose a splitting $\pi : \mathcal{C}(X,Y) \to \mathcal{C}_S(X,Y)$ of *graded* k-modules. Since ι is a cochain map, i.e., $d \circ \iota = \iota \circ d$, it follows that the differential in $\mathcal{C}_S(X,Y)$ is necessarily given by the composite

$$d = \big[\mathcal{C}_S(X,Y) \xrightarrow{\iota} \mathcal{C}(X,Y) \xrightarrow{d} \mathcal{C}(X,Y) \xrightarrow{\pi} \mathcal{C}_S(X,Y)\big].$$

In particular, the commutation relation $d \circ \iota = \iota \circ d$ takes the form

$$\iota \circ \pi \circ d \circ \iota = d \circ \iota : \mathcal{C}_S(X,Y) \to \mathcal{C}(X,Y).$$

Then the comultiplication Δ' given by (5) is a cochain map. Indeed,

$$\Delta' \circ d = (\pi \otimes \pi) \circ \Delta \circ \iota \circ \pi \circ d \circ \iota = (\pi \otimes \pi) \circ \Delta \circ d \circ \iota.$$

On the other hand,

$$\begin{aligned}(1 \otimes d + d \otimes 1) \circ \Delta' =&(1 \otimes \pi \circ d \circ \iota + \pi \circ d \circ \iota \otimes 1) \circ (\pi \otimes \pi) \circ \Delta \circ \iota \\ =&(\pi \otimes \pi) \circ (1 \otimes d) \circ (1 \otimes \iota \circ \pi) \circ \Delta \circ \iota \\ &+(\pi \otimes \pi) \circ (d \otimes 1) \circ (\iota \circ \pi \otimes 1) \circ \Delta \circ \iota \\ =&(\pi \otimes \pi) \circ (1 \otimes d + d \otimes 1) \circ \Delta \circ \iota\end{aligned}$$

due to equations (3) and (4). Since Δ is a cochain map, it follows that $\Delta' \circ d = (1 \otimes d + d \otimes 1) \circ \Delta'$, thus $(\mathcal{C}_S, \Delta')$ is a differential graded cocategory. The further arguments remain unchanged.

4. Proof of Theorem 1.

Let $\mathcal{C}$, $\mathcal{D}$ be cocomplete cocategories, $f, g \colon \mathcal{C} \to \mathcal{D}$ cocategory homomorphisms. Denote $S = \{X \in \operatorname{Ob}\mathcal{C} | f(X) = g(X)\}$. Suppose $h \colon \mathcal{B} \to \mathcal{C}$ is a cocategory homomorphism such that $f \circ h = g \circ h$. Then for each $W \in \operatorname{Ob}\mathcal{B}$, $f(h(W)) = g(h(W))$, therefore $h(\operatorname{Ob}\mathcal{B}) \subset S$. By the universal property of the cocategory $\mathcal{C}_S$ there exists a unique cocategory homomorphism $\overline{h} \colon \mathcal{B} \to \mathcal{C}_S$ such that $\iota \circ \overline{h} = h$. Therefore, it suffices to construct an equalizer of the pair of cocategory homomorphisms $f \circ \iota, g \circ \iota \colon \mathcal{C}_S \to \mathcal{D}$. Thus, we may assume without loss of generality that $\operatorname{Ob} f = \operatorname{Ob} g$. Let us construct an equalizer

$$\mathcal{E} \xrightarrow{e} \mathcal{C} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{D}$$

in the category of cocomplete cocategories. Put $\operatorname{Ob}\mathcal{E} = \operatorname{Ob}\mathcal{C}$. For $X, Y \in \operatorname{Ob}\mathcal{C}$, denote by $\mathcal{M}(X,Y)$ the k-vector space

$$\bigoplus_{\substack{X=X_0,X_1,\dots,X_p\in\operatorname{Ob}\mathcal{C}\\ Y_0,\dots,Y_{q-1},Y_q=Y\in\operatorname{Ob}\mathcal{C}}}^{p,q\geqslant 0} \mathcal{C}(X,X_1)\otimes\cdots\otimes\mathcal{C}(X_{p-1},X_p)\otimes\mathcal{D}(f(X_p),f(Y_0)) \otimes\mathcal{C}(Y_0,Y_1)\otimes\cdots\otimes\mathcal{C}(Y_{q-1},Y).$$

Define a k-linear map $R\colon \mathcal{C}(X,Y)\to\mathcal{M}(X,Y)$ by

$$R = \sum_{\substack{X=X_0,X_1,\dots,X_p\in\operatorname{Ob}\mathcal{C}\\ Y_0,\dots,Y_{q-1},Y_q=Y\in\operatorname{Ob}\mathcal{C}}}^{p,q\geqslant 0} (1^{\otimes p}\otimes(f-g)\otimes 1^{\otimes q})\circ\Delta^{(p+1+q)}_{X,X_1,\dots,X_p,Y_0,\dots,Y_{q-1},Y}.$$

It is well defined since $\mathcal{C}$ is cocomplete. Let $\mathcal{E}(X,Y) = \operatorname{Ker} R$, so that we have an exact sequence

$$0\to\mathcal{E}(X,Y)\xrightarrow{e}\mathcal{C}(X,Y)\xrightarrow{R}\mathcal{M}(X,Y).$$

Choose a splitting $p\colon \mathcal{C}(X,Y)\to\mathcal{E}(X,Y)$ such that $p\circ e = \mathrm{id}_{\mathcal{E}(X,Y)}$. Suppose that $X, Y, Z \in \operatorname{Ob}\mathcal{C}$. Then the composite

$$\big[\mathcal{E}(X,Z)\xrightarrow{e}\mathcal{C}(X,Z)\xrightarrow{\Delta_{X,Y,Z}}\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z)\xrightarrow{R\otimes 1}\mathcal{M}(X,Y)\otimes\mathcal{C}(Y,Z)\big]$$

vanishes. Indeed, by coassociativity, the composite $(R\otimes 1)\circ\Delta_{X,Y,Z}$ equals

$$\sum_{\substack{X=X_0,X_1,\dots,X_p\in\operatorname{Ob}\mathcal{C}\\ Y_0,\dots,Y_{q-1},Y_q=Y\in\operatorname{Ob}\mathcal{C}}}^{p,q\geqslant 0} (1^{\otimes p}\otimes(f-g)\otimes 1^{\otimes q+1}) \circ(\Delta^{(p+1+q)}_{X,X_1,\dots,X_p,Y_0,\dots,Y_{q-1},Y}\otimes 1)\circ\Delta_{X,Y,Z}$$

$$= \sum_{\substack{X=X_0,X_1,\dots,X_p\in\operatorname{Ob}\mathcal{C}\\ Y_0,\dots,Y_{q-1},Y_q=Y\in\operatorname{Ob}\mathcal{C}}}^{p,q\geqslant 0} (1^{\otimes p}\otimes(f-g)\otimes 1^{\otimes q+1})\circ\Delta^{(p+1+q+1)}_{X,X_1,\dots,X_p,Y_0,\dots,Y_{q-1},Y,Z}:$$

$$\mathcal{C}(X,Z)\to \bigoplus_{\substack{X=X_0,X_1,\dots,X_p\in\operatorname{Ob}\mathcal{C}\\ Y_0,\dots,Y_{q-1},Y_q=Y\in\operatorname{Ob}\mathcal{C}}}^{p,q\geqslant 0} \mathcal{C}(X,X_1)\otimes\cdots\otimes\mathcal{C}(X_{p-1},X_p)\otimes\mathcal{D}(f(X_p),f(Y_0)) \otimes\mathcal{C}(Y_0,Y_1)\otimes\cdots\otimes\mathcal{C}(Y_{q-1},Y)\otimes\mathcal{C}(Y,Z).$$

It follows from the definition of $\mathcal{E}$ that

$$(1^{\otimes p}\otimes(f-g)\otimes 1^{\otimes q+1})\circ\Delta^{(p+1+q+1)}_{X,X_1,\dots,X_p,Y_0,\dots,Y_{q-1},Y,Z}\circ e = 0,$$

for all $X, X_1, \dots, X_p, Y_0, \dots, Y_{q-1}, Y, Z \in \operatorname{Ob}\mathcal{C}$, therefore $(R\otimes 1)\circ\Delta_{X,Y,Z}\circ e = 0$. Since the sequence

$$0\to\mathcal{E}(X,Y)\otimes\mathcal{C}(Y,Z)\xrightarrow{e\otimes 1}\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z)\xrightarrow{R\otimes 1}\mathcal{M}(X,Y)\otimes\mathcal{C}(Y,Z)$$

is exact, it follows that the map $\Delta_{X,Y,Z} \circ e$ factors through $\mathcal{E}(X,Y) \otimes \mathcal{C}(Y,Z)$. In other words, there exists a unique k-linear map $\psi\colon \mathcal{E}(X,Z) \to \mathcal{E}(X,Y) \otimes \mathcal{C}(Y,Z)$ such that $(e \otimes 1) \circ \psi = \Delta_{X,Y,Z} \circ e$. Since $e \otimes 1$ is an embedding split by $p \otimes 1$, the map ψ is necessarily given by the composite

$$\bigl[\mathcal{E}(X,Z) \xrightarrow{e} \mathcal{C}(X,Z) \xrightarrow{\Delta_{X,Y,Z}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \xrightarrow{p\otimes 1} \mathcal{E}(X,Y) \otimes \mathcal{C}(Y,Z)\bigr].$$

In particular, the equation $(e \otimes 1) \circ \psi = \Delta_{X,Y,Z} \circ e$ takes the form

$$(e \circ p \otimes 1) \circ \Delta_{X,Y,Z} \circ e = \Delta_{X,Y,Z} \circ e\colon \mathcal{E}(X,Z) \to \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z).$$

Similarly, the following equation holds true:

$$(1 \otimes e \circ p) \circ \Delta_{X,Y,Z} \circ e = \Delta_{X,Y,Z} \circ e\colon \mathcal{E}(X,Z) \to \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z).$$

Combining these equations yields

$$(e \otimes e) \circ (p \otimes p) \circ \Delta_{X,Y,Z} \circ e = \Delta_{X,Y,Z} \circ e\colon \mathcal{E}(X,Z) \to \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z).$$

Define

$$\tilde{\Delta}_{X,Y,Z} = \bigl[\mathcal{E}(X,Z) \xrightarrow{e} \mathcal{C}(X,Z) \xrightarrow{\Delta_{X,Y,Z}} \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \xrightarrow{p\otimes p} \mathcal{E}(X,Y) \otimes \mathcal{E}(Y,Z)\bigr].$$

As in the case of $\mathcal{C}_S$, one shows that $\tilde{\Delta}$ turns $\mathcal{E}$ into a cocomplete cocategory, and that e becomes a cocategory homomorphism.

Suppose $h\colon \mathcal{B} \to \mathcal{C}$ is a cocategory homomorphism such that $f \circ h = g \circ h$. Then $R \circ h = 0\colon \mathcal{B}(X,Y) \to \mathcal{M}(h(X),h(Y))$. Indeed, by identity (2)

$$\begin{aligned}\Delta^{(p+1+q)}_{h(X),X_1,\dots,X_p,Y_0,\dots,Y_{q-1},h(Y)} \circ h \\ = \sum_{\substack{h(U_1)=X_1,\dots,h(U_p)=X_p \\ h(V_0)=Y_0,\dots,h(V_{q-1})=Y_{q-1}}} (h_{X,U_1} \otimes \dots \otimes h_{U_p,V_0} \otimes \dots \otimes h_{V_{q-1},Y}) \\ \circ \Delta^{(p+1+q)}_{X,U_1,\dots,U_p,V_0,\dots,V_{q-1},Y}.\end{aligned}$$

It follows that

$$\begin{aligned}R \circ h = \sum_{\substack{h(U_1)=X_1,\dots,h(U_p)=X_p \\ h(V_0)=Y_0,\dots,h(V_{q-1})=Y_{q-1}}} (h^{\otimes p} \otimes (f-g) \circ h \otimes h^{\otimes q}) \circ \Delta^{(p+1+q)}_{X,U_1,\dots,U_p,V_0,\dots,V_{q-1},Y} \\ =0,\end{aligned}$$

for $X_1,\dots,X_p,Y_0,\dots,Y_{q-1} \in \operatorname{Ob}\mathcal{C}$. Therefore $h\colon \mathcal{B}(X,Y) \to \mathcal{C}(h(X),h(Y))$ factors through $e\colon \mathcal{E}(h(X),h(Y)) \to \mathcal{C}(h(X),h(Y))$, i.e., $h = e \circ j$ for some k-linear map $j\colon \mathcal{B}(X,Y) \to \mathcal{E}(h(X),h(Y))$. The morphism of k-quivers $j\colon \mathcal{B} \to \mathcal{E}$ with $\operatorname{Ob} j = \operatorname{Ob} h$ is a cocategory homomorphisms since

$$(e \otimes e) \circ (j \otimes j) \circ \Delta = (h \otimes h) \circ \Delta = \Delta \circ h = \Delta \circ e \circ j = (e \otimes e) \circ \Delta \circ j,$$

and $e \otimes e$ is an embedding split by $p \otimes p$. Uniqueness of j is obvious since e is an embedding. □

Remark 4. The theorem is true for cocomplete graded (resp. differential graded) cocategories as well, with appropriate modifications in the proof similar to those made in Remark 3.

Remark 5. The same proof shows the existence of equalizers in the category of cocomplete coalgebras, which are just cocategories with only one object. The intermediate step described in Section 3 becomes superfluous.

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