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NOTE ON COMMUTATIVITY IN DOUBLE SEMIGROUPS AND TWO-FOLD MONOIDAL CATEGORIES

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(*communicated by Ronald Brown*)

Abstract

A concrete computation — twelve slidings with sixteen tiles — reveals that certain commutativity phenomena occur in every double semigroup. This can be seen as a sort of Eckmann-Hilton argument, but it does not use units. The result implies in particular that all cancellative double semigroups and all inverse double semigroups are commutative. Stepping up one dimension, the result is used to prove that all strictly associative two-fold monoidal categories (with weak units) are degenerate symmetric. In particular, strictly associative one-object, one-arrow 3-groupoids (with weak units) cannot realise all simply-connected homotopy 3-types.

1. Introduction and results

1.1. The Eckmann-Hilton argument. In 1932, Čech introduced the higher homotopy groups $\pi_i, i > 1$ in a contribution submitted to the International Congress of Mathematicians in Zürich. His paper was received by Alexandrov and Hopf who quickly realised that all these groups are abelian (or perhaps Čech had noticed this himself), and for this reason they felt it could not be the correct notion. They convinced Čech to withdraw his paper, and in the final proceedings only a very short communication of Čech was included [**3**].[1]

The natural generality of the commutativity argument, known as the Eckmann-Hilton argument [**5**], is an elementary statement about double monoids (although asssociativity is not essential). Recall that a double monoid is a set S equipped with two compatible monoid structures, i.e. two associative and unitary multiplications $*_h$ and $*_v$ satisfying the interchange law

$$(x *_h y) *_v (z *_h w) = (x *_v z) *_h (y *_v w)$$

for all $x, y, z, w \in S$. If $*_h$ is depicted horizontally and $*_v$ vertically, then the interchange law says that in a composite composition

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[1] I learned this story from Ronnie Brown, who in turn got it from Eldon Dyer.

$$\begin{array}{|c|c|}\hline z & w \\ \hline x & y \\ \hline\end{array}$$

it doesn't matter whether the vertical or the horizontal composition is performed first.

It is automatic from the interchange law that the two units coincide. This unit, which we denote by 1, obviously commutes with every other element, for the horizontal as well as the vertical composition. This innocent-looking special case of commutativity in fact forces the two composition laws to be commutative, and to coincide. This is the Eckmann-Hilton argument:

$$\begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \quad = \quad \begin{array}{|c|c|}\hline 1 & b \\ \hline a & 1 \\ \hline\end{array} \quad = \quad \begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} \quad = \quad \begin{array}{|c|c|}\hline b & 1 \\ \hline 1 & a \\ \hline\end{array} \quad = \quad \begin{array}{|c|c|}\hline b & a \\ \hline\end{array}$$

(Since the homotopy group π_2 is defined in terms of maps from squares, which can be composed in two compatible ways (horizontally and vertically), the commutativity of π_2 as well as all higher homotopy groups follows readily [**5**].)

1.2. Double semigroups. It is clear that the existence of the unit is a key ingredient in the Eckmann-Hilton argument. For double semigroups (i.e. sets with two compatible non-unital associative multiplications), the argument does not work, and indeed it is easy to exhibit examples of double semigroups which are not commutative.

The main discovery of the present note is that certain commutativity phenonema do arise even in double semigroups without units. These phenomena occur in expressions with many terms, where the interchange law and associativity combined give rise to some rearrangements of terms. One such commutativity is expressed by

Proposition 2.3. *For any sixteen elements $a, b, \ldots$ in any double semigroup, this equation holds:*

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & a & b & \\ \hline & & & \\ \hline & & & \\ \hline\end{array} \quad = \quad \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & b & a & \\ \hline & & & \\ \hline & & & \\ \hline\end{array}$$

(The empty boxes represent fourteen nameless elements, the same on each side of the equation, and in the same order.)

The proof is an elementary computation exploiting the geometrical representation of the two multiplication laws. It would be quite cumbersome to write it out algebraically.

1.3. Cancellative double semigroups and inverse double semigroups are commutative. In double semigroups with some further cancellation properties, the blanks in the equation can be cancelled away and we get absolute commutativity. In particular, every cancellative double semigroup is commutative (Corollary 2.5). It is also shown that every inverse double semigroup is commutative (Proposition 2.8).

1.4. Two-fold monoidal categories and braidings. The notion of double monoid as well as the Eckmann-Hilton argument make sense in any monoidal category in place of the category of sets. A double monoid in ***Cat*** is the same thing as a category with two compatible strict monoidal structures, and the Eckmann-Hilton argument shows that such are commutative.

It is natural to consider the non-strict version of this situation, i.e. a category equipped with two (non-strict) monoidal structures which are compatible up to coherent isomorphism. In this case the Eckmann-Hilton argument consists of a sequence of specific isomorphisms, and it turns out to define a braiding. This was observed by Joyal and Street [**8**] in 1985, and in fact was one of the motivating examples for their discovery of the notion of braided monoidal category. (Conversely a braiding on a monoidal category can be used to construct a second, weakly compatible, monoidal structure.) In this way, the Eckmann-Hilton argument is directly related to core subtleties of higher category theory.

1.5. Strictifications. One of the key themes in higher category theory is strictification. Finding strict or semi-strict models for weak structure often amounts to powerful coherence results. It is well known that every monoidal category is equivalent to a strict one [**14**] [**15**, XI.3], but the argument of the previous paragraph shows that not every two-fold monoidal category is equivalent to a strict one — which is just another expression of the fact that not every braided monoidal category is braided equivalent to a commutative monoid in ***Cat***. In fact, two-fold monoidal categories can be seen as tricategories with one object and one arrow, and the observation is the simplest case of the fact that not all tricategories are equivalent to strict 3-categories [**6**].

As a rule of thumb, one can strictify one level of weak structure, but in general not two levels at the same time; see for example Paoli [**16**] who studies two different one-level strictifications of weak 3-groupoids in the sense of Tamsamani [**18**], and compares with cat^2-groups [**13**] in the path-connected case. However, it is sometimes possible to strictify one level of structure and parts of other levels. For example, every two-fold monoidal category is equivalent to a two-fold strict monoidal category with iso-interchange [**1**] (corresponding to the fact that every braided monoidal category is equivalent to a braided strict monoidal category).

The braiding that results from the ***Cat*** version of the Eckmann-Hilton argument is a composite of unit structure isomorphisms and interchange isomorphisms. As just explained, it is possible to strictify the units if the interchange law is kept weak. It is a natural question whether it would be possible to strictify instead the interchange law while keeping the unit weak. This idea is related to Simpson's conjecture:

1.6. Simpson's conjecture. Based on a careful analysis of strict 3-groupoids, and the observation that the units play a key role in the Eckmann-Hilton argument, Simpson [**17**] was led to suspect that units (= identity arrows) can account for all higher homotopical data in higher categories. A strong version of his conjecture states roughly that every weak n-category is equivalent to one where only the units are weak. (See [**17**] and [**10**] for more formal statements of the conjecture.) Simpson's conjecture is highly surprising and goes against all trends in higher category theory, where the emphasis was always on the composition laws.

A weaker version of the conjecture states that strictly associative n-groupoids with weak units is a model for homotopy n-types. In contrast, completely strict n-groupoids can model only homotopy n-types whose higher Whitehead brackets (i.e., those beyond the action of π_1) are zero. The first interesting case is that of 1-connected homotopy 3-types, since this is the first appearance of a non-trivial higher Whitehead bracket, $\pi_2 \otimes \pi_2 \to \pi_3$, which in turn is closely related to the braiding. In fact, 1-connected 3-types can be realised by braided categorical groups, cf. Joyal-Tierney [**9**]; see also Brown-Gilbert [**2**] for a closely related model. A version of Simpson's conjecture in this case was proved in Joyal-Kock [**7**]: one-object 3-groupoids which are strict in all respects except that there are only weak identity arrows (forming a contractible space) can model all 1-connected homotopy 3-types.

1.7. Strictification of composition and interchange? It is tempting to reformulate (and distort) Simpson's conjecture by saying that composition and interchange can always be strictified if just the units remain weak. One might think that every two-fold monoidal category is equivalent to one with strict compositions and strict interchange. However, this is false: in Section 3 of this note, the commutativity in Proposition 2.3 is used to show that

> **Proposition 3.2.** *If $\mathscr{C}$ is a two-fold monoidal category, strictly associative and with strict interchange, then $\mathscr{C}$ is degenerate symmetric (i.e. has a symmetry σ such that $\sigma_{X,X} = id_{X\otimes X}$).*

Recall from [**1**] that braided monoidal categories correspond to two-fold loop spaces (via group completion of the nerve) and arbitrary 1-connected homotopy 3-types [**8**], while symmetric monoidal categories correspond to infinite loop spaces. Since infinite loop spaces have vanishing Whitehead bracket $\pi_2 \otimes \pi_2 \to \pi_3$, we find the following corollary.

> **Corollary 3.4.** *Strictly associative one-object, one-arrow* 3*-groupoids (but still with weak units) cannot realise all simply-connected homotopy* 3*-types.*

Acknowledgements. The sliding argument of Proposition 2.3 was discovered after conversations with André Henriques, whom I thank for precious input. I am also thankful to Ronnie Brown for some pertinent comments.

2. Double semigroups

A *double semigroup* is a set equipped with two compatible associative multiplication laws. In other words, it is just like a double monoid, except that there is no unit.

2.1. Example. A double semigroup is not necessarily commutative: take any set with at least two elements, and let both composition laws be the 'K-combinator': $x *_v y := x *_h y := x$. Clearly, this is associative, and satisfies the interchange law, but it is not commutative.

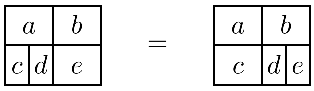
2.2. Sliding tiles. When writing the graphical representation of some product in a double semigroup, it is important to note that there is a certain freedom in where to set the 'walls' — this comes about because of associativity. For example, in the product

$$(c *_h d *_h e) *_v (a *_h b)$$

there are two ways of setting parentheses in $(c *_h d *_h e)$:

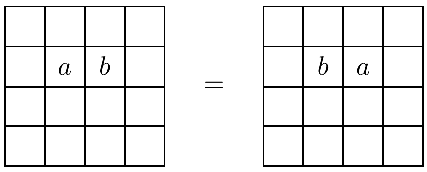
$$((c *_h d) *_h e) *_v (a *_h b) = (c *_h (d *_h e)) *_v (a *_h b).$$

Hence graphically we get



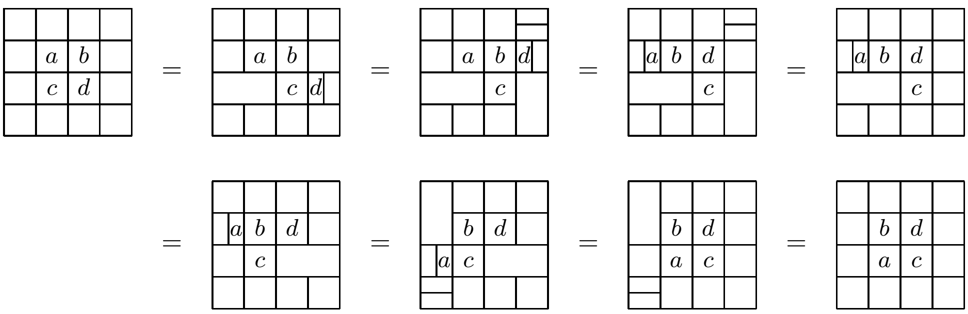
The upshot is that in the graphical representation, sliding the inner walls of a given rectangle does not change the corresponding algebraic expression. It is clear that such slidings can never change the order of the elements that touch the border of the expression. Hence in the picture above, the order of the five elements will always be c, d, e, b, a, walking around counter-clockwise. But in expressions with more elements, nontrivial permutations can take place, as the following computation shows.

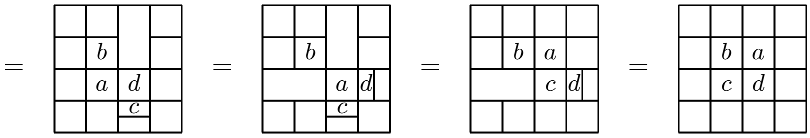
Proposition 2.3. *For any sixteen elements $a, b, \ldots$ in any double semigroup, this equation holds:*



(The empty boxes represent fourteen nameless elements, the same on each side of the equation, and in the same order.)

Proof. We shall perform twelve slidings, each representing a strict equality. We only label the middle four elements, since anyway the configuration of the elements touching the border is fixed by any sliding.





(Note that the first eight moves effectuate a cyclic permutation of the four middle elements, and that the last four moves permutes three of the middle elements.) ☐

The remainder of this section explores a few consequences of the commutativity phenomenon.

2.4. Cancellative double semigroups. A double semigroup is *cancellative* if $x * c = y * c$ implies $x = y$, for multiplication with any c from any of the four sides. The following corollary is immediate.

Corollary 2.5. *A cancellative double semigroup is commutative.* ☐

More generally it is sufficient that there exists one cancellable element c whose powers in both directions are also cancellable: then place this c in the empty boxes of the argument and cancel.

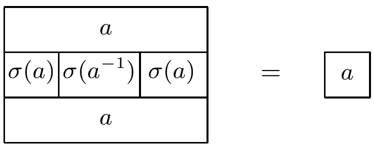
2.6. Inverse double semigroups. Recall that two elements x and y in a semigroup are said to be each other's inverses if $xyx = x$ and $yxy = y$, and that an *inverse semigroup* is one where every element has a unique inverse. This can also be described as a universal algebraic structure: an associative multiplication, together with a unary 'inverse' operation $x \mapsto x^{-1}$, satisfying $(xy)^{-1} = y^{-1}x^{-1}$, and $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$. Inverse semigroups are important in many areas of mathematics and arise notably as semigroups of partial symmetries, cf. Clifford-Preston [**4**]. See also Lawson [**12**], who explains the equivalence (due to Ehresmann and Schein) between inverse semigroups and certain ordered groupoids.

An *inverse double semigroup* is a double semigroup both of whose semigroup structures are inverse. We shall see in a moment that inverse double semigroups are commutative.

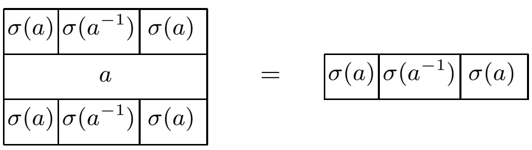
Lemma 2.7. *In an inverse double semigroup, the horizontal and vertical inverse operations commute.*

Proof. Given an element $a \in S$, let a^{-1} denote the horizontal inverse, and let $\sigma(a)$ denote the vertical inverse. We need to show that $\sigma(a^{-1}) = \sigma(a)^{-1}$.

Claim 1:



Claim 2:



Both claims follow by starting on the left-hand side by rewriting $a = aa^{-1}a$. Then use the interchange law, and compute each column.

These two equations show that $\sigma(a)\sigma(a^{-1})\sigma(a)$ is the vertical inverse to a, in other words

$$\sigma(a)\sigma(a^{-1})\sigma(a) = \sigma(a).$$

Now repeat the arguments with a^{-1} and a interchanged, to show also

$$\sigma(a^{-1})\sigma(a)\sigma(a^{-1}) = \sigma(a^{-1}).$$

These two equations show that $\sigma(a^{-1})$ is the horizontal inverse to $\sigma(a)$. That is, $\sigma(a^{-1}) = \sigma(a)^{-1}$ as we wanted to show. ☐

In view of the lemma, we can adopt the following notation: given an element a, let a^{-1} denote the horizontal inverse, and let A denote the vertical inverse. Then by the lemma, the horizontal inverse of A is the vertical inverse of a^{-1}, and we can denote this element by A^{-1} without unambiguity.

Proposition 2.8. *Every inverse double semigroup is commutative.*

Proof. Let A and B be two elements in an inverse double semigroup. The proof consists in writing a big multiplication where AB appears in the middle, as inverse to the outer factors. Then commute A and B using Proposition 2.3. By uniqueness of inverses we can then conclude that $AB = BA$. In fact this argument is needed four times: one for each way one element can be inverse to another. Here goes:

Claim 1:

$$\begin{array}{|c|}\hline ab \\ \hline ABA^{-1}B^{-1}AB \\ \hline ab \\ \hline \end{array} \quad = \quad \boxed{ab}$$

Claim 2:

$$\begin{array}{|c|}\hline ABA^{-1}B^{-1}AB \\ \hline ab \\ \hline ABA^{-1}B^{-1}AB \\ \hline \end{array} \quad = \quad \boxed{ABA^{-1}B^{-1}AB}$$

Claim 1 and 2 together say that $ABA^{-1}B^{-1}AB$ is the vertical inverse to ab. That is,

$$AB\ A^{-1}B^{-1}\ AB = AB.$$

Repeating the arguments with $()^{-1}$ on every symbol we find also

$$A^{-1}B^{-1}\ AB\ A^{-1}B^{-1} = A^{-1}B^{-1}.$$

These two equation together say that AB is the horizontal inverse to $A^{-1}B^{-1}$. But so is BA, and we are done, provided we can prove the two claims.

Let us just prove Claim 1. The second claim is analogous.

$$\begin{array}{|c|}\hline ab \\ \hline ABA^{-1}B^{-1}AB \\ \hline ab \\ \hline \end{array} \quad \overset{(1)}{=} \quad \begin{array}{|c|}\hline abb^{-1}a^{-1}ab \\ \hline ABA^{-1}B^{-1}AB \\ \hline abb^{-1}a^{-1}ab \\ \hline \end{array}$$

$$
\overset{(2)}{=} \quad
\begin{array}{|c|c|c|c|c|c|}
\hline
a & b & b^{-1} & a^{-1} & a & b \\ \hline
A & B & A^{-1} & B^{-1} & A & B \\ \hline
a & b & b^{-1} & a^{-1} & a & b \\ \hline
A & B & B^{-1} & A^{-1} & A & B \\ \hline
a & b & b^{-1} & a^{-1} & a & b \\ \hline
\end{array}
$$

$$
\overset{(3)}{=} \quad
\begin{array}{|c|c|c|c|c|c|}
\hline
a & b & b^{-1} & a^{-1} & a & b \\ \hline
A & B & B^{-1} & A^{-1} & A & B \\ \hline
a & b & b^{-1} & a^{-1} & a & b \\ \hline
A & B & B^{-1} & A^{-1} & A & B \\ \hline
a & b & b^{-1} & a^{-1} & a & b \\ \hline
\end{array}
$$

$$
\overset{(4)}{=} \quad \boxed{ab}
$$

Step (1) is to rewrite using $ab = ab(ab)^{-1}ab = abb^{-1}a^{-1}ab$. In Step (2), the bottom row is expanded into three rows. Step (3) is the crucial commutation, justified by Proposition 2.3. In Step (4), each column is reduced to a single symbol, and the resulting six-fold horizontal product is resolved. ☐

3. Two-fold monoidal categories

3.1. Double semigroups in *Cat*. As remarked, Proposition 2.3 holds true for double semigroups in any monoidal category. We shall now be concerned with double semigroups in ***Cat***; these are categories equipped with two strictly associative multiplication laws satisfying the strict interchange law. Observe that Proposition 2.3 holds also when the symbols a and b represent arrows.

As in the previous section, in order to get a useful and generic statement out of Proposition 2.3, we need the presence of some cancellability. One interesting case (the only one we consider) is when the two multiplication laws possess weak units, that is when we have a two-fold monoidal category, strictly associative and with strict interchange. Weak units are in particular weakly cancellable, which is what we need. The weakness means that we do not get strict commutativity as conclusion, but we do get a symmetry, and in fact a degenerate one. We say that a symmetry σ on a monoidal category is *degenerate* if for every object X we have $\sigma_{X,X} = \mathrm{id}_{X\otimes X}$. The result is this:

Proposition 3.2. *Let $\mathscr{C}$ be a two-fold monoidal category, strictly associative and with strict interchange, but only weak units. Then $\mathscr{C}$ is degenerate symmetric.*

The proof starts on the next page.

Note that degeneracy is stable under braided monoidal equivalence, and in particular most braided monoidal categories are not equivalent to degenerate symmetric ones. Hence we get this corollary:

Corollary 3.3. *Not every two-fold monoidal category is equivalent to one with strict associativity and strict interchange.* □

¿From standard facts about braided categorical groups and homotopy 3-types, as explained in the Introduction, we then get:

Corollary 3.4. *Strictly associative one-object, one-arrow* 3*-groupoids (but still with weak units) cannot realise all simply-connected homotopy* 3*-types.* □

3.5. Weak units. A weak unit for a category with (strict) multiplication is an object I equipped with natural isomorphisms $\lambda_X : I \otimes X \xrightarrow{\sim} X$ and $\rho_X : X \otimes I \xrightarrow{\sim} X$ satisfying $\mathrm{id}_X \otimes \lambda_Y = \rho_X \otimes \mathrm{id}_Y$. Alternatively [**11**], a weak unit can be characterised as an object I equipped with a single isomorphism $\alpha : I \otimes I \xrightarrow{\sim} I$ and having the property that tensoring with it from either side is an equivalence of categories. (The left and right constraints can be canonically constructed from α, cf. [**11**].) The property that tensoring with I from either side is an equivalence is the crucial property for the present purposes — we shall say that I is *cancellable.*

3.6. Two-fold strictly associative monoidal categories. Let $(\mathscr{C}, \otimes_h, \otimes_v)$ denote a strict double semigroup in ***Cat***, where each multiplication has a weak unit, denoted respectively (I_h, α_h) and (I_v, α_v). Certain compatibilities could be required of the two structures, for example commutativity of the square ✠ in the next proof, but we shall not need any such further conditions.

Lemma 3.7. *The two units are isomorphic as objects in* $\mathscr{C}$: $I_h \simeq I_v$.

Proof. In fact there are two different isomorphisms, depending on which route we take in the diagram

$$\begin{tikzcd}
 & \boxed{\begin{array}{c|c} I_h & I_v \\ \hline I_v & I_h \end{array}} \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "\maltese"] & \boxed{\begin{array}{c} I_v \\ \hline I_v \end{array}} \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \boxed{I_v} \\
\boxed{I_h} \arrow[r, phantom, "\simeq"] & \boxed{\begin{array}{c|c} I_h & I_h \end{array}} \arrow[r, phantom, "\simeq"] & \boxed{\begin{array}{c|c} I_v & I_h \\ \hline I_h & I_v \end{array}} &
\end{tikzcd}$$

□

We shall not need any specific isomorphism — just from its existence we conclude that I_h is cancellable also vertically. So tensoring with I_h from any of the four sides is an equivalence of categories, that's all we need to know. We now set

$$I := I_h,$$

and forget about the vertical unit.

Proof of Proposition 3.2. Step 1: the functor

$$\begin{aligned} F : \mathscr{C} &\longrightarrow \mathscr{C} \\ X &\longmapsto \begin{matrix} I & I & I \\ I & X & I \\ I & I & I \\ I & I & I \end{matrix} \end{aligned}$$

is an equivalence of categories. This is clear since it is the composite of tensoring with I from each of the four sides.

Step 2: the functor F has a strong multiplicative structure (with respect to $\otimes := \otimes_h$). In other words, there are natural isomorphisms $FX \otimes FY \xrightarrow{\sim} F(X \otimes Y)$ satisfying the usual associativity condition [**15**, XI.2]. Specifically,

$$\begin{matrix} I & I & I & I & I & I \\ I & X & I & I & Y & I \\ I & I & I & I & I & I \\ I & I & I & I & I & I \end{matrix} \quad \xrightarrow{\sim} \quad \begin{matrix} I & I & I \\ I & XY & I \\ I & I & I \\ I & I & I \end{matrix}$$

is defined row-wise as the composite of horizontal left and right constraints, multiplying together horizontally the four middle columns. By coherence for the horizontal unit, it does not matter how these multiplications are effectuated.

Step 3: F has the property that for any pair of objects (or arrows) $X, Y \in \mathscr{C}$, we have $FX \otimes FY = FY \otimes FX$. This follows immediately from Proposition 2.3.

Now we define a symmetry σ on $\otimes$ component-wise by transporting these equalities back along the multiplicative equivalence F. Explicitly, by fully faithfulness and strong multiplicativity of F we have bijections

$$\mathscr{C}(X \otimes Y, Y \otimes X) \simeq \mathscr{C}(F(X \otimes Y), F(Y \otimes X)) \simeq \mathscr{C}(FX \otimes FY, FY \otimes FX)$$

and in this last hom set we have a distinguished element, namely the identity arrow. Define $\sigma_{X,Y}$ to be the arrow $X \otimes Y \to Y \otimes X$ corresponding to the identity arrow under this bijection. It is easy to see that the $\sigma_{X,Y}$ are natural in X and Y, and that they satisfy the axioms for a symmetry [**8**, Def. 2.1]. (Note that compatibility with the unit is automatic from the axioms, cf. [**8**, Prop. 2.1].) It is also clear that σ is degenerate. $\square$

References

[1] Cornel Bălteanu, Zbigniew Fiedorowicz, Roland Schwänzl, and Rainer Vogt. *Iterated monoidal categories.* Adv. Math. **176** (2003), 277–349. ArXiv:math.AT/9808082.

[2] Ronald Brown and Nicholas D. Gilbert. *Algebraic models of* 3*-types and automorphism structures for crossed modules.* Proc. London Math. Soc. (3) **59** (1989), 51–73.

[3] Eduard Čech. *Höherdimensionale Homotopiegruppen.* In *Verhandlungen des internationalen Mathematiker-Kongresses, Zürich 1932*, vol. 2, page 203. Walter Saxer, Zürich, 1932. Reprint, Kraus, Nendeln, Liechtenstein, 1967.

[4] Alfred H. Clifford and Gordon B. Preston. *The algebraic theory of semigroups. Vol. I+II.* No. 7 in Mathematical Surveys. American Mathematical Society, Providence, R.I., 1961;1967.

[5] Beno Eckmann and Peter J. Hilton. *Group-like structures in general categories I: multiplication and comultipliplcation.* Math. Ann. **145** (1962), 227–255.

[6] Robert Gordon, A. John Power, and Ross Street. *Coherence for tricategories.* Mem. Amer. Math. Soc. **117** (1995), vi+81pp.

[7] André Joyal and Joachim Kock. *Weak units and homotopy* 3*-types.* To appear in the StreetFest Proceedings (Contemp. Math., edited by M. Batanin, A. Davydov, M. Johnson, S. Lack, and A. Neeman). ArXiv:math.CT/0602084,

[8] André Joyal and Ross Street. *Braided tensor categories.* Adv. Math. **102** (1993), 20–78. Formerly Macquarie Math. Reports No. 850067 (Dec. 1985) and No. 860081 (Nov. 1986).

[9] André Joyal and Myles Tierney. *Algebraic homotopy types.* Unpublished manuscript (1984).

[10] Joachim Kock. *Weak identity arrows in higher categories.* IMRP Internat. Math. Res. Papers **2006** (2006), 1–54. ArXiv:math.CT/0507116.

[11] Joachim Kock. *Elementary remarks on units in monoidal categories.* Preprint, arXiv:math.CT/0507349.

[12] Mark V. Lawson. *Inverse semigroups. The theory of partial symmetries.* World Scientific Publishing Co. Inc., River Edge, NJ, 1998.

[13] Jean-Louis Loday. *Spaces with finitely many nontrivial homotopy groups.* J. Pure Appl. Alg. **24** (1982), 179–202.

[14] Saunders Mac Lane. *Natural associativity and commutativity.* Rice Univ. Studies **49** (1963), 28–46.

[15] Saunders Mac Lane. *Categories for the working mathematician, second edition.* No. 5 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[16] Simona Paoli. *Semistrict models of connected 3-types and Tamsamani's weak 3-groupoids.* Preprint, arXiv:math.AT/0607330.

[**17**] Carlos Simpson. *Homotopy types of strict 3-groupoids.* Preprint, arXiv:math.CT/9810059.

[**18**] Zouhair Tamsamani. *Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux.* K-Theory **16** (1999), 51–99. (alg-geom/9512006 and alg-geom/9607010).

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