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EQUIVARIANT STABLE STEMS FOR PRIME ORDER GROUPS

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(*communicated by Lionel Schwartz*)

Abstract

For groups of prime order, equivariant stable maps between equivariant representation spheres are investigated using the Borel cohomology Adams spectral sequence. Features of the equivariant stable homotopy category, such as stability and duality, are shown to lift to the category of modules over the associated Steenrod algebra. The dependence on the dimension functions of the representations is clarified.

Introduction

Along with the conceptual understanding of stable homotopy theory, the ability to do computations has always been of major importance in that part of algebraic topology. For example, the Adams spectral sequence has been used to compute stable homotopy groups of spheres, also known as stable stems, in a range that by far exceeds the geometric understanding of these groups, as discussed in [**15**]. In contrast to that, the focus of equivariant stable homotopy theory has mostly been on structural results, which – among other things – compare the equivariant realm with the non-equivariant one. These results are of course helpful for calculations as well, but nevertheless some fundamental computations have not been done yet. In this text equivariant stable stems are investigated from the point of view of the Adams spectral sequence based on Borel cohomology.

Let us assume that p is an odd prime number. (The final section contains the changes necessary for the even prime.) The group G in question will always be the cyclic group $C_p = \{ z \in \mathbb{C} \mid z^p = 1 \}$ of order p. For finite G-CW-complexes X and Y, based as always in this text, let $[X, Y]^G$ denote the corresponding group of stable G-equivariant maps from X to Y with respect to a complete G-universe. (Some references for equivariant stable homotopy theory are [**17**], [**20**] and [**14**]; in contrast to those, [**1**] does without spectra.) The G-spheres considered here are one-point-compactifications S^V of real G-representations V, with the point at infinity as base-point. Thus, the equivariant stable stems are the groups $[S^V, S^W]^G$ for real G-representations V and W.

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The groups $[S^V, S^W]^G$ depend up to (albeit non-canonical) isomorphism only on the class $\alpha = [V] - [W]$ in the Grothendieck group $RO(G)$ of real representations. The isomorphism type will sometimes be denoted by $[S^0, S^0]^G_\alpha$. Attention will often be restricted to those α in $RO(G)$ with $|\dim_{\mathbb{R}}(\alpha) - \dim_{\mathbb{R}}(\alpha^G)| \leqslant 2p \pm c$ (for some integer c not depending on p) as that facilitates the computations and the presentation of the results. More precisely, this ensures that the power operation are trivial on the free parts of the representation spheres, see Proposition 2. Given a class α, the groups $[S^0, S^0]^G_{\alpha+*}$ will be zero in small degrees and complicated in large degrees. At least the first $2p-2$ interesting groups will be described, counted from the first non-zero one. There, the first extension problem appears which could not be solved, see Figure 6. All differentials vanish in this range.

The main computations are presented in Figures 4, 6, and 8. (In the labeling of the figures, the symbol of a vector space will stand for its real dimension, so that V is an abbreviation for $\dim_{\mathbb{R}}(V)$, for example.) The general results proven on the way may be of independent interest.

In Section 1 the main tool used here is described, namely the Adams spectral sequence based on Borel cohomology. This has been introduced by Greenlees, and one may refer to [**8**], [**9**] and [**11**] for its properties. For finite G-CW-complexes X and Y, that spectral sequence converges to the p-adic completion of $[X, Y]^G_*$. This gives the information one is primarily interested in, since localisation may be used to compute $[X, Y]^G$ away from p. (See for example Lemma 3.6 on page 567 in [**16**].) As a first example – which will be useful in the course of the other computations – the Borel cohomology Adams spectral sequence for $[S^0, S^0]^G_*$ will be discussed. In Section 2, the Borel cohomology of the spheres S^V will be described. This will serve as an input for the spectral sequence. It will turn out that the groups on the E_2-page of the Borel cohomology Adams spectral sequence which computes $[S^V, S^W]^G$ only depend on the dimension function of $\alpha = [V] - [W]$, i.e. on the two integers $\dim_{\mathbb{R}}(\alpha)$ and $\dim_{\mathbb{R}}(\alpha^G)$, implying that it is sufficient to consider the cases $[S^V, S^0]^G$ and $[S^0, S^W]^G$. This is done in Sections 3 and 4, respectively.

Not only are most of the computations new (for odd primes at least – see below), the approach via the Borel cohomology Adams spectral sequence gives a bonus: it automatically incorporates the book-keeping for p-multiplication, and the corresponding filtration eases the study of induced maps. This can be helpful in the study of other spaces which are built from spheres. (See [**22**], which has been the motivation for this work, where this is used.) To emphasise this point: the results on the E_2-terms are more fundamental than the – in our cases – immediate consequences for the equivariant stable stems.

The final section deals with the even prime. This has been the first case ever to be considered, by Bredon [**4**], and some time later by Araki and Iriye [**3**]. In this case, our method of choice is applied here to the computations of $[S^0, S^0]^G_*$, $[S^L, S^0]^G_*$ and $[S^0, S^L]^G_*$ in the range $* \leqslant 13$, where L is a non-trivial real 1-dimensional representation. However, in this case, only the results on the E_2-terms are new; the implications of our charts for the equivariant stable stems at $p = 2$ can also be extracted from [**3**].

1. The Borel cohomology Adams spectral sequence

In this section, some basic facts about Borel cohomology and the corresponding Adams spectral sequence will be presented. The fundamental reference is [**9**]. In addition to that, [**8**], [**10**], [**11**], and [**12**] might be helpful. See also [**19**] for a different approach to the construction of the spectral sequence.

Let p be an odd prime number, and write G for the group C_p. Let H^* denote (reduced) ordinary cohomology with coefficients in the field $\mathbb{F}$ with p elements. For a finite G-CW-complex X, let

$$b^*X = H^*(EG_+ \wedge_G X)$$

denote the Borel cohomology of X. The coefficient ring

$$b^* = b^*S^0 = H^*(BG_+)$$

is the mod p cohomology ring of the group. Since p is odd, this is the tensor product of an exterior algebra on a generator σ in degree 1 and a polynomial algebra on a generator τ in degree 2.

$$H^*(BG_+) = \Lambda(\sigma) \otimes \mathbb{F}[\tau]$$

The generator τ is determined by the embedding of C_p into the group of units of $\mathbb{C}$, and σ is determined by the requirement that it is mapped to τ by the Bockstein homomorphism.

The Borel cohomology Adams spectral sequence

For any two finite G-CW-complexes X and Y, there is a Borel cohomology Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{b^*b}(b^*Y, b^*X) \Longrightarrow [X \wedge EG_+, Y \wedge EG_+]^G_{t-s}. \qquad (1)$$

Before explaining the algebra b^*b in the next subsection, let me spend a few words on the target.

There are no essential maps from the free G-space $X \wedge EG_+$ to the cofibre of the projection from $Y \wedge EG_+$ to Y, which is contractible. Therefore, the induced map

$$[X \wedge EG_+, Y \wedge EG_+]^G_* \longrightarrow [X \wedge EG_+, Y]^G_*$$

is an isomorphism. On the other hand, the map

$$[X, Y]^G_* \longrightarrow [X \wedge EG_+, Y]^G_*$$

is p-adic completion: this is a corollary of the completion theorem (formerly the Segal conjecture), see for example [**6**]. In this sense, the Borel cohomology Adams spectral sequence (1) converges to the p-adic completion of $[X, Y]^G_*$.

Gradings

Let me comment on the grading conventions used. The extension groups for the Adams spectral sequences will be graded homologically, so that homomorphisms of degree t *lower* degree by t. This means that

$$\mathrm{Hom}^t_{R^*}(M^*, N^*) = \mathrm{Hom}^0_{R^*}(M^*, \Sigma^t N^*)$$

if M^* and N^* are graded modules over the graded ring R^*. This is the traditional convention, implying for example that the ordinary Adams spectral sequence reads

$$\mathrm{Ext}^{s,t}_{A^*}(H^*Y, H^*X) \Longrightarrow [X, Y]_{t-s}.$$

But, sometimes it is more natural to grade cohomologically, so that homomorphisms of degree t *raise* degree by t.

$$\mathrm{Hom}^t_{R^*}(M^*, N^*) = \mathrm{Hom}^0_{R^*}(\Sigma^t M^*, N^*)$$

Using cohomological grading for the extension groups, the Adams spectral sequence would read

$$\mathrm{Ext}^{s,t}_{A^*}(H^*Y, H^*X) \Longrightarrow [X, Y]^{s+t}.$$

In the present text, unless otherwise stated, the grading of the groups Hom and Ext over A^* and b^*b will be homological, whereas over b^* it will be cohomological.

The structure of b^*b

The mod p Steenrod algebra A^* has an element β in degree 1, namely the Bockstein homomorphism. For $i \geqslant 1$, there are elements P^i in degree $2i(p-1)$, the Steenrod power operations. By convention, P^0 is the unit of the Steenrod algebra. Often the total power operation

$$P = \sum_{i=0}^{\infty} P^i$$

will be used, which is a ring endomorphism on cohomology algebras. This is just a rephrasing of the Cartan formula. As an example, the A^*-action on the coefficient ring $b^* = b^*S^0$ is given by

$$\begin{aligned} \beta(\sigma) &= \tau, \\ \beta(\tau) &= 0, \\ P(\sigma) &= \sigma \text{ and} \\ P(\tau) &= \tau + \tau^p. \end{aligned}$$

As a vector space, b^*b is the tensor product $b^* \otimes A^*$. The multiplication is a twisted product, the twisting being given by the A^*-action on b^*: for elements a in A^* and B in b^*, the equation

$$(1 \otimes a).(B \otimes 1) = \sum_a (-1)^{|a_2|\cdot|B|}(a_1 B) \otimes a_2$$

holds. Here and in the following the Sweedler convention for summation (see [**23**]) will be used, so that

$$\sum_a a_1 \otimes a_2$$

is the coproduct of an element a in A^*.

Changing rings

If M^* and N^* are modules over b^*b, they are also modules over b^*. Using the antipode S of A^*, the vector space $\mathrm{Hom}_{b^*}(M^*, N^*)$ is acted upon by A^* via

$$(a\phi)(m) = \sum_a a_1 \phi(S(a_2)m).$$

For example, evaluation at the unit of b^* is an isomorphism

$$\mathrm{Hom}_{b^*}(b^*, N^*) \xrightarrow{\cong} N^* \tag{2}$$

of A^*-modules. The A^*-invariant elements in $\mathrm{Hom}_{b^*}(M^*, N^*)$ are just the b^*b-linear maps from M^* to N^*. Therefore, evaluation at a unit of the ground field $\mathbb{F}$ is an isomorphism

$$\mathrm{Hom}^t_{A^*}(\mathbb{F}, \mathrm{Hom}_{b^*}(M^*, N^*)) \xrightarrow{\cong} \mathrm{Hom}^t_{b^*b}(M^*, N^*), \tag{3}$$

using cohomological grading throughout. The associated Grothendieck spectral sequence takes the form of a change-of-rings spectral sequence

$$\mathrm{Ext}^r_{A^*}(\mathbb{F}, \mathrm{Ext}^s_{b^*}(M^*, N^*)) \Longrightarrow \mathrm{Ext}^{r+s}_{b^*b}(M^*, N^*).$$

This is a spectral sequence of graded $\mathbb{F}$-vector spaces. In the case where M^* is b^*-projective, the spectral sequence collapses and the isomorphism (3) passes to an isomorphism

$$\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, \mathrm{Hom}_{b^*}(M^*, N^*)) \xrightarrow{\cong} \mathrm{Ext}^{s,t}_{b^*b}(M^*, N^*).$$

In particular, the 0-line of the Borel cohomology Adams spectral sequence for groups of the form $[X, S^0]^G_*$ consists of the A^*-invariants of b^*X.

Again a remark on the gradings: all the extension groups in this subsection have been cohomologically graded so far. If one wants to use the spectral sequence to compute the input of an Adams spectral sequence, one should convert the grading on the *outer* extension groups into a homological grading. The spectral sequence then reads

$$\mathrm{Ext}^{r,t}_{A^*}(\mathbb{F}, \mathrm{Ext}^s_{b^*}(M^*, N^*)) \Longrightarrow \mathrm{Ext}^{r+s,t}_{b^*b}(M^*, N^*),$$

and only the grading on the inner $\mathrm{Ext}^s_{b^*}(M^*, N^*)$ is cohomological then.

An example: $[S^0, S^0]^G_*$

As a first example, one may now calculate the groups $[S^0, S^0]^G_*$ in a reasonable range.

Since b^* is a free b^*-modules and $\mathrm{Hom}_{b^*}(b^*, b^*) \cong b^*$ as A^*-modules, one sees that the groups on the E_2-page are

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*) \cong \mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, \mathrm{Hom}_{b^*}(b^*, b^*)) \cong \mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, b^*).$$

These groups are the same as those for the E_2-term of the ordinary Adams spectral sequence for $[BG_+, S^0]$, which might have been expected in view of the Segal conjecture: the p-completions of the targets are the same.

As for the calculation of the groups $\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, b^*)$, there is an isomorphism

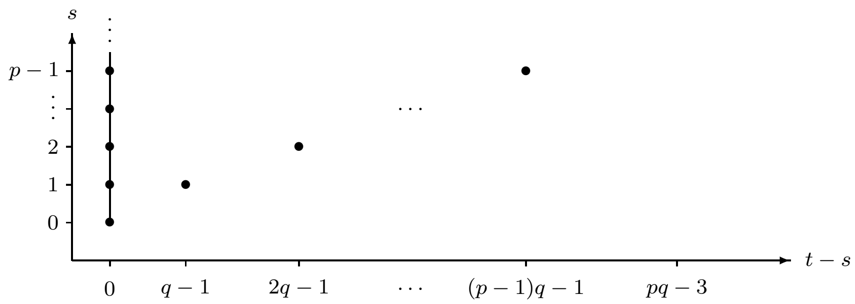
$$\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, b^*) \cong \mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, \mathbb{F}) \oplus \mathrm{Ext}^{s-1,t-1}_{A^*}(b^*, \mathbb{F}). \tag{4}$$

This is proved in [**2**]. The isomorphism (4) can be thought of as an algebraic version of the geometric splitting theorem, which says that $[S^0, S^0]^G_*$ is isomorphic to a direct sum $[S^0, S^0]_* \oplus [S^0, BG_+]_*$.
The groups $\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F},\mathbb{F})$ and $\mathrm{Ext}^{s-1,t-1}_{A^*}(b^*,\mathbb{F})$ on the right hand side of (4) can be calculated in a reasonable range using standard methods. Here, the results will be presented in the usual chart form. (A dot represents a group of order p. A line between two dots represents the multiplicative structure which leads to multiplication with p in the target.) For example, some of the groups on the E_2-page of the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F},\mathbb{F}) \Longrightarrow [S^0, S^0]_{t-s}$$

are displayed in Figure 1.

Figure 1: The ordinary Adams spectral sequence for $[S^0, S^0]_*$

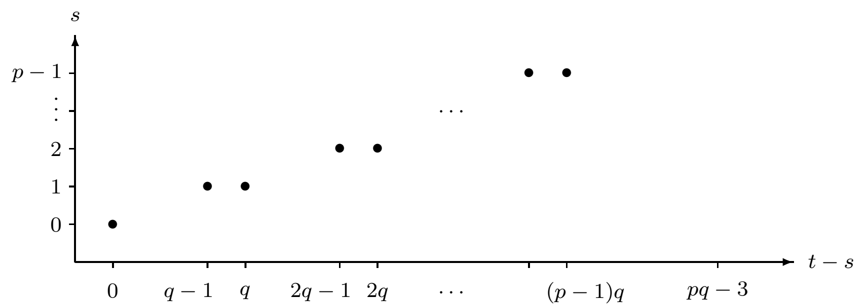


The single dots represent the elements $\alpha_1, \dots, \alpha_{p-1}$ from the image of the J-homomorphism at the odd prime in question; α_j lives in degree $jq - 1$, where $q = 2(p-1)$ as usual. The chart stops right before the β-family would appear with β_1 in degree $pq - 2$ and the next α-element α_p, the first divisible one, in degree $pq - 1$. This is reflected on the E_2-term, where calculations with the Steenrod algebra become more complicated in cohomological degree $t = pq = 2p(p-1)$, when the p-th power of P^1 vanishes and the next indecomposable P^p appears. See [**21**] for more on all of this. For the rest of this text, only the groups for $t - s \leqslant 4p - 6$ are relevant.
Now let $\mathbb{M}(\beta)$ be the 2-dimensional A^*-module on which β acts non-trivially, the generator sitting in degree zero. This is an extension

$$0 \longleftarrow \mathbb{F} \longleftarrow \mathbb{M}(\beta) \longleftarrow \Sigma\mathbb{F} \longleftarrow 0,$$

which represents the dot at the spot $(s,t) = (1,1)$ in Figure 1. This A^*-module is the cohomology of the Moore spectrum $M(p)$, the cofibre of the degree p self-map of S^0. Figure 2 shows the beginning of the Adams spectral sequence for $[\,S^0, M(p)\,]$, which has the groups $\mathrm{Ext}^{s,t}_{A^*}(\mathbb{M}(\beta),\mathbb{F})$ on its E_2-page. Again, only the groups for $t - s \leqslant 4p - 6$ are relevant in the following.
Now one may turn attention to the second summand $\mathrm{Ext}^{s-1,t-1}_{A^*}(b^*,\mathbb{F})$ in (4). Since $b^* = H^*BG_+$, the vector space $\mathrm{Ext}^{s-1,t-1}_{A^*}(b^*,\mathbb{F})$ decomposes into a sum

Figure 2: The ordinary Adams spectral sequence for $[S^0, M(p)]_*$



of $\mathrm{Ext}^{s-1,t-1}_{A^*}(\mathbb{F},\mathbb{F})$ on the one hand and $\mathrm{Ext}^{s-1,t-1}_{A^*}(H^*BG,\mathbb{F})$ on the other. As an A^*-module, H^*BG decomposes into the direct sum of submodules,

$$H^*BG = M_1 \oplus M_2 \oplus \cdots \oplus M_{p-1},$$

where M_j is concentrated in degrees $2j-1$ and $2j$ modulo $2(p-1)$. The generators of M_1 as an $\mathbb{F}$-vector space are $\sigma, \tau, \sigma\tau^{p-1}, \tau^p, \sigma\tau^{2p-2}, \tau^{2p-1}, \ldots$. As $\sigma\tau^{2p-2}$ sits in degree $4p-3$, we will focus on the terms in degrees at most $4p-4$ throughout the calculation. In that range, M_1 has a resolution

$$M_1 \longleftarrow A^*\langle h_1\rangle \oplus A^*\langle h_{2p-1}\rangle \longleftarrow A^*\langle e_{2p-1}\rangle \oplus A^*\langle e_{2p}\rangle,$$

with $A^*\langle x_d\rangle$ a free A^*-module with a generator named x_d in degree d. The maps are given by

$$\begin{aligned} h_1 &\mapsto \sigma \\ h_{2p-1} &\mapsto \sigma\tau^{p-1} \\ e_{2p-1} &\mapsto P^1 h_1 \\ e_{2p} &\mapsto P^1\beta h_1 - \beta P^1 h_{2p-1}. \end{aligned}$$

For $j = 2, \ldots, p-1$, the generators of the A^*-module M_j as an $\mathbb{F}$-vector space are $\sigma\tau^{j-1}, \tau^j, \sigma\tau^{j+p-2}, \tau^{j+p-1}, \ldots$. In our range, M_j has a resolution

$$M_j \longleftarrow A^*\langle h_{2j-1}\rangle \longleftarrow A^*\langle e_{2j+2(p-1)}\rangle,$$

where the maps are given by

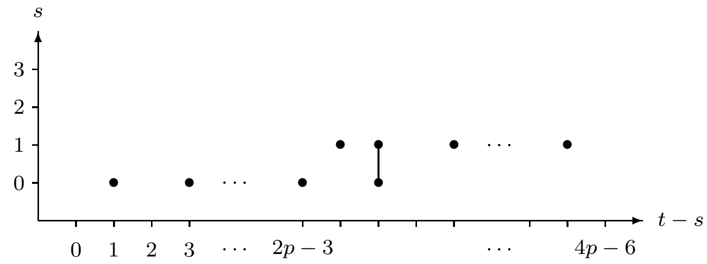
$$\begin{aligned} h_{2j-1} &\mapsto \sigma\tau^{j-1} \\ e_{2j+2(p-1)} &\mapsto (j\beta P^1 - (j-1)P^1\beta)h_{2j-1}. \end{aligned}$$

Together with Adams' vanishing line, this leads to the groups displayed in Figure 3, which is complete in degrees $t-s \leqslant 4p-6$.

The extension can be established in the following manner: A generator for the group at the spot $(t-s, s) = (2p-1, 0)$ is given by a homomorphism from H^*BG to $\Sigma^{2p-1}\mathbb{F}$ which sends $\sigma\tau^{p-1}$ to a generator. Since

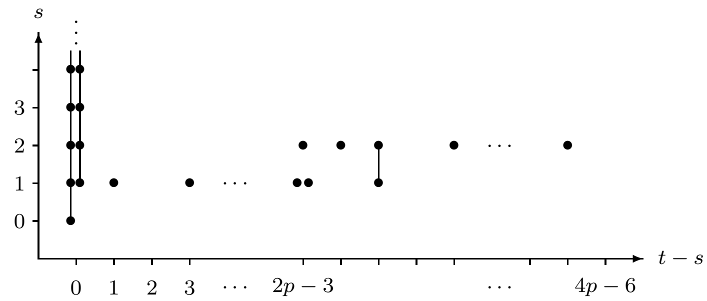
$$\beta(\sigma\tau^{p-1}) = \tau^p = P^1(\tau),$$

Figure 3: The ordinary Adams spectral sequence for $[S^0, BC_p]_*$



this does not factor through $\Sigma^{2p-1}\mathbb{M}(\beta)$.
Assembling the information as required by (4), one gets the Borel cohomology Adams spectral sequence for $[S^0, S^0]^G_*$, see Figure 4. There are no non-trivial differentials possible in the range displayed. Thus, one may easily read off the p-completions of the groups $[S^0, S^0]^G_*$ in that range.

Figure 4: The Borel cohomology Adams spectral sequence for $[S^0, S^0]^G_*$



This finishes the discussion of the Borel cohomology Adams spectral sequence for $[S^0, S^0]^G_*$. Later, the reader's attention will also be drawn to the Borel cohomology Adams spectral sequence for $[G_+, S^0]^G_*$ when this will seem illuminating. Also the Borel cohomology Adams spectral sequence for $[S^0, G_+]^G_*$ will be studied and used.

2. The Borel cohomology of spheres

Let p and G be as in the previous section. Let V be a real G-representation. This section provides a description of the Borel cohomology b^*S^V. This will later serve as an input for the Borel cohomology Adams spectral sequence.
To start with, if V^G is the fixed subrepresentation, the b^*-module $b^*S^{V^G}$ is free over b^* on a generator in degree $\dim_{\mathbb{R}}(V^G)$. This follows from the suspension theo-

rem. Also b^*S^V is free over b^* on a generator in degree $\dim_{\mathbb{R}}(V)$. This follows from the generalised suspension isomorphism: the Thom isomorphism. Note that

$$EG_+ \wedge_G S^V = (EG \times_G S^V)/(EG \times_G \{\infty\})$$

is the Thom space of the vector bundle $EG \times_G V$ over BG. This vector bundle is orientable, and the Thom isomorphism implies that b^*S^V is a free b^*-module on one generator.

Isolating the isotropy

Let $F(V)$ be the fibre of the inclusion of S^{V^G} into S^V, so that there is a cofibre sequence

$$F(V) \longrightarrow S^{V^G} \longrightarrow S^V \longrightarrow \Sigma F(V). \tag{5}$$

Of course, if the complement of V^G in V is denoted by $V - V^G$, the relation

$$F(V) \simeq_G \Sigma^{V^G} F(V - V^G)$$

holds, so one may assume $V^G = 0$. In that case, $S^{V^G} = S^0$ and the fibre of the inclusion is just the sphere $S(V)_+$ inside V with a disjoint base-point added. The quotient space $Q(V)$ of $F(V)$ is then a lens space with a disjoint base-point added. In general, it is a suspension of that.

If F is a free G-space with quotient Q, the map $EG_+ \to S^0$, which sends EG to 0, induces a G-equivalence $EG_+ \wedge F \to F$, which in turn induces an isomorphism from H^*Q to b^*F. This isomorphism will often be used to identify the two groups. As $F(V)$ is G-free, the groups $b^*F(V) \cong H^*Q(V)$ vanish above the dimension of the orbit space $Q(V) = F(V)/G$, that is for $* \geqslant \dim_{\mathbb{R}}(V)$. It now follows (by downward induction on the degree) that the inclusion $S^{V^G} \subset S^V$ induces an inclusion in Borel cohomology. This implies

Proposition 1. *There is a short exact sequence*

$$0 \longleftarrow b^*F(V) \longleftarrow b^*S^{V^G} \longleftarrow b^*S^V \longleftarrow 0$$

of b^*-modules.

In particular, the graded vector space $b^*F(V)$ is 1-dimensional for the degrees $\dim_{\mathbb{R}}(V^G) \leqslant * < \dim_{\mathbb{R}}(V)$ and zero otherwise. As a b^*-module it is cyclic, generated by any non-zero element in degree $\dim_{\mathbb{R}}(V^G)$.

The action of the Steenrod algebra

It remains to discuss the A^*-action on the b^*-modules in sight. On $b^*S^{V^G}$ it is clear by stability. On b^*S^V it can be studied by including b^*S^V into $b^*S^{V^G}$. The A^*-action on $b^*F(V)$ also follows from the short exact sequence in Proposition 1, since that displays $b^*F(V)$ as the quotient A^*-module of $b^*S^{V^G}$ by b^*S^V.

If V is a real G-representation, there is an integer $k(V) \geqslant 0$ such that

$$\dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(V^G) = 2k(V).$$

For example, $k(\mathbb{R}G) = (p-1)/2$ and $k(\mathbb{C}G) = p-1$. The assumption in the following proposition ensures that the action of the power operations on $b^*F(V)$ is trivial.

Proposition 2. *If $k(V) \leqslant p$, there is an isomorphism*

$$b^*F(V) \cong \Sigma^{\dim_{\mathbb{R}}(V^G)} \left(\mathbb{F} \oplus \left(\bigoplus_{j=1}^{k(V)-1} \Sigma^{2j-1}\mathbb{M}(\beta) \right) \oplus \Sigma^{2k(V)-1}\mathbb{F} \right)$$

of A^*-modules.

Algebraic stability

One may now provide an algebraic version of the stability in the stable homotopy category.

Proposition 3. *Let V be a real G-representation. For G-spaces X and Y there is an isomorphism*

$$\operatorname{Ext}^s_{b^*}(b^*Y, b^*X) \xrightarrow{\cong} \operatorname{Ext}^s_{b^*}(b^*\Sigma^V Y, b^*\Sigma^V X)$$

of A^*-modules.

Proof. One has $b^*\Sigma^V X \cong b^*S^V \otimes_{b^*} b^*X$ since b^*S^V is a free b^*-module. The b^*-module b^*S^V is invertible. (This can be seen in more than one way. On the one hand, there is a spectrum S^{-V} such that $S^V \wedge S^{-V} \simeq_G S^0$. Therefore, b^*S^{-V} is the required inverse. On the other hand, one might describe the inverse algebraically by hand, imitating b^*S^{-V} and avoiding spectra.) Therefore, tensoring with b^*S^V is an isomorphism

$$\operatorname{Hom}_{b^*}(b^*Y, b^*X) \xrightarrow{\cong} \operatorname{Hom}_{b^*}(b^*S^V \otimes_{b^*} b^*Y, b^*S^V \otimes_{b^*} b^*X)$$

of A^*-modules. The result follows by passage to derived functors. □

Chasing the isomorphism from the preceding proposition through the change-of-rings spectral sequence, one obtains, as a corollary, that there is also an isomorphism

$$\operatorname{Ext}^{s,t}_{b^*b}(b^*Y, b^*X) \xrightarrow{\cong} \operatorname{Ext}^{s,t}_{b^*b}(b^*\Sigma^V Y, b^*\Sigma^V X).$$

This is the desired analogue on the level of E_2-pages of the suspension isomorphism

$$[X, Y]^G_* \cong [\Sigma^V X, \Sigma^V Y]^G_* \tag{6}$$

on the level of targets.

Dependence on the dimension function

As mentioned in the introduction, the suspension isomorphism (6) implies that the isomorphism type of $[S^V, S^W]^G$ only depends on the class $\alpha = [V] - [W]$ in the representation ring $RO(G)$. But, for the groups on the E_2-pages of the Borel cohomology Adams spectral sequences even more is true: up to isomorphism, they only depend on the dimension function of α, i.e. on the two integers $\dim_{\mathbb{R}}(\alpha)$ and $\dim_{\mathbb{R}}(\alpha^G)$. This is the content of the following result.

Proposition 4. *If two G-representations V and W have the same dimension function, the b^*b-modules b^*S^V and b^*S^W are isomorphic.*

Proof. Recall from Proposition 1 that the inclusion of S^{V^G} into S^V induces an isomorphism from b^*S^V with its image in $b^*S^{V^G}$, which is the part of degree at least $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W)$. Of course, the same holds for W in place of V. Since also $\dim_{\mathbb{R}}(V^G) = \dim_{\mathbb{R}}(W^G)$, one can use an isomorphism $b^*S^{V^G} \cong b^*S^{W^G}$ to identify the two images. □

For example, if L and M are non-trivial irreducible G-representations which are not isomorphic, the groups on the E_2-page of the Borel cohomology Adams spectral sequence for $[S^L, S^M]^G_*$ are up to isomorphism just those for $[S^0, S^0]^G_*$. An isomorphism $b^*S^L \leftarrow b^*S^M$ represents a G-map $S^L \to S^M$ which has degree coprime to p. But, this can not be a stable G-equivalence, since it is *not* true that S^L is stably G-equivalent to S^M, see [**7**].

Now, given any $\alpha = [V] - [W]$ in $RO(G)$, one would like to know the groups on the E_2-term for $[S^V, S^W]^G_*$. Using Proposition 3 above, one may assume $V = V^G$ or $W = W^G$. Since the integer grading takes care of the trivial summands, one might just as well suppose that $V = 0$ or $W = 0$ respectively. Thus, it is enough to know the groups on the E_2-terms for $[S^V, S^0]^G_*$ and $[S^0, S^W]^G_*$. In the following two sections, these will be calculated for some V and W.

3. Cohomotopy groups of spheres

In this section, a calculation of some of the groups $[S^V, S^0]^G_*$ will be presented if V is a G-representation with $k(V)$ small. The tool will be the Borel cohomology Adams spectral sequence, and the starting point will be the short exact sequence induced by the cofibre sequence (5). The fixed point case $[S^{V^G}, S^0]^G_*$ – which up to re-indexing is the case $[S^0, S^0]^G_*$ – has already been dealt with as an example in the first section. One may turn towards the free points now.

Cohomotopy groups of free G-spaces in general

Let F be a finite free G-CW-complex. The groups on the E_2-page of the Borel cohomology Adams spectral sequence for $[F, S^0]^G_*$ are

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*F) \cong \mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, \mathrm{Hom}_{b^*}(b^*, b^*F)) \cong \mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, b^*F).$$

If Q is the orbit space of F, one may identify b^*F and H^*Q. Thus, the groups on the E_2-page of the Borel cohomology Adams spectral sequence for $[F, S^0]^G_*$ are really the same as the groups on the E_2-page of the ordinary Adams spectral sequence for $[Q, S^0]_*$. This might not be surprising: the targets are isomorphic. Note that the preceding discussion applies (in particular) to $[G_+, S^0]^G_*$.

Cohomotopy groups of $F(V)$

Let V be a G-representation. In the case $k(V) \leqslant p$, a splitting of $b^*F(V)$ as an A^*-module has been described in Proposition 2 above. The groups $\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, b^*F(V))$ split accordingly. It is more convenient to pass to the duals. If M^* is an A^*-module,

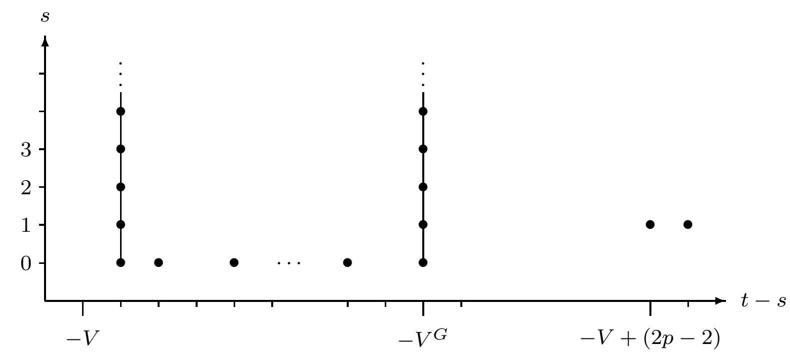
$$DM^* = \mathrm{Hom}(M^*, \mathbb{F})$$

is its dual. For example, one has $D\mathbb{M}(\beta) \cong \Sigma^{-1}\mathbb{M}(\beta)$. Thus, the Ext-groups above are isomorphic to $\mathrm{Ext}^{s,t}_{A^*}(Db^*F(V), \mathbb{F})$. Since

$$Db^*F(V) \cong \Sigma^{-\dim_{\mathbb{R}}(V^G)} \left(\mathbb{F} \oplus \left(\bigoplus_{j=1}^{k(V)-1} \Sigma^{-2j}\mathbb{M}(\beta) \right) \oplus \Sigma^{1-2k(V)}\mathbb{F} \right),$$

one may use the data collected about $\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, \mathbb{F})$ and $\mathrm{Ext}^{s,t}_{A^*}(\mathbb{M}(\beta), \mathbb{F})$ in Figures 1 and 2, respectively, to assemble the E_2-term. This is displayed in Figure 5 for $k(V) \leqslant p-2$. Note that in that case,the number $-\dim_{\mathbb{R}}(V^G)$ is strictly less than the number $-\dim_{\mathbb{R}}(V) + (2p-2)$. The series of dots in the 1-line continues to the right until and including the case $t-s = -\dim_{\mathbb{R}}(V^G) + (2p-3)$, followed by zeros until $-\dim_{\mathbb{R}}(V) + (4p-6)$. By multiplicativity, there are no non-trivial differentials. Hence it is easy to read off the p-completions of the groups $[F(V), S^0]^G_*$ in the range considered. Note that these are isomorphic to the p-completions of the groups $[Q(V), S^0]_*$ and therefore also computable with non-equivariant methods.

Figure 5: The Borel cohomology Adams spectral sequence for $[F(V), S^0]^G_*$



Cohomotopy groups of S^V

If V is a non-trivial G-representation, the short exact sequence from Proposition 1 leads to a long exact sequence of extension groups:

$$\leftarrow \mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^{V^G}) \leftarrow \mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^V) \leftarrow \mathrm{Ext}^{s-1,t}_{b^*b}(b^*, b^*F(V)) \leftarrow$$

This will allow the determination of $\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^V)$ in a range.

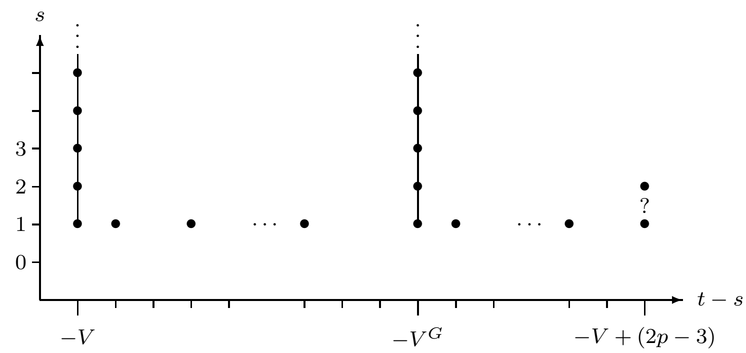
The starting point is the computation of the 0-line, which consists of the A^*-invariants in b^*S^V: Inspection of the A^*-action shows that, since $V^G \neq V$ by hypothesis, one has

$$\mathrm{Ext}^{0,t}_{b^*b}(b^*, b^*S^V) = \mathrm{Hom}^t_{b^*b}(b^*, b^*S^V) = 0 \tag{7}$$

for all integers t.

With the information on the 0-line just described, it is not hard to use the previous computations as summarized in Figure 4 and 5, and the long exact sequence above to calculate the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^V)$ in a range. Figure 6 displays the result for $k(V) \leqslant p-2$. By multiplicativity, there are no non-trivial differentials. Hence, one can immediately read off the p-completions of the groups $[S^V, S^0]^G_*$ in the range considered. The group at the spot $(t-s, s) = (-\dim_{\mathbb{R}}(V) + (2p-3), 2)$ survives, since the group at the spot $(t-s, s) = (-\dim_{\mathbb{R}}(V) + (2p-2), 0)$ is trivial by (7). As the question mark indicates, the extension problem has not been solved in general yet.

Figure 6: The Borel cohomology Adams spectral sequence for $[S^V, S^0]^G_*$



4. Homotopy groups of spheres

In this section, a calculation of some of the groups $[S^0, S^W]^G_*$ will be presented, if W is a G-representation with $k(W)$ small. The tool will again be the Borel cohomology Adams spectral sequence, and the starting point will again be the cofibre sequence (5). The fixed points have already been dealt with as an example in the first section. One may turn towards the free points now.

Homotopy groups of free G-spaces in general

Let F be a finite free G-CW-complex. The groups on the E_2-page of the Borel cohomology Adams spectral sequence for $[S^0, F]^G_*$ are $\mathrm{Ext}^{s,t}_{b^*b}(b^*F, b^*)$. These can in be computed with the change-of-rings spectral sequence. In order to do so, one has to know the A^*-modules $\mathrm{Ext}^s_{b^*}(b^*F, b^*)$.

The case $F = G_+$ might illustrate what happens. Using the standard minimal free resolution of $\mathbb{F}$ as a b^*-module, or otherwise, one computes

$$\mathrm{Ext}^s_{b^*}(\mathbb{F}, b^*) \cong \begin{cases} \Sigma^{-1}\mathbb{F} & s = 1 \\ 0 & s \neq 1. \end{cases}$$

Therefore, the groups on the E_2-page of the Borel cohomology Adams spectral sequence for $[S^0, G_+]^G_*$ are $\operatorname{Ext}^{s-1,t-1}_{A^*}(\mathbb{F}, \mathbb{F})$. These are – up to a filtration shift – those on the E_2-page of the Adams spectral sequence for $[S^0, S^0]_*$. This might be what one expects: the targets are isomorphic, but an isomorphism uses the transfer. The preceding example has the following application.

Proposition 5. *If M^* is a finite b^*b-module and d is an integer such that $M^t = 0$ holds for all $t < d$, then $t - s < d$ implies $\operatorname{Ext}^{s,t}_{b^*b}(M^*, b^*) = 0$.*

Proof. This can be proven by induction. If M^* is concentrated in dimension $e \geqslant d$, the module M^* is a sum of copies of $\Sigma^e\mathbb{F}$. In this case the result follows from the example which has been discussed before. If M^* is not concentrated in some dimension, let e be the maximal degree such that $M^e \neq 0$. Then M^e is a submodule. There is a short exact sequence

$$0 \longrightarrow M^e \longrightarrow M^* \longrightarrow M^*/M^e \longrightarrow 0.$$

The result holds for M^e by what has been explained before and for M^*/M^e by induction. It follows for M^* by inspecting the long exact sequence induced by that short exact sequence. $\square$

If F is a finite free G-CW-complex, the hypothesis in the previous proposition is satisfied for $M^* = b^*F$ and some d.

Corollary 6. *If W is a G-representation, then the vector spaces $\operatorname{Ext}^{s,t}_{b^*b}(b^*F(W), b^*)$ vanish in the range $t - s < \dim_{\mathbb{R}}(W^G)$. The same holds for $\operatorname{Ext}^{s,t}_{b^*b}(b^*S^W, b^*)$.*

Proof. For $F(W)$ it follows immediately from the previous proposition. Using this, the obvious long exact sequence shows that the inclusion of S^{W^G} into S^W induces an isomorphism $\operatorname{Ext}^{s,t}_{b^*b}(b^*S^{W^G}, b^*) \cong \operatorname{Ext}^{s,t}_{b^*b}(b^*S^W, b^*)$. This gives the result for S^W. $\square$

Propositions 3 and 4 now imply the following.

Corollary 7. *Let V and W be G-representations such that the dimension function of $[W] - [V]$ is non-negative. Then the groups $\operatorname{Ext}^{s,t}_{b^*b}(b^*S^W, b^*S^V)$ vanish in the range $t - s < \dim_{\mathbb{R}}(W^G) - \dim_{\mathbb{R}}(V^G)$.*

Of course, similar results for the *targets* of the spectral sequences follow easily from the dimension and the connectivity of the spaces involved. The point here was to prove them for the E_2-pages of the spectral sequences.
The example $F = G_+$ above suggests the following result.

Proposition 8. *Let F be a finite free G-CW-complex. If Q denotes the quotient, then there is an isomorphism $\operatorname{Ext}^1_{b^*}(b^*F, b^*) \cong \Sigma^{-1}DH^*Q$, and $\operatorname{Ext}^s_{b^*}(b^*F, b^*)$ is zero for $s \neq 1$.*

Proof. Since there is an injective resolution

$$0 \longrightarrow b^* \longrightarrow b^*[1/\tau] \longrightarrow b^*[1/\tau]/b^* \longrightarrow 0 \tag{8}$$

of b^* as a graded b^*-module, only the two cases $s = 0$ and $s = 1$ need to be considered. For any finite b^*-module M^* such as $b^*F \cong H^*Q$, both $\mathrm{Hom}_{b^*}(M^*, b^*)$ and $\mathrm{Hom}_{b^*}(M^*, b^*[1/\tau])$ are trivial. By the injectivity of the b^*-module $b^*[1/\tau]$, the boundary homomorphism in the long exact sequence associated to (8) is an isomorphism between the vector spaces $\mathrm{Hom}_{b^*}(M^*, b^*[1/\tau]/b^*)$ and $\mathrm{Ext}^1_{b^*}(M^*, b^*)$. This implies that the latter is zero. Finally, note that $\mathrm{Hom}_{b^*}(M^*, b^*[1/\tau]/b^*)$ is isomorphic to $\mathrm{Hom}_{\mathbb{F}}(M^*, \Sigma^{-1}\mathbb{F}) = \Sigma^{-1}DM^*$. $\square$

By the previous proposition, the change-of-rings spectral sequence converging to $\mathrm{Ext}^{s,t}_{b^*b}(b^*F, b^*)$ has only one non-trivial row, namely the one for $s = 1$, and it collapses. Consequently,

$$\begin{aligned}\mathrm{Ext}^{s,t}_{b^*b}(b^*F, b^*) &\cong \mathrm{Ext}^{s-1,t}_{A^*}(\mathbb{F}, \Sigma^{-1}DH^*Q)\\ &\cong \mathrm{Ext}^{s-1,t-1}_{A^*}(\mathbb{F}, DH^*Q)\\ &\cong \mathrm{Ext}^{s-1,t-1}_{A^*}(H^*Q, \mathbb{F}).\end{aligned}$$

Again, the groups on the E_2-page of the Borel cohomology Adams spectral sequence for $[S^0, F]^G_*$ are isomorphic to those on the E_2-page of the ordinary Adams spectral sequence for $[S^0, Q]_*$, up to a shift.

Homotopy groups of $F(W)$

Now let us consider the G-space $F(W)$ for some G-representation W. Proposition 8 may be used to determine the A^*-module $\mathrm{Ext}^1_{b^*}(b^*F(W), b^*)$. If $k(W) \leqslant p$, it is isomorphic to

$$\Sigma^{-\dim_{\mathbb{R}}(W)}\left(\mathbb{F} \oplus \left(\bigoplus_{j=1}^{k(W)-1} \Sigma^{2j-1}\mathbb{M}(\beta)\right) \oplus \Sigma^{2k(W)-1}\mathbb{F}\right),$$

and the vector space $\mathrm{Ext}^s_{b^*}(b^*F(W), b^*)$ is zero for $s \neq 1$. (One may also compute that – more elementary – using Proposition 1.) Using this, one may assemble the E_2-page for $[S^0, F(W)]^G_*$ without further effort. The Figure 7 shows the result with the hypothesis $k(W) \leqslant p-2$, ensuring $\dim_{\mathbb{R}}(W) - 1 < \dim_{\mathbb{R}}(W^G) + (2p-3)$. The series of dots in the 2-line continues on the right until $t - s = \dim_{\mathbb{R}}(W) + (2p-4)$, followed by zeros until $t - s = \dim_{\mathbb{R}}(W^G) + (4p-7)$. There are no non-trivial differentials in the displayed range.

Homotopy groups of S^W

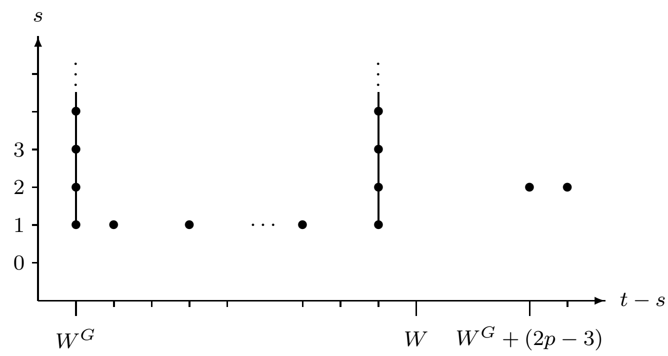
Let W be a non-trivial G-representation. Trying to compute $[S^0, S^W]^G_*$, one might be tempted to use the geometric splitting theorem and the ordinary Adams spectral sequence. While this could also be done, here the use of the Borel cohomology Adams spectral sequence will be illustrated again.

As in the computation of the E_2-term for $[S^V, S^0]^G_*$, in order to get started, one computes the 0-line by hand as follows.

Proposition 9. *There are isomorphisms*

$$\mathrm{Ext}^{0,t}_{b^*b}(b^*S^W, b^*) = \mathrm{Hom}^t_{b^*b}(b^*S^W, b^*) \cong \begin{cases}\mathbb{F} & t = \dim_{\mathbb{R}}(W^G)\\ 0 & t \neq \dim_{\mathbb{R}}(W^G)\end{cases}$$

Figure 7: The Borel cohomology Adams spectral sequence for $[S^0, F(W)]^G_*$



for any G-representation W.

Proof. Here homological grading is used, since one is computing the E_2-page of an Adams spectral sequence. So one should be looking at the vector space of all degree-preserving b^*-linear maps from b^*S^W into $\Sigma^t b^*$ which are also A^*-linear. The one map which immediately comes into mind is the map

$$b^*S^W \longrightarrow b^*S^{W^G} \cong \Sigma^{\dim_{\mathbb{R}}(W^G)} b^* \tag{9}$$

induced by the inclusion. The claim is that (up to scalars) this is the only non-zero one.

The b^*-linear maps are easily classified: since b^*S^W is a free b^*-module, the vector space $\mathrm{Hom}_{b^*}(b^*S^W, \Sigma^t b^*)$ is 1-dimensional for $t \leqslant \dim_{\mathbb{R}}(W^G)$ and zero otherwise. Let $\theta(W)$ and $\theta(W^G)$ be generators for the b^*-modules b^*S^W and $b^*S^{W^G}$, respectively. Write $k = k(W)$. Then the map (9) sends $\theta(W)$ to some scalar multiple of $\tau^k\theta(W^G)$.

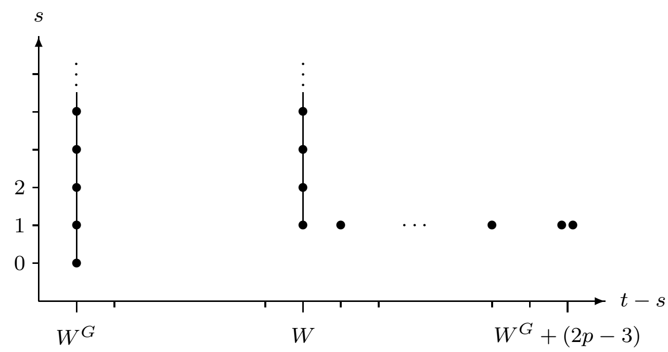
Any map from b^*S^W to b^* of some degree sends the basis element $\theta(W)$ to some scalar multiple of $\sigma^\lambda\tau^l\theta(W^G)$ for some λ in $\{0, 1\}$ and some non-negative integer l. If this map is A^*-linear, A^* must act on $\sigma^\lambda\tau^l$ as it acts on τ^k. But this implies that $\sigma^\lambda\tau^l = \tau^k$: the action of β shows that $\lambda = 0$, and the operation $P^{\max\{k,l\}}$ in A^* distinguishes τ^k and τ^l for $k \neq l$. This argument shows that any A^*-linear map $b^*S^W \to b^*$ has to have degree $\dim_{\mathbb{R}}(W^G)$. □

Using this information on the 0-line, one has a start on the long exact sequence

$$\to \mathrm{Ext}^{s,t}_{b^*b}(b^*S^{W^G}, b^*) \to \mathrm{Ext}^{s,t}_{b^*b}(b^*S^W, b^*) \to \mathrm{Ext}^{s+1,t}_{b^*b}(b^*F(W), b^*) \to$$

induced by the short exact sequence from (5). (In order to use the results obtained for $\mathrm{Ext}^{s+1,t}_{b^*b}(b^*F(W), b^*)$ earlier in this section, the restriction $k(W) \leqslant p - 2$ will have to be made.) This allows to determine the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*S^W, b^*)$ in a range, as displayed in Figure 8.

Figure 8: The Borel cohomology Adams spectral sequence for $[S^0, S^W]^G_*$



5. The prime two

In this final section, the even prime $p = 2$ will be dealt with. Consequently the group G is C_2. Let L denote a non-trivial 1-dimensional real G-representation. The E_2-pages of the Borel cohomology Adams spectral sequences converging to the 2-completions of the groups $[S^0, S^0]^G_*$, $[S^L, S^0]^G_*$, and $[S^0, S^L]^G_*$ will be described in the range $t - s \leqslant 13$. The reader might want to compare the implications for the targets with those obtained by Araki and Iriye (in [**3**]) using different methods. The methods used here are very much the same as in the previous sections, so barely more information than the relevant pictures will be given. The main difference is that the fibre of the inclusion of the fixed points S^0 in S^L is G_+. Therefore, the cofibre sequence (5) induces a short exact sequence of the form

$$0 \longleftarrow \mathbb{F} \longleftarrow b^* \longleftarrow b^*S^L \longleftarrow 0 \qquad (10)$$

in this case.

Computing $[S^0, S^0]^G_*$

To start with, one needs charts of the ordinary Adams spectral sequences for $[S^0, S^0]_*$ and $[S^0, BC_2]_*$ at the prime 2. The information in Figures 9 and 10 is taken from Bruner's tables [**5**]. Lines of slope 1 indicate the multiplicative structure which leads to multiplication with η in the target.

As for S^0, it is classical that the first differential in the ordinary Adams spectral sequence is between the columns $t - s = 14$ and 15. Therefore, there are no differentials in the displayed range. As for BC_2, by the geometric Kahn-Priddy theorem, its homotopy surjects onto that of the fibre of the unit $S^0 \to H\mathbb{Z}$ of the integral Eilenberg-MacLane spectrum $H\mathbb{Z}$. This is reflected in the displayed data, and can be used to infer the triviality of the differentials in the given range. Note that there is also an algebraic version of the Kahn-Priddy theorem, see [**18**].

Using the algebraic splitting (4), the Borel cohomology Adams spectral sequence for $[S^0, S^0]^G$ can then be assembled as for the odd primes, see Figure 11. Since $[S^0, S^0]^G_* \cong [S^0, S^0] \oplus [S^0, BG_+]$ holds by the (geometric) splitting theorem, there

Figure 9: The ordinary Adams spectral sequence for $[S^0, S^0]_*$

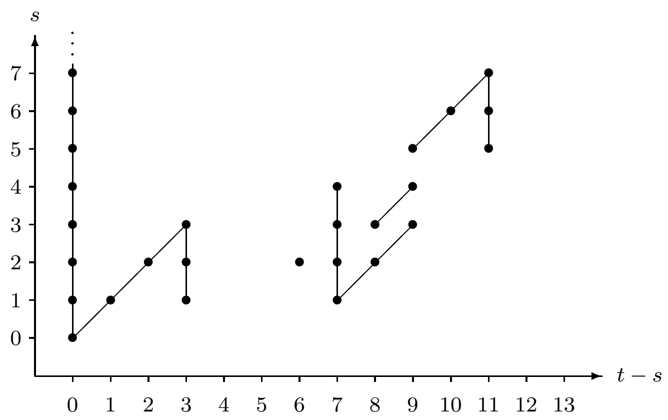
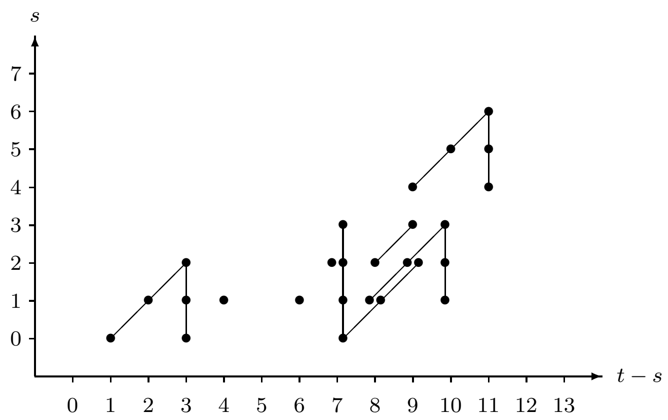


Figure 10: The ordinary Adams spectral sequence for $[S^0, BC_2]_*$



can be no non-trivial differentials in this range: all elements have to survive.

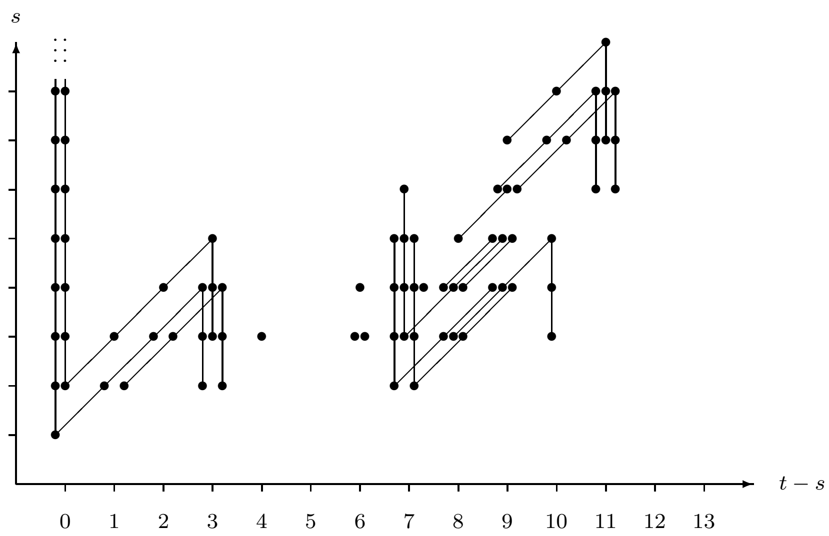
Computing $[S^L, S^0]^G_*$

The Borel cohomology Adams spectral sequence for $[G_+, S^0]^G_*$ has the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*, \mathbb{F}) \cong \mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, \mathbb{F})$ on its E_2-page. Using the long exact sequence

$$\leftarrow \mathrm{Ext}^{s,t}_{b^*b}(b^*, \mathbb{F}) \leftarrow \mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*) \leftarrow \mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^L) \leftarrow$$

associated to the short exact sequence (10), one can now proceed as before to compute the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^L)$ on the E_2-page of the Borel cohomology Adams spectral sequence for $[S^L, S^0]^G_*$ using that the groups on the 0-line must be trivial. One sees that the homomorphism $\mathrm{Ext}^{s,t}_{b^*b}(b^*, \mathbb{F}) \leftarrow \mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*)$ are always surjec-

Figure 11: The Borel cohomology Adams spectral sequence for $[S^0, S^0]^G$



tive so that the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^L)$ are just the kernels.

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*S^L) \cong \mathrm{Ext}^{s-1,t-1}_{A^*}(\mathbb{F}, \mathbb{F}) \oplus \mathrm{Ext}^{s-1,t-1}_{A^*}(H^*BC_2, \mathbb{F})$$

The chart is displayed in Figure 12. Since the differentials in the spectral sequence are natural, the long exact sequence above shows that they must be trivial.

Computing $[S^0, S^L]^G_*$

The Borel cohomology Adams spectral sequence for $[S^0, G_+]^G_*$ has the groups $\mathrm{Ext}^{s,t}_{b^*b}(\mathbb{F}, b^*)$ on its E_2-page. These can be computed by the change-of-rings spectral sequence. One needs to know $\mathrm{Ext}^s_{b^*}(\mathbb{F}, b^*)$ for that. But, the short exact sequence (10) is a free resolution of the b^*-module $\mathbb{F}$ which can be used to compute these extension groups. As for the odd primes, it follows that $E_2^{s,t}$ is isomorphic to $\mathrm{Ext}^{s-1,t-1}_{A^*}(\mathbb{F}, \mathbb{F})$. Using the short exact sequence (10), one may then compute some of the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*S^L, b^*)$ as cokernels of the induced maps.

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*S^L, b^*) \cong \mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}, \mathbb{F}) \oplus \mathrm{Ext}^{s-1,t-1}_{A^*}(H^*BC_2, \mathbb{F})$$

The result is displayed in Figure 13. Again, the differentials vanish in the displayed range.

Acknowledgment. I would like to thank John Greenlees for helpful remarks. In particular, the idea for the proof of Proposition 8 is due to him. In addition, the referee deserves thanks. Her or his report has led to great improvements.

Figure 12: The Borel cohomology Adams spectral sequence for $[S^L, S^0]^G$

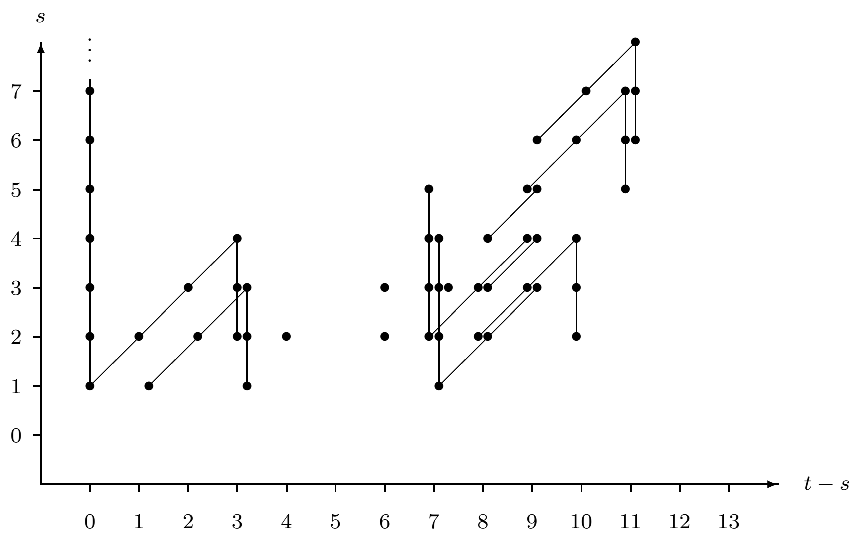
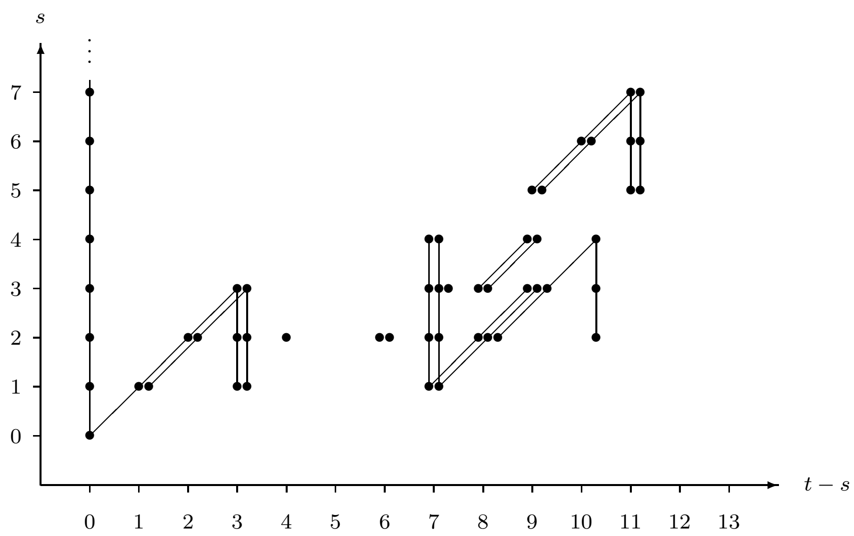


Figure 13: The Borel cohomology Adams spectral sequence for $[S^0, S^L]^G$



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