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HOMOTOPY-THEORETIC ASPECTS OF 2-MONADS

STEPHEN LACK

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Abstract

We study 2-monads and their algebras using a **Cat**-enriched version of Quillen model categories, emphasizing the parallels between the homotopical and 2-categorical points of view. Every 2-category with finite limits and colimits has a canonical model structure in which the weak equivalences are the equivalences; we use these to construct more interesting model structures on 2-categories, including a model structure on the 2-category of algebras for a 2-monad T, and a model structure on a 2-category of 2-monads on a fixed 2-category $\mathscr{K}$.

1. Introduction

1.1. There are obvious connections between 2-category theory and homotopy theory. It is possible, for instance, to construct a 2-category whose objects are spaces, whose morphisms are paths parametrized by intervals of variable length, and whose 2-cells are suitably defined equivalence classes of homotopies between paths. On the other hand, in 2-category theory one tends to say that arrows are isomorphic rather than equal, and that objects are equivalent rather than isomorphic, typically with some coherence conditions involved, and this is analogous to working "up to all higher homotopies".

In both these cases, the 2-categorical picture is somewhat simpler than the homotopical one. In the former case, when one says "up to all higher homotopies", this process does not extend up very far: there are isomorphisms between arrows, and equations between the isomorphisms, but that is as far as it goes with 2-categories. For still higher homotopies, one needs to use n-categories for higher n, or possibly ω-categories. This "degeneracy" of 2-categories, from the homotopy point of view, is closely related to the equivalence relations one must impose on homotopies in order to obtain, as in the previous paragraph, a 2-category of spaces, paths, and homotopies.

1.2. Under this analogy, 2-categories correspond to spaces whose homotopy groups π_n are trivial for $n > 2$, while mere categories would correspond to spaces

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with π_n trivial for $n > 1$. It is, however, possible to model all spaces using just categories, via the nerve construction. This point of view on categories was prominent in work of Quillen [**15**] and Segal [**16**], and is the basis of the theorem of Thomason [**17**] that there is a model structure on the category **Cat** of small categories which is Quillen equivalent to the standard model structure on the category **SSet** of simplicial sets. There is also a corresponding result for **2-Cat** [**18**]. Under this point of view, categories are regarded as being the same if their nerves are homotopy equivalent; this is a much coarser notion than the usual categorical point of view, adopted here, that they are the same if they are equivalent as categories.

1.3. In this paper we pursue the point of view that 2-category theory can be seen as a slightly degenerate part of homotopy theory, and explore various consequences, mostly within the area of two-dimensional monad theory, as developed for example in [**3**]. This point of view builds upon the earlier papers [**12, 13**] which constructed Quillen model structures on the categories **2-Cat** of 2-categories and 2-functors, and **Bicat** of bicategories and strict homomorphisms, investigated their homotopy-theoretic properties, and related these to the existing theory of 2-categories.

The difference between this paper and the earlier ones is that before we looked at a model structure for the category of all 2-categories, whereas here we consider model structures on particular 2-categories. There is a notion of model structure on an enriched category, where the base $\mathscr{V}$ for the enrichment itself has a suitable model structure, and we use this in the case $\mathscr{V} = \mathbf{Cat}$, so that a $\mathscr{V}$-category is a 2-category. We therefore speak of a *model 2-category*, and these are the tools for our analysis of the homotopy-theoretic aspects of 2-monad theory.

1.4. A model 2-category has three specified classes of morphisms, called the cofibrations, the weak equivalences, and the fibrations. They satisfy all the usual properties of model categories, as well as a strengthened version of the lifting properties, which provides the relationship between the model structure and the enrichment. We describe the details in Section 3.

1.5. It turns out that every 2-category with finite limits and colimits has a canonical model structure in which the weak equivalences are the equivalences. The details are described in Section 4. This can be seen as a 2-categorical analogue of the fact that every category has a model structure, called the "trivial" structure, in which the weak equivalences are the isomorphisms. This trivial structure is in practice almost never compatible with the enrichment, and so there is little harm in speaking of the "trivial model structure on the 2-category $\mathscr{K}$" to mean this one with the equivalences as weak equivalences; for a more precise statement, see Remarks 3.5 and 4.16. Just as in the case of ordinary categories, for the trivial model structure on a 2-category, all objects are fibrant and cofibrant. The factorizations can be constructed in a uniform way using limits and colimits.

1.6. The model structures of real interest are not the trivial ones; rather they can be constructed from the trivial ones via a lifting process. If $\mathscr{K}$ is a locally presentable 2-category, and T is a 2-monad on $\mathscr{K}$, there is a 2-category $T\text{-Alg}_s$ of strict T-algebras, strict T-morphisms, and T-transformations, and a forgetful 2-functor $U_s : T\text{-Alg}_s \to \mathscr{K}$ with a left adjoint $F_s \dashv U_s$. A variety of examples will be discussed in the following paragraphs; all of these 2-monads and more can be found in the final section of [**3**]. We use this adjunction to define a model structure on $T\text{-Alg}_s$, in which a strict T-morphism f is a fibration or weak equivalence in $T\text{-Alg}_s$ if and only if the underlying $U_s f$ is one in $\mathscr{K}$, where $\mathscr{K}$ is equipped with the trivial model structure. The resulting model structure on $T\text{-Alg}_s$ is not itself trivial; in particular it has weak equivalences which are not equivalences in $T\text{-Alg}_s$. The details of the process are developed in Section 5. The following section provides an overview of various examples of such 2-monads T, and of the remainder of the paper.

1.7. This paper has had a long gestation period, with the basic results dating back to 2002. I am grateful to the participants of the seminars at which it was presented — the Australian Category Seminar (2002 and 2006) and the Chicago Category Seminar (2006) — for their interest and for various helpful comments. Part of the writing up was done during a visit to Chicago in May 2006, and I am very grateful to Peter May, Eugenia Cheng, and the members of the topology/categories group for their hospitality.

2. Overview of examples

2.1. An important class of examples is obtained by taking $\mathscr{K} = \mathbf{Cat}$. Then T describes some sort of algebraic structure borne by categories. For example this could be monoidal categories, strict monoidal categories, symmetric monoidal categories, categories with finite limits, categories with finite products and coproducts satisfying the distributive law, and so on. In each case, an algebra will consist of a category equipped with a specific choice of all elements of the structure (for example, a specific choice of the product $X \times Y$ of two objects, if the structure involves binary products), and the strict T-morphisms will be the functors which strictly preserve these choices. Such strict morphisms are of theoretical importance only; usually one would consider the pseudo T-morphisms, which preserve the structure up to suitably coherent isomorphisms. We write T-Alg for the 2-category of strict T-algebras, pseudo T-morphisms, and T-transformations, and usually speak just of T-morphisms, with the "pseudo" variety of morphism being the default. There is a sense, made precise in Theorem 5.15 below, in which T-Alg is the "homotopy 2-category" of $T\text{-Alg}_s$.

It is familiar in homotopy theory that up-to-homotopy morphisms from A to B can be identified, in an up-to-homotopy sense, with ordinary morphisms from a cofibrant replacement of A to a fibrant replacement of B. There is a corresponding, but rather tighter result here. Every object is already fibrant, and for any A there is a particular cofibrant replacement A' of A for which the pseudomorphisms from A to B are in bijection with the strict ones from A' to B. In the 2-categorical context the cofibrant objects are usually called flexible.

2.2. There are other algebraic structures borne by categories which cannot be described in terms of 2-monads on **Cat**, but can be described by 2-monads on the 2-category $\mathbf{Cat}_g$ of categories, functors, and natural *isomorphisms*. A typical example is the structure of monoidal closed category. The point is that the internal hom is covariant in one variable but contravariant in the other, and there is no way to describe operations $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ using 2-monads on **Cat**. But if we work with $\mathbf{Cat}_g$ then there is no problem: the internal hom is then seen as an operation $\mathscr{C}_{\mathrm{iso}} \times \mathscr{C}_{\mathrm{iso}} \to \mathscr{C}_{\mathrm{iso}}$, where $\mathscr{C}_{\mathrm{iso}}$ is the subcategory of $\mathscr{C}$ consisting of the isomorphisms, and as such the operation is perfectly well-defined. There are subtleties involved, in that more work is required to encode the functoriality of the tensor product: see [**3**]. Similarly such structures as symmetric monoidal closed categories, compact closed categories, cartesian closed categories, or toposes can be described by 2-monads on $\mathbf{Cat}_g$.

2.3. Another important class of examples arises on taking $\mathsf{ob}\mathscr{C}$ to be the objects of a small 2-category $\mathscr{C}$, and $\mathscr{K}$ to be the 2-category $[\mathsf{ob}\mathscr{C}, \mathbf{Cat}]$ of $\mathsf{ob}\mathscr{C}$-indexed families of categories. Then there is a 2-monad T on $\mathscr{K}$ for which $T\text{-Alg}_s$ is the presheaf 2-category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$, consisting of 2-functors, 2-natural transformations, and modifications. The pseudomorphisms in this case are the pseudonatural transformations. In Section 6 we study this example, thinking of presheaves as being the weights for weighted colimits (or limits). The cofibrant objects are once again important: they are the weights for flexible colimits. Flexible colimits are those which can be constructed out of four basic types: coproducts, coinserters, coequifiers, and splittings of idempotents. The coinserters and coequifiers are 2-dimensional colimits which universally add or make equal 2-cells between given 1-cells. In the context of **Cat**, this corresponds to adding or making equal morphisms of a category, without changing the objects. One could use these colimits to force two objects to be isomorphic, but not to be equal.

It is known that the flexible algebras in $T\text{-Alg}_s$ are closed under flexible colimits; in fact they are precisely the closure under flexible colimits of the free algebras. In Theorem 6.4 we give a more general reason for this first fact: in any model 2-category, the cofibrant objects are always closed under flexible colimits. This is a **Cat**-enriched version of the fact that in any model category the cofibrant objects are closed under coproducts and retracts.

2.4. The next example involves a 2-category of 2-monads on a fixed base 2-category $\mathscr{K}$, which in this introduction could usefully be taken to be **Cat**. The 2-category $\mathbf{End}_f(\mathscr{K})$ of finitary (=filtered-colimit-preserving) endo-2-functors of $\mathscr{K}$ is locally finitely presentable, and there is a finitary 2-monad M on $\mathbf{End}_f(\mathscr{K})$ whose 2-category $M\text{-Alg}_s$ of algebras is the 2-category $\mathbf{Mnd}_f(\mathscr{K})$ of finitary 2-monads on $\mathscr{K}$. In Section 7 we consider the trivial model structure on $\mathbf{End}_f(\mathscr{K})$ and the lifted model structure on $\mathbf{Mnd}_f(\mathscr{K})$. One reason for considering $\mathbf{Mnd}_f(\mathscr{K})$, is that one can use colimits in $\mathbf{Mnd}_f(\mathscr{K})$ to give presentations for monads (exactly as in the unenriched setting). This depends on the fact that for any object A there is a 2-monad $\langle A, A\rangle$ for which monad morphisms $T \to \langle A, A\rangle$ are in bijection with

T-algebra structures on A. Thus one can gradually build up the structure encoded in the notion of algebra by forming colimits of monads.

The morphisms of M-Alg are the pseudomorphisms of 2-monads. The main reason to consider these is to deal with pseudoalgebras. Whereas for morphisms it is the pseudomorphisms which arise in practice, and the strict ones are largely just a theoretical construct, it is somewhat different for algebras. Particular algebraic structures one might want to consider can most easily be described using strict algebras — for example there is a 2-monad T whose strict algebras are the not-necessarily-strict monoidal categories — but some sorts of formal manipulations one might do fail to preserve strictness, and so even if one starts with a strict T-algebra one might end up with a non-strict one. This distinction is discussed in Remark 5.8.

The connection between pseudomorphisms of monads and pseudoalgebras, is that to give an object A a pseudo T-algebra structure is equivalent to giving a pseudomorphism of monads from T to $\langle A, A\rangle$. As in Section 2.1, there is a cofibrant replacement T' of T with the property that such pseudomorphisms $T \to \langle A, A\rangle$ are in bijection with strict morphisms $T' \to \langle A, A\rangle$; that is, with T'-algebra structures on A. Furthermore, if the 2-monad T is flexible, then any pseudo T-algebra can be replaced by an *isomorphic* strict one. The fact that flexible colimits of flexible monads are flexible gives a useful criterion for when a 2-monad given by a presentation might be flexible.

2.5. The model structure on $\mathbf{Mnd}_f(\mathscr{K})$ can be used to infer "semantic" information about a 2-monad T: that is, information about the 2-category T-Alg of (strict) T-algebras and (pseudo) T-morphisms. The passage from T to T-Alg is 2-functorial: if $\mathbf{2\text{-}CAT}/\mathscr{K}$ denotes the (enormous!) 2-category of possibly large 2-categories equipped with a 2-functor into $\mathscr{K}$ and the morphisms are the commutative triangles, then there is a 2-functor $\mathsf{sem} : \mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}} \to \mathbf{2\text{-}CAT}/\mathscr{K}$ sending a 2-monad T to T-Alg, equipped with the forgetful 2-functor $U : T\text{-Alg} \to \mathscr{K}$; we write $k^* : T\text{-Alg} \to S\text{-Alg}$ for the 2-functor induced by a morphism of 2-monads $k : S \to T$. Section 8 concerns $\mathsf{sem} : \mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}} \to \mathbf{2\text{-}CAT}/\mathscr{K}$.

The definitions of weak equivalence and fibration for 2-functors coming from the model structure on **2-Cat** of [**12**] make perfectly good sense for large 2-categories, and it is only size issues which prevent this from making $\mathbf{2\text{-}CAT}/\mathscr{K}$ into a model category, and sem into a right Quillen functor. For sem sends preserves limits, fibrations, and trivial fibrations (that is, it sends colimits in $\mathbf{Mnd}_f(\mathscr{K})$ to limits in $\mathbf{2\text{-}CAT}/\mathscr{K}$, cofibrations in $\mathbf{Mnd}_f(\mathscr{K})$ to fibrations in $\mathbf{2\text{-}CAT}/\mathscr{K}$, and so on). The extent to which sem preserves general weak equivalences is closely related to the coherence problem for pseudoalgebras (which involves among other things the replacement of a pseudoalgebra by an equivalent strict one).

The 2-functor $k^* : T\text{-Alg} \to S\text{-Alg}$ restricts to a 2-functor $k^*_s : T\text{-Alg}_s \to S\text{-Alg}_s$, which is the right adjoint part of a Quillen adjunction, where $T\text{-Alg}_s$ and $S\text{-Alg}_s$ are given the lifted model structures. We show in Proposition 8.4 that k^*_s is a Quillen equivalence if and only if $k^* : T\text{-Alg} \to S\text{-Alg}$ is a biequivalence; that is, if $\mathsf{sem}(k)$ is a weak equivalence in $\mathbf{2\text{-}CAT}/\mathscr{K}$.

2.6. Instead of monads, another approach to universal algebra is offered by operads. In [**1**] operadic analogues are established to our lifted model structures on T-Alg_s and on $\mathbf{Mnd}_f(\mathscr{K})$, generalizing earlier work by various authors. In one respect, the setting of [**1**] is more general than that here, since they work over an arbitrary monoidal model category $\mathscr{V}$, whereas here we consider only the case $\mathscr{V} = \mathbf{Cat}$. This makes for substantial simplifications, due to the simple nature of the model structure on **Cat**. On the other hand, there are significant simplifications arising from restricting from general monads to operads, essentially because both in the category of operads, and in the category of algebras for a given operad, one has much tighter control over colimits than in the corresponding case for monads. There is also a more technical difference in that, in contrast to the situation in [**1**], the model structures arising here are not generally cofibrantly generated although they are in certain important cases.

In light of this comparison, it is appropriate to give some indication of the greater generality allowed by monads over operads. Structure described by operads can only involve operations of the form $A^n \to A$; or, in the multi-sorted case $A_1^{n_1} \times A_2^{n_2} \times \ldots A_k^{n_k} \to A_m$, where the superscripts are all natural numbers (corresponding to finite discrete categories). In the case of monads, one can also use more general limits such as pullbacks and cotensors. In particular, 2-monads on **Cat** allow considering structures involving maps $A^C \to A$ defined on all diagrams of shape C, for a (not necessarily discrete) category C.

3. Model 2-categories

3.1. The category **Cat** of small categories and functors has a well-known "categorical" or "folklore" model structure in which the weak equivalences are the equivalences of categories, and the fibrations are the *isofibrations*: these are the functors $p : E \to B$ for which if $e \in E$, and $\beta : b \cong pe$ is an isomorphism in B, there exists an isomorphism $\varepsilon : e' \cong e$ in E with $pe' = b$ and $p\varepsilon = \beta$. The model structure is cofibrantly generated, with generating cofibrations $0 \to 1$, $2 \to \nvDash$, and $\nvDash_2 \to \nvDash$, where 2 is the discrete category with two objects, $\nvDash$ is the arrow category, and $\nvDash_2$ is the category with two objects, and a parallel pair of arrows between them. There is a single generating trivial cofibration $1 \to \mathscr{I}$, where $\mathscr{I}$ is the "free-living isomorphism".

3.2. As well as the model structure, **Cat** also has monoidal structure given by the cartesian product. These two structures satisfy the compatibility conditions required to make **Cat** into a monoidal model category. See [**5**] for the precise definition of monoidal model category; the important point is that there is a resulting notion of "**Cat**-model category" (see [**5**] once again). We call such a structure a *model 2-category*; explicitly, this is a 2-category $\mathscr{K}$, with a model structure on the underlying ordinary 2-category $\mathscr{K}_0$ of $\mathscr{K}$, satisfying the following properties. First of all not just $\mathscr{K}_0$ but $\mathscr{K}$ must have finite limits and colimits. This reduces to the further condition that $\mathscr{K}$ have tensors and cotensors by the arrow category $\nvDash$, which means in turn that for every object A there are objects $\nvDash \cdot A$ and $\nvDash \pitchfork A$ with natural isomorphisms

$$\mathscr{K}(\nvDash \cdot A, B) \cong \mathbf{Cat}(\nvDash, \mathscr{K}(A,B) \cong \mathscr{K}(A, \nvDash \pitchfork B).$$

As well as this condition on the 2-category, there is also a compatibility condition on the model structure. Let $i : A \to B$ be a cofibration and $p : C \to D$ a fibration in $\mathscr{K}$. Then there is a commutative square

$$\begin{tikzcd} \mathscr{K}(B,C) \arrow[r, "\mathscr{K}(i,C)"] \arrow[d, "\mathscr{K}(B,p)"'] & \mathscr{K}(A,C) \arrow[d, "\mathscr{K}(A,p)"] \\ \mathscr{K}(B,D) \arrow[r, "\mathscr{K}(i,D)"'] & \mathscr{K}(A,D) \end{tikzcd}$$

in **Cat**, and so an induced functor $[i,p]$ from $\mathscr{K}(B,C)$ to the pullback. The further property required of a **Cat**-model category is that $[i,p]$ be an isofibration in any case, and moreover an equivalence if either i or p is trivial.

The fact that $[i,p]$ is surjective on objects if either i or p is a weak equivalence is just the usual lifting property for the ordinary model category. We still need (i) that $[i,p]$ is fully faithful if either i or p is a weak equivalence, and (ii) that in any case $[i,p]$ is an isofibration.

Condition (i) says that for any $x, y : B \to C$, there is a bijection between 2-cells $x \to y$ and pairs $\alpha : xi \to yi$ and $\beta : px \to py$ with $p\alpha = \beta i$. Condition (ii) says that if $z : B \to C$ is given, and isomorphisms $\alpha : x \cong zi$ and $\beta : y \cong pz$ with $p\alpha = \beta i$, as in

$$\begin{tikzcd} A \arrow[r, "i"'] \arrow[rr, bend left, "x"] & B \arrow[r, "z"'] \arrow[u, phantom, "\Downarrow\alpha"] & C \arrow[r, "p"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "i"'] & B \arrow[r, "z"'] \arrow[rr, bend left, "y"] & C \arrow[r, "p"'] \arrow[u, phantom, "\Downarrow\beta"] & D \end{tikzcd}$$

then there exist a 1-cell $t : B \to C$ and an isomorphism $\tau : t \cong z$ with $p\tau = \beta$ and $\tau i = \alpha$, as in

$$\begin{tikzcd} A \arrow[r, "i"'] & B \arrow[r, "z"'] \arrow[r, bend left=60, "t"] \arrow[r, phantom, bend left=30, "\Downarrow\tau"] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "i"'] \arrow[rr, bend left, "x"] & B \arrow[r, "z"'] \arrow[u, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "z"'] \arrow[r, bend left=60, "t"] \arrow[r, phantom, bend left=30, "\Downarrow\tau"] & C \arrow[r, "p"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} B \arrow[r, "z"'] \arrow[rr, bend left, "y"] & C \arrow[r, "p"'] \arrow[u, phantom, "\Downarrow\beta"] & D \end{tikzcd}$$

The special case $A = 0$ of (i) gives the first half of:

3.3. Proposition: *If B is cofibrant then $\mathscr{K}(B,-) : \mathscr{K} \to$ **Cat** preserves fibrations and trivial fibrations. In particular, if $p : C \to D$ is a trivial fibration, then composition with p induces an equivalence of categories $\mathscr{K}(B,C) \simeq \mathscr{K}(B,D)$. Dually, if E is fibrant, then $\mathscr{K}(-,E) : \mathscr{K}^{\mathrm{op}} \to$ **Cat** preserves cofibrations and trivial cofibrations, and if $j : C \to D$ is a trivial cofibration, then composition with j induces an equivalence $\mathscr{K}(D,E) \simeq \mathscr{K}(C,E)$.*

As a further consequence of the definition we have:

3.4. Proposition: *If $f : B \to B'$ is a weak equivalence and $f \cong g$ then g is a weak equivalence.*

Proof. Let $q : QB \to B$ be a cofibrant replacement of B. Then fq is a weak equivalence and $gq \cong fq$, while gq will be a weak equivalence if and only if g is one. So we may as well suppose that B is cofibrant.

Thus the unique map $0 \to B$ is a cofibration. Condition (ii) now says that for any fibration $p : C \to D$, and any $z : B \to C$, $y : B \to D$, and $\beta : y \cong pz$, there exists a lifting of β to $\tau : t \to z$ with $pt = y$ and $p\tau = \beta$.

Let $\mathscr{I} \cdot B$ be the tensor of B by $\mathscr{I}$, this is the universal object equipped with maps $d, c : B \to \mathscr{I} \cdot B$ and an isomorphism $\lambda : d \to c$. For fixed z, to give y and β is to give a map $h : \mathscr{I} \cdot B \to D$ with $hd = pz$. Thus condition (ii) can equivalently be expressed as saying that $d : B \to \mathscr{I} \cdot B$ has the left lifting property with respect to any fibration; that is, that $d : B \to \mathscr{I} \cdot B$ is a trivial cofibration.

In particular, $d : B \to \mathscr{I} \cdot B$ is a weak equivalence. But there is a unique $e : \mathscr{I} \cdot B \to B$ with $ed = ec = 1$ and $e\lambda = \mathsf{id}$; since d is a weak equivalence, so are e and c.

Now $f \cong g$, so there is a unique induced $h : \mathscr{I} \cdot B \to B'$ with $hd = f$, $hc = g$, and $h\lambda$ equal to the given isomorphism $f \cong g$. Since d and c are weak equivalences it follows that f is one if and only h is one if and only if g is one. $\square$

3.5. Remark: There are three ways of defining a model structure on an arbitrary (unenriched) category with finite limits and colimits: one can take the cofibrations to be the isomorphisms, the weak equivalences to be the isomorphisms, or the fibrations to be the isomorphisms. (In each case the other two classes of maps then must consist of all the morphisms.) It is easy to see that for a 2-category $\mathscr{K}$ with finite limits and colimits, we can take either the fibrations or the cofibrations to be the isomorphisms and obtain a model 2-category. The other case, where the weak equivalences are the isomorphisms is more complicated, but it is easy to see that this gives a model 2-category if the only invertible 2-cells in $\mathscr{K}$ are the identities. Actually the converse is also true. Suppose that $\mathscr{K}$ is a model 2-category in which the weak equivalences are the isomorphisms, and all maps are fibrations and cofibrations. Let $f, g : A \to B$ be given, with $\varphi : f \cong g$. Form $\mathscr{I} \cdot A$ as in the proof of the previous proposition, and the induced $h : \mathscr{I} \cdot A \to B$. Now $e : \mathscr{I} \cdot A \to A$ is a weak equivalence, as in the proof of the proposition, but it is a cofibration by assumption, so is a trivial cofibration. Also B is fibrant, so $h : \mathscr{I} \cdot A \to B$ extends along e as $h = ke$. Now $f = hd = ked = k = kec = hc = g$ and $\varphi = h\lambda = ke\lambda = \mathsf{id}$. Thus the identities are indeed the only invertible 2-cells; it follows that all equivalences are isomorphisms.

3.6. The homotopy category of **Cat** is the category Ho**Cat** of small categories and isomorphism classes of functors. This category has finite products (computed as in **Cat**), and so we can consider categories enriched in it. The canonical map $p : \mathbf{Cat} \to \mathrm{Ho}\mathbf{Cat}$ preserves finite products, and so every 2-category has an associated Ho**Cat**-category, obtained by applying p to each hom-category. (There are corresponding facts with **Cat** replaced by an arbitrary monoidal model category; see [**5**].) The homotopy category of a model 2-category is canonically a Ho**Cat**-category (once again, see [**5**] for the general situation). If $\mathscr{K}$ is a model **Cat**-category, then the unenriched homotopy category is the category of objects of $\mathscr{K}$ and isomorphism classes of morphisms. The enriched homotopy category Ho$\mathscr{K}$ consists of the objects of $\mathscr{K}$, and the category $\mathrm{Ho}(\mathscr{K}(A,B))$ for each $A, B \in \mathscr{K}$. The point is that horizontal composition of 2-cells is now only determined up to isomorphism.

3.7. A right adjoint 2-functor $U : \mathscr{L} \to \mathscr{K}$ between model 2-categories will be called a right Quillen 2-functor if it sends fibrations to fibrations and trivial fibrations to trivial fibrations; given that U and not just its underlying ordinary functor U_0 has a left adjoint, this will be the case if and only if U_0 is a right Quillen functor. There is a derived Ho**Cat**-adjunction between the homotopy Ho**Cat**-categories, just as in the usual case. This derived Ho**Cat**-adjunction is a Ho**Cat**-equivalence if and only if the unit and counit are invertible, but this depends only on the underlying ordinary adjunction between unenriched homotopy categories, so the usual theory of Quillen equivalences applies.

3.8. There is a 2-categorical construction called the pseudolimit of a morphism which will be crucial to many parts of this paper. If $f : A \to B$ is any 1-cell, its *pseudolimit* is the universal diagram of shape

$$\begin{tikzcd} & A \arrow[dd, "f"] \\ L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\lambda\Uparrow"] & \\ & B \end{tikzcd}$$

with λ invertible. Thus if $a : X \to A$ and $b : X \to B$ with $\varphi : b \cong fa$, there is a unique 1-cell $c : X \to L$ with $ux = a$, $vx = b$, and $\lambda x = \varphi$. There is also a 2-dimensional aspect to the universal property, which can most simply be expressed by saying that if $c, c' : X \to L$, then composition with u induces a bijection between 2-cells $c \to c'$ and 2-cells $uc \to uc'$; in other words u is representably fully faithful. Pseudolimits of arrows can be constructed using pullbacks, equalizers, and cotensors with $\nvDash$; or alternatively using pullbacks and cotensors with $\mathscr{I}$, the category with two objects and an isomorphism between them.

By the universal property, there is a unique $d : A \to L$ with $ud = 1$, $vd = f$, and $\lambda d = \mathsf{id}$. By the 2-dimensional aspect of the universal property, there is a unique invertible 2-cell $\delta : ud \cong 1$ with $d\delta = \mathsf{id}$. We say that u is a *retract equivalence*: this means that it has a section d, with $du \cong 1$.

For our first application of the construction we have:

3.9. Proposition: *In a model 2-category, every equivalence is a weak equivalence.*

Proof. Let $f : A \to B$ be an equivalence in a model 2-category $\mathscr{K}$. Form the pseudolimit L of f as above. We know that $f = vd$ with d the section of a retract equivalence. But f is itself an equivalence, so there is a $g : B \to A$ with $\beta : fg \cong 1$, and this induces a unique $c : B \to L$ with $uc = g$, $vc = 1$, $\lambda c = \beta$, as in

$$\begin{tikzcd} & & A \arrow[dd, "f"] \\ B \arrow[r, "L"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "1"'] & L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\lambda\Uparrow"] & {} \\ & & B \end{tikzcd} \quad = \quad \begin{tikzcd} & A \arrow[dd, "f"] \\ B \arrow[ur, "g"] \arrow[dr, "1"'] \arrow[r, phantom, "\beta\Uparrow"] & {} \\ & B. \end{tikzcd}$$

Furthermore $ucv = gv \cong gfu \cong u$, so $cv \cong 1$, and so v is also a retract equivalence. So now it will suffice to show that every retract equivalence is a weak equivalence: for then u and v will be weak equivalences; but $ud = 1$ will imply that d is a weak equivalence and $vd = f$ will imply that f is a weak equivalence.

Suppose then that $f : A \to B$ has $fg = 1$ and $gf \cong 1$. Then gf is a weak equivalence by Proposition 3.4, and f is a retract of gf by a standard argument, so is itself a weak equivalence, since these are closed under retract. □

4. The trivial model structure on a 2-category

4.1. Let $\mathscr{K}$ be a 2-category with finite limits and colimits. In this section we describe a model 2-category structure on $\mathscr{K}$; we call it the *trivial model structure on the 2-category.* An ordinary model category is trivial when the weak equivalences are precisely the isomorphisms; in a trivial model 2-category the weak equivalences are precisely the equivalences, while the fibrations are defined similarly to the (categorical) model structure on **Cat**. Indeed **Cat** will be our first example of a trivial model 2-category. Note that triviality of a model 2-category is not the same as triviality of the underlying model category: **Cat** is trivial as a model 2-category but the underlying ordinary model category is not trivial, since not every equivalence of categories is an isomorphism. On the other hand, if $\mathscr{K}$ is a model 2-category and the underlying ordinary model category $\mathscr{K}_0$ is trivial, then so is $\mathscr{K}$: see Remark 4.16

4.2. The trivial model structure on a 2-category $\mathscr{K}$ can most concisely be described by saying that a morphism $f : A \to B$ is a weak equivalence or fibration in $\mathscr{K}$ if and only if the functor $\mathscr{K}(E, f) : \mathscr{K}(E, A) \to \mathscr{K}(E, B)$ is one in **Cat**, for every object E of $\mathscr{K}$. A morphism is a cofibration if and only if it has the left lifting property with respect to the trivial fibrations.

Most of this section will be devoted to the proof of:

4.3. Theorem: *If $\mathscr{K}$ is a 2-category with finite limits and colimits then the definition just given makes it into a model 2-category. The weak equivalences are precisely the (adjoint) equivalences, and the fibrations are the isofibrations. We call this the* trivial model structure*, to distinguish it from any others which may exist. The factorizations are functorial, and every object is fibrant and cofibrant.*

4.4. First we explicate the definition. A morphism $f : A \to B$ in a 2-category $\mathscr{K}$ is said to be an equivalence if there exists a morphism $g : B \to A$ with $gf \cong 1_A$ and $fg \cong 1_B$. Since any 2-functor sends equivalences to equivalences, the equivalences are certainly weak equivalences. Conversely, if $f : A \to B$ is a weak equivalence, then $\mathscr{K}(B, f) : \mathscr{K}(B, A) \to \mathscr{K}(B, B)$ is an equivalence of categories, so by essential surjectivity there exists a $g : B \to A$ and $\beta : fg \cong 1_B$. Since $\mathscr{K}(A, f) : \mathscr{K}(A, A) \to \mathscr{K}(A, B)$ is also an equivalence of categories, and $\mathscr{K}(A, f)gf = fgf \cong f = \mathscr{K}(A, f)1_A$, via the isomorphism βf, there is a unique isomorphism $\alpha : gf \cong 1_A$ with $f\alpha = \beta f$. Thus f is an equivalence, and so the weak equivalences are precisely the equivalences. (We note in passing the well-known fact that the isomorphisms $gf \cong 1$ and $1 \cong fg$ can always be chosen so as to satisfy the triangle equations, and so give an adjoint equivalence.) The weak equivalences are closed under retracts and satisfy the 2-out-of-3 property.

The fibrations are the *isofibrations*: these are the maps $f : A \to B$ such that for any morphisms $a : X \to A$ and $b : X \to B$, and any invertible 2-cell $\beta : b \cong fa$, there exists a 1-cell $a' : X \to A$ and an invertible 2-cell $\alpha : a' \cong a$ with $fa' = b$ and $f\alpha = \beta$.

It now follows that the trivial fibrations are precisely the retract equivalences; that is, the morphisms $g : B \to A$ for which there exists a morphism $f : A \to B$ with $gf = 1_A$ and $fg \cong 1_B$. Once again, the isomorphism can be chosen so as to give an adjoint equivalence.

We define the trivial cofibrations to be the maps with the left lifting property with respect to the fibrations; of course these will turn out to be precisely those cofibrations which are weak equivalences.

4.5. In the case $\mathscr{K} = \mathbf{Cat}$, this gives the "categorical" or "folklore" model structure, defined in [**6**], for example. In the case $\mathscr{K} = \mathbf{Cat}^X$ for a set X, this gives the pointwise model structure coming from $\mathbf{Cat}$. In the case of $\mathscr{K} = \mathbf{Cat}(\mathscr{E})$ of internal categories in a topos $\mathscr{E}$, this will not in general be the model structure of [**6**], since there the weak equivalences were the internal functors which are (in the suitably internal sense) fully faithful and essentially surjective on objects, and these are more general than the adjoint equivalences unless the axiom of choice holds in $\mathscr{E}$. In the case $\mathscr{K} = \mathbf{Cat}(\mathscr{C})$ for a suitable finitely complete category $\mathscr{C}$, it does agree with the model structure "for the trivial topology" of [**4**].

Once again, the construction of the pseudolimit of an arrow will play a key role:

4.6. PROPOSITION: *If an arrow $f : A \to B$ in a 2-category $\mathscr{K}$ admits a pseudolimit L as in Section 3.8 above, then f is an isofibration if and only if there exists a 1-cell $v' : L \to A$ and an isomorphism $\lambda' : v' \to u$ with $fv' = v$ and $f\lambda' = \lambda$; in other words, if and only if $\mathscr{K}(L, f)$ is an isofibration in* $\mathbf{Cat}$.

Proof. The "only if" part is immediate; as for the "if" part: if $a : C \to A$, $b : C \to B$, and $\beta : b \cong fa$ are given, let $c : C \to L$ be the induced map; then $v'c : C \to A$ and $\lambda' c : v'c \cong uc = a$ provide the required lifting. □

The 1-cells $1 : A \to A$ and $f : A \to B$, and the identity 2-cell $f = f$, induce a unique 1-cell $d : A \to L$ with $ud = 1$, $vd = f$, and $\lambda d = \mathsf{id}_f$. Since $udu = u$, there is a unique invertible 2-cell $\zeta : du \cong 1$ with $u\zeta$ equal to the identity. It easily follows that ζd is also the identity, so that d and ζ exhibit u as a retract equivalence.

Furthermore, the universal property of L implies

4.7. Proposition: *The morphism $v : L \to B$ is a fibration.*

Proof. If $c : C \to L$ and $\gamma : e \cong vc$, then let $(a = uc, b = vc, \beta = \lambda c)$ be the data corresponding to c via the universal property, so that $\gamma : e \cong b$. Composing β and γ gives an isomorphism $e \cong b \cong fa$, which therefore has the form λy for a unique $y : C \to L$, so that in particular $vy = e$ and $uy = a$. Now $uy = a = uc$, so there is a unique $\delta : y \cong c$ with $u\delta$ equal to the identity, and now

$$\begin{tikzcd}
 & & A \arrow[dd, "f"] \\
C \arrow[r, "c"] \arrow[rrd, bend right, "e"'] \arrow[rd, phantom, "\gamma\Uparrow"] & L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\lambda\Uparrow"] & {} \\
 & {} & B
\end{tikzcd}
=
\begin{tikzcd}
 & & A \arrow[dd, "f"] \\
C \arrow[r, "y"] & L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\lambda\Uparrow"] & {} \\
 & & B
\end{tikzcd}$$

$$=
\begin{tikzcd}
 & & A \arrow[dd, "f"] \\
C \arrow[r, bend left, "c"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\delta\Uparrow"] \arrow[rrd, bend right, "e"'] & L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\lambda\Uparrow"] & {} \\
 & & B
\end{tikzcd}$$

and λc is invertible, so that $v\delta = \gamma$, and δ is the required lifting. $\square$

4.8. Observe also that if f is itself an equivalence, then composing with the equivalence u gives an equivalence fu, so v is an equivalence since it is isomorphic to fu. Thus any morphism f can be factorized as a weak equivalence d followed by a fibration v, and the fibration will be trivial if (and only if) f was a weak equivalence.

4.9. Dually, we can form the pseudocolimit of a morphism $f : A \to B$, involving $i : A \to C$, $j : B \to C$, and $\lambda : i \cong jf$, as in

$$\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "i"] & \\
\arrow[r, phantom, "\Downarrow\lambda"] & C \\
B \arrow[ur, "j"'] &
\end{tikzcd}$$

and there is an induced $e : C \to B$ with $ei = f$, $ej = 1_B$, $e\lambda = \mathsf{id}$, and an isomorphism $\varepsilon : je \cong 1$ with εj and $e\varepsilon$ both identity 2-cells. In particular, e is always a trivial fibration.

4.10. LEMMA:

1. *For any f as above, i is a cofibration and e a trivial fibration.*
2. *If f is a weak equivalence, then i is a trivial cofibration.*

Proof. 1. The fact that e is a trivial fibration was observed above. Let's prove that i is a cofibration. Suppose given then a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

with p a trivial fibration. By the universal property of C, to give v is equivalently to give $t : B \to D$ and $\tau : pu \cong tf$.

Since p is a retract equivalence, there exists a 1-cell $s : B \to E$ with $ps = t$. Then $\tau : pu \cong tf = psf$ has the form $p\sigma$ for a unique isomorphism $\sigma : u \cong sf$. By the universal property of C, there is a unique $r : C \to E$ with $ri = u$, $rj = s$, and $r\lambda = \sigma$. If $pr = v$ then r will provide the desired lifting. But $pri = pu = vi$, $prj = ps = t = vj$, and $pr\lambda = p\sigma = \tau = v\lambda$, and so $pr = v$ by the universal property of C once again.

2. We must show that i has the left lifting property with respect to the fibrations. Suppose given a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

with p a fibration. To give v is equivalently to give $t : B \to D$ and $\tau : pu \cong tf$. We are assuming that f is an equivalence, so we can choose $g : B \to A$, $\alpha : 1 \to gf$, and $\beta : fg \to 1$ giving an adjoint equivalence. Now there are 1-cells $ug : B \to E$ and $t : B \to D$, and an isomorphism $t \cong pug$ as in the left hand side of

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "f"] \arrow[dl, phantom, "\Downarrow\beta"] & E \arrow[d, "p"] \arrow[dl, phantom, "\Downarrow\tau"] \\ B \arrow[ur, "g"] \arrow[r, "1"'] & B \arrow[r, "t"'] & D \end{tikzcd}
\quad = \quad
\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, phantom, "\Downarrow\sigma"] & E \arrow[d, "p"] \\ B \arrow[ur, "g"] \arrow[urr, bend right, "s"'] \arrow[r, "1"'] & B \arrow[r, "t"'] & D \end{tikzcd}$$

and so since p is a fibration, there exist an $s : B \to E$ and an isomorphism $\sigma : ug \cong s$, giving the equality displayed above. Now by the universal property of C, there is a unique $r : C \to E$ with $ri = u$, $rj = s$, and

$$
\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "i"] \arrow[drr, bend left, "u"] & & \\
\arrow[r, phantom, "\Downarrow\lambda"] & C \arrow[r, "r"'] & E \\
B \arrow[ur, "j"'] \arrow[urr, bend right, "s"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "1"] \arrow[drr, bend left, "u"] & & \\
\arrow[r, phantom, "\Downarrow\alpha"] & A \arrow[r, "u"] \arrow[d, phantom, "\Downarrow\sigma"] & E \\
B \arrow[ur, "g"'] \arrow[urr, bend right, "s"'] & &
\end{tikzcd}
$$

If $pr = v$ then r will provide the desired lifting. Now $pri = pu = vi$ and $prj = ps = t = vj$, while

$$
\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "i"] & & \\
\arrow[r, phantom, "\Downarrow\lambda"] & C \arrow[r, "r"] & E \arrow[d, "p"] \\
B \arrow[ur, "j"'] & & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "1"] & & \\
\arrow[r, phantom, "\Downarrow\alpha"] & A \arrow[r, "u"] \arrow[d, phantom, "\Downarrow\sigma"] & E \arrow[d, "p"] \\
B \arrow[ur, "g"'] \arrow[ur, bend right, "s"'] & & D
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "1"] & & & \\
\arrow[r, phantom, "\Downarrow\alpha"] & A \arrow[r, "u"] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\beta"] & E \arrow[dr, "p"] \arrow[d, phantom, "\Downarrow\tau"] & \\
B \arrow[ur, "g"'] \arrow[rr, "1"'] & & B \arrow[r, "v"'] & D
\end{tikzcd}
$$

but by one of the triangle equations this last just reduces to τ; that is to $v\lambda$. Thus $pr\lambda = v\lambda$ and so $pr = v$. $\square$

4.11. By the lemma we know that every morphism can be factorized as a cofibration followed by a trivial fibration, and that every weak equivalence can be factorized as a trivial cofibration followed by a trivial fibration. But we saw in 4.8 that every map can be factorized as a weak equivalence followed by a fibration, thus we now have both factorization properties. Notice that in order to obtain the factorization as a trivial cofibration followed by a fibration we have used both the pseudocolimit and the pseudolimit, whereas for the other factorization we only needed the pseudocolimit.

4.12. We now check that the trivial cofibrations are precisely the maps that are both weak equivalences and cofibrations. If $f : A \to B$ is a weak equivalence and a cofibration, then by the lemma we can factorize it as $f = pi$ with i a trivial cofibration and p a trivial fibration. The lifting property for cofibrations and trivial fibrations now makes f a retract of the trivial cofibration i, and so f is itself a trivial cofibration.

If conversely f is a trivial cofibration, then certainly it is a cofibration; we must show that it is a weak equivalence. To do this, factorize it as a weak equivalence d followed by a fibration v, using the pseudolimit of f, and now the lifting property for trivial cofibrations and fibrations shows that f is a retract of the weak equivalence d, and so a weak equivalence.

This completes the verification of the model category axioms; the factorizations were explicitly constructed and clearly functorial, and every object is fibrant and cofibrant. It remains to check that this gives a model 2-category.

4.13. We now check the conditions (i) and (ii) in the definition of model 2-category 3.2. For the first of these, let $i : A \to B$ be a cofibration and $p : C \to D$ a fibration in $\mathscr{K}$. Let $x, y : B \to C$ be given, with 2-cells $\alpha : xi \to yi$ and $\beta : px \to py$ satisfying $p\alpha = \beta i$. If p is a trivial fibration, then in particular it is an equivalence, and so there is a unique $\gamma : x \to y$ satisfying $p\gamma = \beta$. Furthermore the 2-cells $\gamma i, \alpha : xi \to yi$ satisfy $p\gamma i = \beta i = p\alpha$, and so $\gamma i = \alpha$. Similarly if i is an equivalence then there is a unique $\gamma : x \to y$ satisfying $\gamma i = \alpha$, and it is also the case that $p\gamma = \beta$. This proves (i).

As for condition (ii), let $x : A \to C$, $y : B \to D$, $z : B \to C$, $\alpha : x \cong zi$, and $\beta : y \cong pz$ be given with $p\alpha = \beta i$. We must find $t : B \to C$ and $\tau : t \cong x$ satisfying $p\tau = \beta$ and $\tau i = \alpha$. We do this first under the assumption that i appears in a pseudocolimit

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & E \arrow[dl, "j"] \\ & B. & \end{tikzcd}$$

The general case will then follow since a general cofibration i' can be factorized as such an i followed by a trivial fibration, as in Lemma 4.10, thus by the lifting property i' is a retract of i, whence the general result.

Suppose then that i does indeed arise in such a pseudocolimit. By the universal property of B, to give $y : B \to D$ is to give $y_1 : A \to D$, $y_2 : E \to D$, and an isomorphism $\eta : y_1 \cong y_2 f$. Similarly, to give $z : B \to C$ is to give $z_1 : A \to C$, $z_2 : E \to C$, and $\zeta : z_1 \cong z_2 f$. To give $\beta : y \cong pz$ is just to give $\beta_2 : y_2 \cong pz_2$ (with $\beta_2 = \beta j$ and $\beta i = pz\lambda^{-1}.\beta_2 f.y\lambda = p\zeta^{-1}.\beta_2 f.\eta$). Thus since $\alpha : x \cong zi = z_1$ satisfies $p\alpha = \beta i$ we have $p\alpha = p\zeta^{-1}.\beta_2 f.\eta$ or equivalently $p\zeta.p\alpha = \beta_2 f.\eta$.

Since p is a fibration, there exist a 1-cell $t_2 : E \to C$ and an isomorphism $\tau_2 : t_2 \cong z_2$ as in

$$\begin{tikzcd} & C \arrow[d, "p"] \\ E \arrow[ur, bend left=40, "z_2"] \arrow[ur, bend right=20, "t_2"'] \arrow[ur, phantom, "\Leftarrow\tau_2"] \arrow[r, "y_2"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} & C \arrow[d, "p"] \\ E \arrow[ur, bend left=40, "z_2"] \arrow[r, "y_2"'] \arrow[ur, phantom, "\Leftarrow\beta_2"] & D \end{tikzcd}$$

with $pt_2 = y_2$ and $p\tau_2 = \beta_2$. The 1-cells $x : A \to C$ and $t_2 : E \to C$, and the isomorphism

$$x \xrightarrow{\alpha} zi = z_1 \xrightarrow{\zeta} z_2 f \xrightarrow{\tau_2^{-1} f} t_2 f$$

has the form $t\varphi$ for a unique $t : B \to C$ by the universal property of the pseudocolimit B. Now $tj = t_2$ and $zj = z_2$, so the isomorphism $\tau_2 : t_2 \cong z_2$ extends to a unique isomorphism $\tau : t \cong z$ with $\tau j = \tau_2$. We shall show that t and τ have the required properties.

We have $ti = x$ by construction; we show that $pt = y$ using the universal property of the pseudocolimit B. Now $pti = px = yi$ and $ptj = pt_2 = y_2 = yj$, while

$$pt\varphi = (p\tau_2^{-1}f).(p\zeta).(p\alpha) = \beta_2^{-1}f.\beta_2 f.\eta = \eta = y\varphi$$

so that $pt = y$ as required.

Thus $p\tau$ and β both go from $pt = y$ to pz. Since j is an equivalence, they will be equal if and only if $p\tau j = \beta j$; but $p\tau j = p\tau_2 = \beta_2 = \beta j$.

It remains to show that $\tau i = \alpha$. To do so, observe that $z\varphi.\tau i = \tau jf.t\varphi$ by the middle-four interchange law applied to

$$\begin{tikzcd} A \arrow[r, bend left, "i"] \arrow[r, bend right, "jf"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \arrow[r, bend left, "t"] \arrow[r, bend right, "z"'] \arrow[r, phantom, "\Downarrow\tau"] & C, \end{tikzcd}$$

while $\tau jf.t\varphi = \tau_2 f.t\varphi = \zeta.\alpha = z\varphi.\alpha$, so that $z\varphi.\tau i = z\varphi.\alpha$, but $z\varphi$ is invertible, so $\tau i = \alpha$ as desired.

This completes the proof of Theorem 4.3.

4.14. Example: If $\mathscr{K}$ is a locally discrete 2-category, meaning that the only 2-cells are the identities, we may identify it with its underlying ordinary category. The resulting model structure is the trivial model structure on the category: the weak equivalences are just the isomorphisms, and all maps are both fibrations and cofibrations.

4.15. Example: If $\mathscr{K}$ is locally chaotic, so that between any two parallel arrows $f, g : A \to B$, there is a unique 2-cell $f \to g$ (necessarily invertible), then once again we can identify $\mathscr{K}$ (in a different way) with its underlying category. This time a map $f : A \to B$ is a weak equivalence if and only if there exists an arbitrary map $B \to A$. The trivial fibrations are precisely the retractions.

4.16. Remark: Let $\mathscr{K}$ be a 2-category with finite limits and colimits, and consider the trivial model category structure on the underlying ordinary category $\mathscr{K}_0$ of $\mathscr{K}$. As observed in Remark 3.5 above, this will give a model 2-category if and only if the only invertible 2-cells in $\mathscr{K}$ are identities. But in this case the equivalences are exactly the isomorphisms, and every morphism is an isofibration, so $\mathscr{K}$ will also be trivial as a model 2-category. Thus a model 2-category which is trivial as an ordinary model category is also trivial as a model 2-category; in contrast, as observed above, a trivial model 2-category need not have an underlying ordinary model category which is trivial.

4.17. Finally we observe that even for extremely well-behaved 2-categories, the trivial model structure need not be cofibrantly generated. We write $\mathbf{Cat}^{\nvDash}$ for the 2-category of arrows in **Cat**: an object is a functor $a : A \to A'$, a morphism from $a : A \to A'$ to $b : B \to B'$ is a commutative square, involving $f : A \to B$ and

$f' : A' \to B'$, and a 2-cell from (f, f') to (g, g') consists of 2-cells $\alpha : f \to g$ and $\alpha' : f' \to g'$ satisfying $b\alpha = \alpha a'$.

4.18. PROPOSITION: *The trivial model structure on* $\mathbf{Cat}^{\nvDash}$ *is not cofibrantly generated.*

Proof. Let $\mathscr{K}$ be the category $\mathbf{Set}^{\nvDash}$ of arrows in $\mathbf{Set}$, seen as a locally chaotic 2-category. There is a fully faithful 2-functor $U : \mathscr{K} \to \mathbf{Cat}^{\nvDash}$ sending a function $f : X \to Y$ to the corresponding functor between the chaotic categories on X and Y. This 2-functor has a left adjoint, which sends a functor to the corresponding function between object-sets. The trivial model 2-category structure on $\mathbf{Set}^{\nvDash}$ can be obtained as the lifting of the trivial model 2-category structure on $\mathbf{Cat}^{\nvDash}$. Now if $\mathbf{Cat}^{\nvDash}$ were cofibrantly generated, then so would be $\mathbf{Set}^{\nvDash}$, but it is not. For if $(m_i : A_i \to A_i + B_i)_{i \in I}$ were a small family of generating cofibrations (here A_i and B_i denote objects of $\mathbf{Set}^{\nvDash}$; that is, functions) then so would be $(0 \to B_i)_{i \in I}$. Now every object of $\mathbf{Set}^{\nvDash}$ is cofibrant, so would be a retract of a coproduct of B_i's. By the exactness properties of $\mathbf{Set}$, it would then follow that every object was a coproduct of retracts of B_i's. Now the closure under retracts of the B_i's is still small, so there would be a small full subcategory $\mathscr{G}$ of $\mathbf{Set}^{\nvDash}$ such that every object was a coproduct of objects in $\mathscr{G}$. But now consider the objects of the form $X \to 1$. These constitute a large family, and none of them can be decomposed non-trivially as a coproduct. Thus there cannot be a small family of generating cofibrations. $\square$

4.19. This should not perhaps be too surprising. We have defined the weak equivalences and fibrations by lifting through the representables $\mathscr{K}(C, -)$ for arbitrary C. Since this is generally a large set of objects it is not surprising that it would not lead to a cofibrantly generated structure. In some cases however a small set of objects will suffice, and then the structure will be cofibrantly generated. In particular, if $\mathscr{K} = \mathbf{Cat}$, then it suffices to use just the single representable $\mathbf{Cat}(1, -)$ (which is the identity 2-functor on $\mathbf{Cat}$).

5. The lifted model structure on a 2-category of algebras

5.1. Suppose now that $\mathscr{K}$ is a locally presentable 2-category, endowed with the trivial model structure as in the previous section. Suppose that T is a 2-monad on $\mathscr{K}$ with rank (it preserves α-filtered colimits for some α), and that $T\text{-Alg}_s$ is the 2-category of strict T-algebras, strict morphisms, and T-transformations. Then the forgetful 2-functor $U_s : T\text{-Alg}_s \to \mathscr{K}$ has a left 2-adjoint F_s, with unit $n : 1 \to U_s F_s$ and counit $e : F_s U_s \to 1$. We shall use this adjunction to construct a "lifted" model structure on $T\text{-Alg}_s$. A morphism f in $T\text{-Alg}_s$ is defined to be a fibration or weak equivalence if and only if $U_s f$ is one in $\mathscr{K}$, while a morphism is a cofibration if and only if it has the left lifting property with respect to the trivial fibrations (the maps which are both fibrations and weak equivalences), and a trivial cofibration if and only if it has the left lifting property with respect to the fibrations. The fact that this is a model 2-category will follow immediately from the usual model category axioms thanks to the 2-dimensional aspect of the 2-adjunction.

There exist many theorems about lifting model structures, but they generally depend upon the lifted model structure being cofibrantly generated, which we are not assuming here.

5.2. Given a strict morphism $f : A \to B$ we can form in $T\text{-Alg}_s$ the pseudolimit L of f, with projections $u : L \to A$ and $v : L \to B$ and isomorphism $\lambda : v \cong fu$; since $U_s : T\text{-Alg}_s \to \mathscr{K}$ preserves limits, U_sL will also be the limit in $\mathscr{K}$, and so $U_sv : U_sL \to U_sB$ will be a fibration in $\mathscr{K}$ by Proposition 4.7, and so in turn v will be a fibration in $T\text{-Alg}_s$. Furthermore, we have the unique induced $d : A \to L$ with $ud = 1$ and λd the identity, and just as before this d is an equivalence in $T\text{-Alg}_s$, and so in particular a weak equivalence. This proves that every map can be factorized as a (weak) equivalence followed by fibration. This already implies that every trivial cofibration is a weak equivalence, by the same argument used in Section 4.12.

5.3. Let $f : A \to B$ be an arbitrary strict morphism. Factorize $U_sf : U_sA \to U_sB$ in $\mathscr{K}$ as a cofibration $i_1 : U_sA \to X_1$ followed by a trivial fibration $p_1 : X_1 \to U_sB$. Pushout F_si_1 along the counit $eA : F_sU_sA \to A$, and form the induced map f_1 as in

$$\begin{tikzcd} F_sU_sA \arrow[r, "F_si_1"] \arrow[d, "eA"'] & F_sX_1 \arrow[r, "F_sp_1"] \arrow[d, "c_1"] & F_sU_sB \arrow[d, "eB"] \\ A \arrow[r, "j_1"'] & C_1 \arrow[r, "f_1"'] & B \end{tikzcd}$$

where the left square is the pushout, and $f_1j_1 = f$. Now i_1 is a trivial cofibration in $\mathscr{K}$, so F_si_1 is a trivial cofibration in $T\text{-Alg}_s$, and so in turn is its pushout j_1. There is not so much we can say about f_1 at this stage, but we do know that U_sf_1 has a section, for if s_1 is a section of the trivial fibration p_1, then $U_sf_1.U_sc_1.U_sF_ss_1.nU_sB = U_seB.U_sF_sp_1.U_sF_ss_1.nU_sB = U_seB.nU_sB = 1$.

If f is in fact a weak equivalence, then since $f = f_1j_1$ and j_1 is a weak equivalence, f_1 will be one too. But now U_sf_1 is a weak equivalence with a section, hence a trivial fibration, and so finally f_1 is a trivial fibration. Thus every weak equivalence factorizes as trivial cofibration followed by a trivial fibration. Combined with the factorization, given in Section 5.2, of any map into a weak equivalence followed by a fibration, this now proves that any map can be factorized as a trivial cofibration followed by a fibration. It now follows that the trivial cofibrations are precisely the weak equivalences which are cofibrations, by the argument used in Section 4.12.

5.4. So far things have gone essentially as usual. It remains to show the existence of the other factorization: cofibration followed by trivial fibration. This is the most technical part of the proof. Suppose again then that f is arbitrary, and factorize it as $f = f_1j_1$ as above. We know that U_sf_1 has a section. If we could show that for any two maps $x, y : U_sB \to U_sC_1$ with $U_sf_1.x \cong U_sf_1.y$ we have $x \cong y$, then U_sf_1 would be a trivial fibration, and we would be done; but in general there is no reason why this should be true, and there is more work to be done. Factorize f_1 as f_2j_2

via the same process, also involving $i_2 : U_sC_1 \to X_2$ and $p_2 : X_2 \to U_sB$, and now iterate to obtain a trivial cofibration $j_{n+1} : C_n \to C_{n+1}$ and map $f_{n+1} : C_{n+1} \to B$ for any n. Continue transfinitely, setting $C_m = \mathsf{colim}_{n<m}C_n$ for any limit ordinal m. Any transfinite composite of the j's will be a trivial cofibration, and each f_n will have the property that U_sf_n has a section. So if we can find an n such that for any $x, y : U_sB \to U_sC_n$ with $U_sf_n.x \cong U_sf_n.y$ we have $x \cong y$, then f_n will be a trivial fibration, and we will be done.

Let α be a regular cardinal with the property that T preserves α-filtered colimits and that U_sB is α-presentable in $\mathscr{K}$. Since T preserves α-filtered colimits, so does U_s. Let $x, y : U_sB \to C_\alpha$ be given with $U_sf_\alpha.x \cong U_sf_\alpha.y$. Now $C_\alpha = \mathsf{colim}_{n<\alpha}C_i$, so $U_sC_\alpha = \mathsf{colim}_{n<\alpha}U_sC_i$, and so there exists an $n < \alpha$ such that x and y land in U_sC_n, say via $x', y' : U_sB \to U_sC_n$. Now $U_sf_n.x' = U_sf_\alpha.x \cong U_sf_\alpha.y = U_sf_n.y'$, and $U_sf_n = p_{n+1}.i_{n+1}$, with p_{n+1} a trivial fibration, so the isomorphism lifts through p_{n+1} to give $i_{n+1}.x' \cong i_{n+1}.y'$. Now

$$j_{n+1}.eC_n.F_sx' = c_{n+1}.F_si_{n+1}.F_sx' \cong c_{n+1}.F_si_{n+1}.F_sy' = j_{n+1}.eC_n.F_sy'$$

and j_{n+1} is a trivial cofibration, so has a retraction, and so $eC_n.F_sx' \cong eC_n.F_sy'$, but by adjointness this is just $x' \cong y'$, which finally gives $x \cong y$ as required.

This proves the existence of the model structure; it is automatically a model 2-category, via the 2-dimensional aspect of the adjunction.

5.5. Theorem: *For a 2-monad T with rank, on a locally finitely presentable 2-category $\mathscr{K}$, the category T-Alg_s of strict T-algebras and strict T-morphisms has a model 2-category structure for which the weak equivalences are the maps which are equivalences in $\mathscr{K}$, and the fibrations are the maps which are isofibrations in $\mathscr{K}$.*

5.6. Remark: Observe that the class of all strict T-morphisms of the form $Ti : TX \to TZ$ with $i : X \to Z$ a cofibration in $\mathscr{K}$, while not small, does nonetheless generate the cofibrations of T-Alg_s. Furthermore, we can even restrict to those Ti for which there exist $f : X \to Y$, $j : Y \to Z$, and $\lambda : i \cong fj$, such that i, j, and λ exhibit Z is as the pseudocolimit in $\mathscr{K}$ of f, for every cofibration in $\mathscr{K}$ is a retract of one of these.

Once again every object is fibrant, but it is no longer the case that every object is cofibrant; we shall see below that the cofibrant objects are precisely the *flexible* ones, in the sense of [**3**].

5.7. The strict morphisms, as in T-Alg_s, are very useful for theoretical reasons, but in practice they are rare. More common are the *pseudo* T-morphisms, which preserve the structure only up to coherent isomorphism. Since we are treating this as the basic notion of morphism, we call them simply T-morphisms. They are the morphisms of a 2-category T-Alg of (still strict) T-algebras, T-morphisms, and T-transformations. The inclusion 2-functor T-$\mathrm{Alg}_s \to T$-Alg is the identity on objects, and fully faithful on the hom-categories.

5.8. Remark: There is also a notion of *pseudo T-algebra* for a 2-monad T, in which the usual laws for T-algebras are replaced by coherent isomorphisms; these are considered in Section 7 below. The pseudo algebras are less important than the strict ones for two reasons. First there is a "theoretical" reason, discussed in Section 7, that for a 2-monad T with rank on a locally presentable 2-category $\mathscr{K}$, the pseudo T-algebras are just the strict algebras for a different 2-monad T'. There is also a more practical reason, which we illustrate with the example of monoidal categories. There is a 2-monad T on **Cat** whose strict algebras are the strict monoidal categories. It is true that "up to equivalence" the pseudo T-algebras are the same as the (not necessarily strict) monoidal categories, but this is a relatively hard fact. Much easier is the fact that there is a 2-monad S whose strict algebras are precisely the monoidal categories (we sketch in Section 7.4 below the slightly simpler case of "semigroupoidal categories"). The situation is similar but more pronounced in the case of more complicated structures than monoidal categories. The reason for considering pseudoalgebras at all is that some natural constructions on algebras only produce pseudoalgebras, even if one starts with a strict one.

5.9. We write $U : T\text{-Alg} \to \mathscr{K}$ for the forgetful map, to distinguish it from $U_s : T\text{-Alg}_s \to \mathscr{K}$. The evident inclusion $J : T\text{-Alg}_s \to T\text{-Alg}$ clearly satisfies $UJ = U_s$. It was proved in [**3**] that $J : T\text{-Alg}_s \to T\text{-Alg}$ has a left 2-adjoint, sending an algebra A to an algebra A', with counit a strict map $q : A' \to A$ and with unit a pseudomap $p : A \to A'$ satisfying $qp = 1$ and $pq \cong 1$. Thus q is a trivial fibration. The universal property of A' asserts among other things that for any algebra B, composition with p induces a bijection between strict maps $A' \to B$ and pseudo maps $A \to B$.

5.10. Proposition: *For a strict morphism $f : A \to B$, the following are equivalent:*

(i) f is a weak equivalence;

(ii) $U_s f$ is an equivalence in $\mathscr{K}$;

(iii) Jf is an equivalence in T-Alg.

Proof. The equivalence of (i) and (ii) holds by definition of weak equivalences in $T\text{-Alg}_s$. The equivalence of (ii) and (iii) is a routine (but important) exercise. □

5.11. Proposition: *For a strict morphism $f : A \to B$, the following are equivalent:*

(i) f is a fibration;

(ii) $U_s f$ is an isofibration in $\mathscr{K}$;

(iii) Jf is an isofibration in T-Alg.

Proof. The equivalence of (i) and (ii) holds by definition of weak equivalences in $T\text{-Alg}_s$. The equivalence of (ii) and (iii) is a straightforward consequence of Proposition 4.6. For let $u : L \to A$, $v : L \to B$, and $\lambda : v \cong fu$ exhibit L as the pseudolimit of f (in T-Alg_s, and so also in T-Alg and in $\mathscr{K}$). Then by Proposition 4.6, condition (ii) holds if and only if there are a $v' : L \to A$ and $\lambda' : v' \cong u$ in $\mathscr{K}$ with $f\lambda' = \lambda$ and $fv' = v$, while (iii) holds if and only if there exist such a v' and λ' in T-Alg; but the pseudomorphism structure of u can be transported across λ' to give a pseudomorphism structure on v', and so these are indeed equivalent. □

Notice in particular that $q : A' \to A$ is a trivial fibration for any algebra A, since we have $qp = 1$ and $pq \cong 1$ in T-Alg (and in $\mathscr{K}$). Thus for any algebra A, the map $q : A' \to A$ has a section in T-Alg; if it has a section in T-Alg_s— that is, a *strict* map $r : A \to A'$ with $qr = 1$ — then A is said to be *flexible* [**3**].

5.12. Theorem: *The cofibrant objects of T-Alg_s are precisely the flexible algebras; in particular, any algebra of the form A' is cofibrant, and so a cofibrant replacement for A. Every free algebra is flexible.*

Proof. Since $q : A' \to A$ is a trivial fibration, then certainly it will have a section if A is cofibrant. Thus cofibrant objects are flexible. For the converse, it will suffice to show that each A' is cofibrant, for any retract of a cofibrant object is cofibrant.

Suppose then that $t : E \to B$ is a trivial fibration in T-Alg_s, and $v : A' \to B$ an arbitrary strict map. We must show that it lifts through t. There is a pseudomorphism $s : B \to E$ with $ts = 1$, and so $tsv = v$. But the pseudomorphism $svp : A \to E$ has the form up for a unique strict map $u : A' \to E$. Now the strict maps tu and v from A' to B satisfy $tup = tsvp = vp$, and so $tu = v$ by the universal property of A', which gives the required lifting.

To see that free algebras are flexible, observe that any object $X \in \mathscr{K}$ is cofibrant, but for the lifted model structure the left adjoint preserves cofibrant objects, so the free algebra TX on X is cofibrant, and so flexible. □

The same relationship between flexibility and cofibrancy was observed in [**12**]. The following result was in [**3**]:

5.13. Proposition: *Any pseudomorphism with flexible domain is isomorphic to a strict morphism.*

Proof. Let $r : A \to A'$ be a strict morphism which is a section of $q : A' \to A$. Now $qrq = q = q1$, and q is an equivalence in T-Alg, so $rq \cong 1$ in T-Alg; but rq and 1 are in T-Alg_s, and the inclusion $J : T\text{-}\mathrm{Alg}_s \to T\text{-}\mathrm{Alg}$ is locally fully faithful (fully faithful on 2-cells) and so $rq \cong 1$ in T-Alg_s. This also implies that $r = rqp \cong p$ in T-Alg.

Now suppose that $f : A \to B$ is a pseudomorphism. It can be written as $f = gp$ for a unique strict morphism $g : A' \to B$, and now $gr \cong gp = f$, so that the pseudomorphism f is isomorphic to the strict morphism gr. □

5.14. The homotopy category of T-Alg_s is easily described: it is the category of strict T-algebras, and isomorphism classes of pseudo T-morphisms. As explained in Section 3.6, it has a canonical enrichment to a Ho**Cat**-category. But the resulting

Ho**Cat**-category can also be seen as the Ho**Cat**-category underlying the 2-category T-Alg (again in the sense of Section 3.6). Thus T-Alg is a kind of "homotopy 2-category" of $T\text{-Alg}_s$. This point of view is reinforced by the following proposition, which describes a universal property of T-Alg.

For 2-categories $\mathcal{M}$ and $\mathscr{L}$ we write $\mathsf{Ps}(\mathcal{M},\mathscr{L})$ for the 2-category of 2-functors, pseudonatural transformations, and modifications, from $\mathcal{M}$ to $\mathscr{L}$.

5.15. Theorem: *Let $\mathscr{L}$ be any 2-category. Composition with $J : T\text{-Alg}_s \to T\text{-Alg}$ induces a biequivalence of 2-categories between $\mathsf{Ps}(T\text{-Alg},\mathscr{L})$ and the full sub-2-category $\mathsf{Ps}_w(T\text{-Alg}_s,\mathscr{L})$ consisting of those 2-functors $T\text{-Alg}_s \to \mathscr{L}$ sending weak equivalences to equivalences.*

Proof. First observe $J : T\text{-Alg}_s \to T\text{-Alg}$ sends weak equivalences to equivalences, by Proposition 5.10, and pseudofunctors preserve equivalences, so composition with J does indeed induce a 2-functor $R : \mathsf{Ps}(T\text{-Alg},\mathscr{L}) \to \mathsf{Ps}_w(T\text{-Alg}_s,\mathscr{L})$. If $F : T\text{-Alg}_s \to \mathscr{L}$ sends weak equivalences to equivalences, then in particular it sends each $q_A : A' \to A$ to a weak equivalence. Now q is the counit of the adjunction $L \dashv J$, so Fq is an equivalence in $\mathsf{Ps}(T\text{-Alg}_s,\mathscr{L})$, and so $F \simeq FLJ = R(FL)$, and R is biessentially surjective on objects. To see that it is an equivalence on hom-categories, and so a biequivalence, observe that for $M, N : T\text{-Alg} \to \mathscr{L}$ we have

$$\begin{aligned}\mathsf{Ps}_w(T\text{-Alg}_s,\mathscr{L})(MJ,NJ) &= \mathsf{Ps}(T\text{-Alg}_s,\mathscr{L})(MJ,NJ)\\ &\simeq \mathsf{Ps}(T\text{-Alg},\mathscr{L})(M,NJL)\\ &\simeq \mathsf{Ps}(T\text{-Alg},\mathscr{L})(M,N)\end{aligned}$$

using adjointness and the fact that the unit $1 \to JL$ is an equivalence. □

We cannot expect this to work using 2-natural transformations. It is clear from the proof that rather than sending all weak equivalences to equivalences, we could ask only that trivial fibrations be sent to equivalences; but by Ken Brown's lemma [**5**, 1.1.2] and the fact that all objects of $T\text{-Alg}_s$ are fibrant, any 2-functor $T\text{-Alg}_s \to \mathscr{L}$ sending all trivial fibrations to equivalences must in fact send all weak equivalences to equivalences.

6. Flexible colimits

6.1. In this section we consider $T\text{-Alg}_s$ and its model structure for a particular case of T, relevant to (weighted) colimits in 2-categories. Recall that if $S : \mathscr{C} \to \mathscr{K}$ and $J : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ are 2-functors, with $\mathscr{C}$ small, we write $J * S$ for the J-weighted colimit of S, defined by an isomorphism

$$\mathscr{K}(J * S, A) \cong [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](J, \mathscr{K}(S,A))$$

natural in A, where $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ is the 2-category of 2-functors, 2-natural transformations, and modifications. The presheaf J is called the weight. We shall describe a 2-category $\mathscr{K}$ and a 2-monad T on $\mathscr{K}$ for which $T\text{-Alg}_s$ is precisely this 2-category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$.

6.2. Let $\mathscr{C}$ be a small 2-category, and write $\mathsf{ob}\mathscr{C}$ for its set of objects. Our base 2-category $\mathscr{K}$ will be $[\mathsf{ob}\mathscr{C}, \mathbf{Cat}]$; this is just the set of $\mathsf{ob}\mathscr{C}$-indexed families of categories. The 2-category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ has an evident forgetful 2-functor $U_s : [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}] \to [\mathsf{ob}\mathscr{C}, \mathbf{Cat}]$ given by restriction along the inclusion $\mathsf{ob}\mathscr{C} \to \mathscr{C}^{\mathrm{op}}$, and U_s has left and right adjoints given by left and right Kan extension along the inclusion. It is now straightforward to verify using (an enriched variant of) Beck's theorem that U_s is monadic, so that $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ has the form $T\text{-Alg}_s$ for a 2-monad T on $[\mathsf{ob}\mathscr{C}, \mathbf{Cat}]$. Since U_s has a right as well as a left adjoint, it follows that T too has a right adjoint, so preserves all colimits, thus in particular is finitary. The corresponding 2-category T-Alg is $\mathsf{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$, consisting of the 2-functors, pseudonatural transformations, and modifications. A flexible algebra for this 2-monad is called a *flexible weight* [**2**], and colimits weighted by flexible weights are called *flexible colimits*.

Just as all ordinary colimits can be computed using coproducts and coequalizers, all flexible colimits can be computed using four basic types of flexible colimit: coproducts, coinserters, coequifiers, and splittings of idempotents [**2**]. For a good introduction to these various limit notions, see [**8**]. Here we simply recall that all these notions are defined by universal properties involving 2-natural isomorphisms, and that the coinserter of a pair $f, g : A \to B$ is the universal map $p : B \to C$ equipped with a 2-cell $pf \to pg$; while the coequifier of a parallel pair α and β of 2-cells between 1-cells $f, g : A \to B$ is the universal $p : B \to C$ for which $p\alpha = p\beta$.

6.3. A fundamental result [**2**, Theorem 4.9 and Proposition 4.10] is that the flexible algebras are closed under flexible colimits (in $T\text{-Alg}_s$). This is equivalent to being closed under these four types of colimit. Here we offer an alternative viewpoint on this fundamental result, based on the fact that the flexible algebras are precisely the cofibrant objects in a model 2-category. In any model category the cofibrant objects are closed under coproducts and splittings of idempotents; but in a model 2-category we have:

6.4. Theorem: *In a model 2-category, the cofibrant objects are closed under flexible colimits.*

Proof. Since cofibrant objects are always closed under coproducts and splittings of idempotents (retracts), it remains to show that they are closed under coinserters and coequifiers.

Let $f, g : F \to A$ be morphisms between cofibrant objects, and let $i : A \to B$ and $\alpha : if \to ig$ exhibit B as the coinserter of f and g. We shall show that i is a cofibration, and so that B is cofibrant. Suppose then that

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "y"'] & D \end{tikzcd}$$

is a commutative square with p a trivial fibration. Regarding p and x as fixed, to give y is just to give $\varphi : pxf \to pxg$. Since A is cofibrant there is by Proposition 3.3 a unique 2-cell $\psi : xf \to xg$ with $p\psi = \varphi$. By the universal property of the coinserter there is now a unique $z : B \to C$ with $zi = x$ and $z\alpha = \psi$. On the other hand $pzi = px = yi$ and $pz\alpha = p\psi = \varphi = y\alpha$ so $pz = y$ by the uniqueness part of the universal property. Thus z is the required fill-in and so i is a cofibration.

Now we turn to coequifiers. Let $f, g : F \to A$ be morphisms between cofibrant objects, and let $i : A \to B$ be the coequifier of 2-cells $\alpha, \beta : f \to g$. We shall show that i is a cofibration and so that B is cofibrant. Consider a square as above; this time y is uniquely determined by px, and its existence is equivalent to the equation $px\alpha = px\beta$. Since F is cofibrant and p is a trivial fibration, we have $x\alpha = x\beta$ by Proposition 3.3 once again, and so a unique $z : B \to C$ with $zi = x$. The equation $pz = y$ is an immediate consequence using the universal property of B once again, so z provides the fill-in for the square, and i is once again a cofibration. $\square$

6.5. In the case of the lifted model structure on T-Alg_s, the flexible (=cofibrant) algebras are precisely the closure under flexible colimits of the free algebras. On the one hand, all objects of $\mathscr{K}$ are cofibrant, so all free algebras are cofibrant, and we have seen that cofibrant objects are closed under flexible colimits. For the converse, it was shown in [**11**] that for any algebra A, the algebra A' can be constructed from free algebras using coinserters and coequifiers (in T-Alg_s); since the flexible algebras are the retracts of the A', it follows that they are flexible colimits of free algebras.

7. Flexible monads

7.1. In this section we study a certain 2-category of 2-monads, and a lifted model structure coming from the underlying 2-category of endo-2-functors. We continue to consider a fixed locally finitely presentable 2-category $\mathscr{K}$; for this section the most important case is $\mathscr{K} = \mathbf{Cat}$. A 2-functor $T : \mathscr{K} \to \mathscr{K}$ is said to be *finitary* if it preserves filtered colimits; or, equivalently, if it is the left Kan extension of its restriction to the full sub-2-category $\mathscr{K}_f$ of $\mathscr{K}$ consisting of the finitely presentable objects. Since the composite of finitary 2-functors is clearly finitary, and identity 2-functors are so, one obtains a strict monoidal category $\mathbf{End}_f(\mathscr{K})$ of finitary endo-2-functors of $\mathscr{K}$. As a category, it is equivalent to the 2-functor category $[\mathscr{K}_f, \mathscr{K}]$: the equivalence sends a finitary 2-functor T to its composite TJ with the inclusion $J : \mathscr{K}_f \to \mathscr{K}$, and sends $S : \mathscr{K}_f \to \mathscr{K}$ to the left Kan extension $\mathsf{Lan}_J(S)$. The strict monoidal structure on $\mathbf{End}_f(\mathscr{K})$ transports across the equivalence to give a (no longer strict) monoidal structure on $[\mathscr{K}_f, \mathscr{K}]$: the tensor product $S \circ R$ is given by $\mathsf{Lan}_J(S)R$, and the unit is J. We sometimes identify 2-functors $\mathscr{K}_f \to \mathscr{K}$ with the corresponding finitary endo-2-functors of $\mathscr{K}$.

7.2. A 2-monad on $\mathscr{K}$ consists of a 2-functor $T : \mathscr{K} \to \mathscr{K}$ equipped with 2-natural transformations $m : T^2 \to T$ and $i : 1 \to T$ satisfying the usual monad equations. It is said to be finitary if the endo-2-functor T is so. (We often allow ourselves to speak of "the 2-monad T", leaving the multiplication m and unit i understood.)

There is now a category $\mathbf{Mnd}_f(\mathscr{K})$ of finitary 2-monads on $\mathscr{K}$ and *strict* morphisms; it is the category of monoids in $\mathbf{End}_f(\mathscr{K})$ or equivalently in $[\mathscr{K}_f, \mathscr{K}]$). The forgetful functor $W : \mathbf{Mnd}_f(\mathscr{K}) \to [\mathscr{K}_f, \mathscr{K}]$ has a left adjoint H and is monadic [**10**].

7.3. Monad morphisms are useful for describing algebras for monads. Recall from [**7**] or [**9**, Section 2] that if A and B are objects of $\mathscr{K}$ then there is a 2-functor $\langle A, B\rangle : \mathscr{K}_f \to \mathscr{K}$ which sends a finitely presentable object C to the cotensor $\mathscr{K}(C, A) \pitchfork B$, and now to give a 2-natural transformation $T \to \langle A, B\rangle$ is equivalently to give a map $TA \to B$ in $\mathscr{K}$; more precisely, we have an isomorphism of categories $[\mathscr{K}_f, \mathscr{K}](T, \langle A, B\rangle) \cong \mathscr{K}(TA, B)$. Furthermore, if $A = B$, then there is a 2-monad structure on $\langle A, A\rangle$, such that for a 2-monad T, a 2-natural transformation $T \to \langle A, A\rangle$ is a monad map if and only if the corresponding $TA \to A$ makes A into a T-algebra. This observation illustrates the importance of colimits in $\mathbf{Mnd}_f(\mathscr{K})$: it shows, for example, that an algebra for the coproduct $S + T$ of monads S and T is just an object equipped with an S-algebra structure and a T-algebra structure. We shall see further examples below.

First observe, following [**7, 9**] once again, that if $f, g : A \to B$ we may form the comma-object

$$\begin{tikzcd} \{f,g\} \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\swarrow\lambda"] & \langle A,A\rangle \arrow[d, "\langle A,g\rangle"] \\ \langle B,B\rangle \arrow[r, "\langle f,B\rangle"'] & \langle A,B\rangle \end{tikzcd}$$

in $\mathbf{End}_f(\mathscr{K})$, and now to give a 2-natural transformation $\gamma : T \to \{f, g\}$ is equivalently to give morphisms $a : TA \to A$ (corresponding to $c\gamma$) and $b : TB \to B$ (corresponding to $d\gamma$), and an invertible 2-cell $b.Tf \to ga$. Once again, if $f = g$, then there is a trivial 2-monad structure on $\{f, f\}$ such that if T is a 2-monad, then γ is a monad map if and only if (A, a) and (B, b) are T-algebras and the induced 2-cell $\overline{f} : b.Tf \to fa$ makes $(f, \overline{f})$ into a T-morphism. Thus once again we can analyze the (pseudo)morphisms of algebras for colimits of 2-monads. Finally, if $\rho : f \to g$ is a 2-cell in $\mathscr{K}$, then we may form the pullback

$$\begin{tikzcd} {[\rho,\rho]} \arrow[r] \arrow[d] & \{f,f\} \arrow[d, "\{f,\rho\}"] \\ \{g,g\} \arrow[r, "\{\rho,g\}"'] & \{f,g\} \end{tikzcd}$$

and this has a canonical monad structure for which monad maps $T \to [\rho, \rho]$ correspond to 2-cells in T-Alg.

7.4. This allows us to give *presentations* for 2-monads, as in [**10**]. For example, take $\mathscr{K} = \mathbf{Cat}$, and let $E : \mathbf{Cat} \to \mathbf{Cat}$ be the finitary 2-functor sending a category $\mathscr{C}$ to $\mathscr{C} \times \mathscr{C}$. An algebra for the free monad HE on E is just a category $\mathscr{C}$ with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. A (pseudo)morphism is a functor between such categories which preserves the "tensor product" up to an arbitrary natural isomorphism: there are no coherence conditions at this stage.

Now let $D : \mathbf{Cat} \to \mathbf{Cat}$ be the finitary 2-functor sending $\mathscr{C}$ to $\mathscr{C} \times \mathscr{C} \times \mathscr{C}$, and HD the free monad on D. Since for any HE-algebra $\mathscr{C}$ there are two trivial HD-algebra structures (corresponding to the two bracketings of a triple product), there are two induced monad maps $f, g : HD \to HE$. We can form the *co-isoinserter* of these maps, which is the universal monad map $r : HE \to S$ equipped with a monad isomorphism $\rho : rf \cong rg$. An S-algebra is now a category $\mathscr{C}$, with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, and a natural isomorphism $\alpha : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. An S-morphism is a functor preserving the tensor product up to coherent isomorphism. Finally we may consider the finitary 2-functor $B : \mathbf{Cat} \to \mathbf{Cat}$ sending $\mathscr{C}$ to $\mathscr{C}^4$. There are two HB-algebra structures on an S-algebra $\mathscr{C}$, involving the derived operations $((A \otimes B) \otimes C) \otimes D$ and $A \otimes (B \otimes (C \otimes D))$, and these induce two monad maps $f', g' : HB \to S$. The two isomorphisms $((A \otimes B) \otimes C) \otimes D \cong A \otimes (B \otimes (C \otimes D))$ which can be built out of α induce two monad transformations $\varphi, \psi : f' \to g'$, and we can now form the *coequifier* of these 2-cells, namely the universal monad map $q : S \to T$ for which $q\varphi = q\psi$. A T-algebra is now exactly what one might call a *semigroupoidal category*: a category $\mathscr{C}$ equipped with a tensor product $\otimes$ which is associative up to coherent isomorphism (coherent in the sense of the Mac Lane pentagon), but not necessarily having a unit. A T-morphism is a strong semigroupoidal functor (one which preserves the tensor product up to coherent isomorphism).

7.5. Since $\mathbf{Mnd}_f(\mathscr{K})$ is monadic over $\mathbf{End}_f(\mathscr{K})$ via a finitary 2-monad M, we have a lifted model structure on $\mathbf{Mnd}_f(\mathscr{K})(= M\text{-Alg}_s)$. As usual the cofibrant objects are the flexible algebras, here called flexible monads.

We know that (i) free monads (on a finitary endo-2-functor) are flexible, and that (ii) flexible colimits of flexible monads are flexible. Since co-isoinserters and coequifiers are both flexible colimits (a co-isoinserter can be constructed out of coinserters and coequifiers) it follows that T is a flexible monad. The key feature of the presentation given above is that it used coinserters and coequifiers but not such "inflexible" colimits as coequalizers. As observed by Kelly, Power, and various collaborators, a 2-monad is always flexible if it can be given by a presentation which "involves no equations between objects". Thus the 2-monad for monoidal categories is flexible, while that for strict monoidal categories is not (it involves the equation $A \otimes (B \otimes C) = (A \otimes B) \otimes C$).

7.6. A monad morphism $k : S \to T$ induces a 2-functor $k_s^* : T\text{-Alg}_s \to S\text{-Alg}_s$ commuting with the forgetful 2-functors into $\mathscr{K}$; there is also an induced 2-functor $T\text{-Alg} \to S\text{-Alg}$, considered in Section 8 below. Explicitly, k_s^* sends a T-algebra $(A, a : TA \to A)$ to the S-algebra (A, a'), where a' is the composite of a and $fA : SA \to TA$. Since fibrations and trivial fibrations in the 2-categories of algebras are defined as in $\mathscr{K}$, and k_s^* commutes with the forgetful 2-functors, k_s^* preserves fibrations and trivial fibrations. It also has a left adjoint [**3**], and so is the right adjoint part of a Quillen adjunction. In Section 8 we shall find conditions under which it is a Quillen equivalence.

7.7. As well as the 2-category $\mathbf{Mnd}_f(\mathscr{K}) = M\text{-Alg}_s$ of 2-monads and strict morphisms, we can also consider the 2-category M-Alg of finitary 2-monads on $\mathscr{K}$ and *pseudomorphisms* of monads. Explicitly, a pseudomorphism from T to S consists of a 2-natural transformation $f : T \to S$ which preserves the unit and multiplication up to isomorphism; these isomorphisms are required to satisfy coherence conditions formally identical to those for a strong monoidal functor. One of the main reason for considering pseudomorphisms is for dealing with *pseudoalgebras*; this technique goes back to [**7**]. A pseudoalgebra for a 2-monad T is an object A equipped with a morphism $a : TA \to A$ satisfying the usual algebra axioms up to coherent isomorphism. This may be expressed by saying that the 2-natural $\alpha : T \to \langle A, A\rangle$ corresponding to A is a pseudomorphism of monads. Thus we have bijective correspondences between pseudo T-algebra structures on A, pseudomorphisms $T \to \langle A, A\rangle$, strict morphisms $T' \to \langle A, A\rangle$, and (strict) T'-algebra structures on A, and in fact $T'\text{-Alg}_s$ is isomorphic to the 2-category Ps-T-Alg_s of pseudo T-algebras and *strict* T-morphisms, and similarly T'-Alg isomorphic to the 2-category Ps-T-Alg of pseudo T-algebras and pseudo T-morphisms.

7.8. Notice that if T is flexible, then by Proposition 5.13 every pseudomorphism $T \to \langle A, A\rangle$ is isomorphic to a strict one. When this is translated into a statement about algebras it states that for a flexible T, every pseudoalgebra structure on an object A is isomorphic in Ps-T-Alg to a strict algebra structure on A via pseudomorphism of the form $(1_A, \varphi)$. In particular, for a flexible monad, every pseudoalgebra is isomorphic to a strict one.

8. Structure and semantics

8.1. We now turn from monads to their algebras. Whereas earlier we considered model structures on $T\text{-Alg}_s$, as a vehicle to understanding the more important 2-category T-Alg, we now focus on T-Alg itself. The passage from a 2-monad T on $\mathscr{K}$ to the 2-category T-Alg with forgetful 2-functor $U : T\text{-Alg} \to \mathscr{K}$ is functorial. Given a (strict) morphism $k : S \to T$ of 2-monads, the 2-functor $k^*_s : T\text{-Alg}_s \to S\text{-Alg}_s$ extends to a 2-functor $k^* : T\text{-Alg} \to S\text{-Alg}$, also commuting with the forgetful 2-functors. In order to capture this situation, we consider the (enormous) 2-category **2-CAT** of (not necessarily small) 2-categories, 2-functors, and 2-natural transformations (ignoring the further structure which makes it into a 3-category), and then the slice 2-category $\mathbf{2\text{-}CAT}/\mathscr{K}$, an object of which is a 2-category $\mathscr{L}$ equipped with a 2-functor $U : \mathscr{L} \to \mathscr{K}$, and a morphism of which is a commutative triangle. If M and N are morphisms from $U : \mathscr{L} \to \mathscr{K}$ to $U' : \mathscr{L}' \to \mathscr{K}$, a 2-cell from M to N is a 2-natural transformation $\rho : M \to N$ whose composite with U' is the identity on U. Then there is a functor $\mathsf{sem} : \mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}} \to \mathbf{2\text{-}CAT}/\mathscr{K}$ which sends a 2-monad T to $U : T\text{-Alg} \to \mathscr{K}$, and a morphism $j : S \to T$ in $\mathbf{Mnd}_f(\mathscr{K})$ to $k^* : T\text{-Alg} \to S\text{-Alg}$.

8.2. Although size problems prevent there from being a model structure on $\mathbf{2\text{-}CAT}/\mathscr{K}$, there are nonetheless obvious notions of fibration and weak equivalence, which we now describe.

There is a Quillen model structure on the category **2-Cat** of small 2-categories and 2-functors, described in [**12**], for which the weak equivalences are the biequivalences: these are the 2-functors $F : \mathscr{K} \to \mathscr{L}$ for which each functor $F : \mathscr{K}(A, B) \to \mathscr{L}(FA, FB)$ is an equivalence and furthermore for each $C \in \mathscr{L}$ there is an $A \in \mathscr{K}$ and an equivalence $C \simeq FA$ in $\mathscr{L}$. A 2-functor $F : \mathscr{K} \to \mathscr{L}$ is a fibration if each $F : \mathscr{K}(A, B) \to \mathscr{L}(FA, FB)$ is a fibration in **Cat**, and moreover equivalences lift through F in a sense made precise in [**13**]. (There is a mistake in the description of fibrations and trivial cofibrations in [**12**]; this is corrected in [**13**], which also provides a model structure on the category of bicategories and strict homomorphisms, and shows that these two model categories are Quillen equivalent.) The cofibrations are of course the maps with the left lifting property with respect to the trivial fibrations; these are characterized in [**12**].

Clearly the definitions of weak equivalence and fibration have nothing to do with size, and one can easily extend them to give notions of fibration and weak equivalence in **2-CAT**.

There is an evident functor $D : \mathbf{2\text{-}CAT}/\mathscr{K} \to \mathbf{2\text{-}CAT}$ sending $U : \mathscr{L} \to \mathbf{Cat}$ to $\mathscr{L}$, and we define a morphism f in $\mathbf{2\text{-}CAT}/\mathscr{K}$ to be a weak equivalence or fibration if and only if Df is one in **2-CAT**.

8.3. Let $k : S \to T$ be a monad morphism. We consider the following induced maps. First of all there is $k_s^* : T\text{-Alg}_s \to S\text{-Alg}_s$, which is a right Quillen 2-functor. Then there is the 2-functor $k^* : T\text{-Alg} \to S\text{-Alg}$ which extends k_s^*. Finally there is the Ho**Cat**-functor $\mathrm{Ho}(k_s^*) : \mathrm{Ho}(T\text{-Alg}_s) \to \mathrm{Ho}(S\text{-Alg}_s)$. Since all objects in $T\text{-Alg}_s$ are fibrant, $\mathrm{Ho}(k_s^*)$ is induced directly from k_s^* without having to use fibrant approximation. Thus $\mathrm{Ho}(k_s^*)$ is simply the underlying Ho**Cat**-functor of $k^* : T\text{-Alg} \to S\text{-Alg}$.

8.4. Proposition: *The following are equivalent:*

(i) $k_s^* : T\text{-Alg}_s \to S\text{-Alg}_s$ *is a Quillen equivalence;*

(ii) $Ho(k_s^*) : Ho(T\text{-Alg}_s) \to Ho(S\text{-Alg}_s)$ *is an equivalence of Ho***Cat***-categories;*

(iii) $k^* : T\text{-Alg} \to S\text{-Alg}$ *is a biequivalence.*

Proof. The equivalence of (i) and (ii) was proved in Section 3.7. For T-algebras A and B, we have $k^* : T\text{-Alg}(A, B) \to S\text{-Alg}(k^*A, k^*B)$ an equivalence if and only if $\mathrm{Ho}(k_s^*) : \mathrm{Ho}T\text{-Alg}_s(A, B) \to \mathrm{Ho}S\text{-Alg}_s(k^*A, k^*B)$ is invertible, while for a T-algebra A and an S-algebra C, we have $k^*A \cong C$ in $\mathrm{Ho}S\text{-Alg}_s$ if and only if $k^*A \simeq C$ in S-Alg. This gives the equivalence between (ii) and (iii). □

8.5. Colimits in $\mathbf{Mnd}_f(\mathscr{K})$ of course become limits in $\mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}}$, and cofibrations and weak equivalences in $\mathbf{Mnd}_f(\mathscr{K})$ become fibrations and weak equivalences in $\mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}}$. Size issues notwithstanding, the functor $\mathsf{sem} : \mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}} \to \mathbf{2\text{-}CAT}/\mathscr{K}$ sends limits to limits, fibrations to fibrations, and trivial fibrations to trivial fibrations, as we verify in Sections 8.6-8.10 below, using the constructions $\langle A, A\rangle$, $\{f, f\}$, and $[\rho, \rho]$ of Section 7.3.

8.6. To say that sem preserves limits is to say that it sends colimits in $\mathbf{Mnd}_f(\mathscr{K})$ to limits in $\mathbf{2\text{-}CAT}/\mathscr{K}$. Consider the case of coproducts. A product in $\mathbf{2\text{-}CAT}/\mathscr{K}$ is just a fibre product in $\mathbf{2\text{-}Cat}$ (over $\mathbf{Cat}$). Suppose then that $(S_i)_{i\in I}$ is a small family of finitary 2-monads on $\mathbf{Cat}$, with coproduct $S = \sum_i S_i$. The product in $\mathbf{2\text{-}CAT}/\mathscr{K}$ of the $\mathsf{sem}(S_i)$, is the 2-category in which an object is a $\mathscr{K}$-object A equipped with an S_i-algebra structure $a_i : S_iA \to A$ for each i; a morphism between two such is a $\mathscr{K}$-morphism $f : A \to B$ equipped with, for each i, an isomorphism $\overline{f}_i : b_i.Sf \cong fa_i$ making f into an S_i-morphism; and a 2-cell between two such is a $\mathscr{K}$-transformation $f \to g$ compatible with the S_i-morphism structure for each i. So to make A into an object of $\prod_i \mathsf{sem}(S_i)$ is to give a monad map $\alpha_i : S_i \to \langle A, A\rangle$ for each i; but this is precisely to give a single monad map $\alpha : S \to \langle A, A\rangle$; that is, an S-algebra $a : SA \to A$ structure for A. The case of morphisms is treated similarly. Let (A, a) and (B, b) be S-algebras, with notation for the other associated maps as above. Then to make a $\mathscr{K}$-morphism $f : A \to B$ into a morphism in $\prod_i \mathsf{sem}(S_i)$ is to give monad maps $\varphi_i : S_i \to \{f, f\}$ for each i, compatible with the $\alpha_i : S_i \to \langle A, A\rangle$ and $\beta_i : S_i \to \langle B, B\rangle$. But by the universal property of S, this amounts to giving a single monad map $\varphi : S \to \{f, f\}$ compatible with α and β; that is, to a single $\overline{f} : b.Sf \cong fa$ making f into an S-morphism. Thus $\mathsf{sem}(S)$ and $\prod_i \mathsf{sem}(S_i)$ have the same objects and morphisms; it remains to check the 2-cells, and this can be done entirely analogously, using the construction $[\rho, \rho]$.

This proves that $\mathsf{sem} : \mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}} \to \mathbf{2\text{-}CAT}/\mathscr{K}$ preserves products; the case of general limits is similar, and left to the reader.

8.7. Let $j : S \to T$ be a trivial cofibration in $\mathbf{Mnd}_f(\mathscr{K})$, and so a trivial fibration in $\mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}}$ from T to S; we shall show that $j^* : T\text{-Alg} \to S\text{-Alg}$ is a trivial fibration in $\mathbf{2\text{-}CAT}$, and so that $\mathsf{sem}(j)$ is a trivial fibration in $\mathbf{2\text{-}CAT}/\mathscr{K}$. Since S (like every other object of $\mathbf{Mnd}_f(\mathscr{K})$) is fibrant, we know by Proposition 3.3 that $\mathbf{Mnd}_f(\mathscr{K})(j, S) : \mathbf{Mnd}_f(\mathscr{K})(T, S) \to \mathbf{Mnd}_f(\mathscr{K})(S, S)$ is a surjective equivalence. Thus there is a monad morphism $g : T \to S$ with $gj = 1$, and a unique isomorphism $\rho : jg \cong 1$ in $\mathbf{Mnd}_f(\mathscr{K})$ with $g\rho = \mathsf{id}$ and $\rho j = \mathsf{id}$. By functoriality of sem, we have $j^*g^* = 1$ and $g^*j^* \cong 1$, so j^* is not just a trivial fibration, but in fact a retract equivalence in $\mathbf{2\text{-}CAT}/\mathscr{K}$. This proves that $\mathsf{sem} : \mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}} \to \mathbf{2\text{-}CAT}/\mathscr{K}$ sends trivial fibrations to trivial fibrations.

8.8. Remark: In fact by the same argument even the trivial cofibrations for the trivial model structure in $\mathbf{Mnd}_f(\mathscr{K})$ are sent to trivial fibrations in $\mathbf{2\text{-}CAT}/\mathscr{K}$.

8.9. The situation with general weak equivalences is more delicate. A morphism $f : S \to T$ in $\mathbf{Mnd}_f(\mathscr{K})$ is a weak equivalence if and only if there exists a 2-natural $g : T \to S$ with $gf \cong 1$ and $fg \cong 1$. This g will automatically be a monad pseudomorphism, but need not in general be a monad morphism, thus it need not induce a 2-functor $g^* : S\text{-Alg} \to T\text{-Alg}$. As a special case, consider the weak equivalence $q : T' \to T$. Then q^* is the inclusion $T\text{-Alg} \to \text{Ps-}T\text{-Alg}$, which is a weak equivalence if and only if every pseudo T-algebra is equivalent to a strict one: the "general coherence problem" for T-algebras. This is still an open problem in the current generality, but has been solved in various special cases — see [**11**] and the references therein.

If the monads S and T are flexible, however, then any weak equivalence $f : S \to T$ does induce a biequivalence $f^* : T\text{-Alg} \to S\text{-Alg}$. This can be proved using the observation of Section 7.8, or it could also be deduced using Ken Brown's lemma [**5**, 1.1.2]. More generally, f^* will be a biequivalence for any weak equivalence $f : S \to T$ for which every pseudo S-algebra is equivalent to a strict one.

On the other hand, the weak equivalences in $\mathbf{Mnd}_f(\mathscr{K})$ for the trivial model structure are just the equivalences in $\mathbf{Mnd}_f(\mathscr{K})$, and these *are* mapped to weak equivalences in $\mathbf{2\text{-}CAT}/\mathscr{K}$.

8.10. The next thing to do is to check whether sem preserves fibrations; that is, whether $j^* : T\text{-Alg} \to S\text{-Alg}$ is a fibration in **2-CAT** whenever $j : S \to T$ is a cofibration in $\mathbf{Mnd}_f(\mathscr{K})$. First we check that equivalences can be lifted through j^*. Suppose then that (A, a) is a T-algebra, that (B, b) is an S-algebra, and that $(f, \overline{f}) : (B, b) \to j^*(A, a)$ is an equivalence of S-algebras; the latter implies in particular that $(f, \overline{f})$ is a morphism of S-algebras with $f : A \to B$ an equivalence in $\mathscr{K}$. We must show that the S-algebra structure on B can be extended to a T-algebra structure in such a way that f becomes a morphism of T-algebras.

To do this, let $\beta : S \to \langle B, B\rangle$ and $\alpha : T \to \langle A, A\rangle$ be the monad morphisms corresponding to $b : SB \to B$ and $a : TA \to A$. We shall need, among other things, to extend β along $j : S \to T$. Let $\varphi : S \to \{f, f\}$ be the monad morphism corresponding to $(f, \overline{f})$. Then the diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "m"'] & \{f, f\} \arrow[d, "d"] \\ T \arrow[r, "\alpha"'] & \langle A, A\rangle \end{tikzcd}$$

of monad morphisms commutes, and the equivalence lifting property for $j^* : T\text{-Alg} \to S\text{-Alg}$ now amounts to the existence of a fill-in. Since j was assumed to be a cofibration, this fill-in will exist provided that $f : \{f, f\} \to \langle A, A\rangle$ is a trivial fibration in $\mathbf{Mnd}_f(\mathscr{K})$, or equivalently in $\mathbf{End}_f(\mathscr{K})$. But $f : A \to B$ was assumed an equivalence, thus $\langle f, B\rangle$ is an equivalence, and the result now follows by the general fact that if

$$\begin{tikzcd} W \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\Downarrow"] & X \arrow[d, "w"] \\ Y \arrow[r] & Z \end{tikzcd}$$

is an iso-comma object (in any 2-category) and w an equivalence then p is a retract equivalence.

This proves the equivalence lifting property. We now turn to the 2-dimensional aspect of the definition of fibration. This asserts that if $(f, \overline{f}) : (A, a) \to (B, b)$ is a T-morphism, and $\rho : (g, \overline{g}) \cong j^*(f, \overline{f})$ is an invertible S-transformation, then we can lift this to an invertible T-transformation. This is entirely straightforward and is true for *any* monad morphism $j : S \to T$.

This completes the verification that $\mathsf{sem} : \mathbf{Mnd}_f(\mathscr{K})^{\mathrm{op}} \to \mathbf{2\text{-}CAT}/\mathscr{K}$ preserves fibrations and trivial fibrations.

8.11. The statement about fibrations is that if $j : S \to T$ is a cofibration in $\mathbf{Mnd}_f(\mathscr{K})$, then $j^* : T\text{-Alg} \to S\text{-Alg}$ is a fibration in **2-CAT**. As a special case, if T is flexible (cofibrant), then the forgetful 2-functor $U : T\text{-Alg} \to \mathscr{K}$ is a fibration in **2-CAT**. This amounts to the facts that (i) if (A, a) is a T-algebra, and $f : B \to A$ an equivalence in $\mathscr{K}$, then there is a T-algebra structure (B, b) on B, for which f can be made into a T-morphism, and (ii) if $(f, \overline{f}) : (A, a) \to (B, b)$ is a T-morphism, and $\varphi : g \cong f$, then g can be made into a T-morphism $(g, \overline{g})$ in such a way that φ is a T-transformation $(g, \overline{g}) \to (f, \overline{f})$. As before, (ii) is true for any 2-monad T, while (i) asserts that T-algebra structure can be transported along equivalences. This is true for flexible monads, but not in general. It is false, for example, in the case of the 2-monad T for strict monoidal categories. Consider the category C of countable sets, viewed as a monoidal category under the cartesian product. This can be replaced by an equivalent strict monoidal category A. Let B be a skeleton of C (choose one set of each countable cardinality), then there exists an equivalence of categories $f : B \to A$. But by an argument due to Isbell (see[**14**, VII.1]), there is no way to transport the strict monoidal structure on A across the equivalence f to obtain a strict monoidal structure on B.

8.12. It is only the hugeness of $\mathbf{2\text{-}CAT}/\mathscr{K}$ that prevents it from having a model structure, and sem from having a left adjoint, and so becoming a right Quillen 2-functor. It would be interesting to find a full sub-2-category of $\mathbf{2\text{-}CAT}/\mathscr{K}$ containing the image of sem, admitting a model structure with the fibrations and weak equivalences defined as in $\mathbf{2\text{-}CAT}/\mathscr{K}$, and on which a left adjoint to sem can be defined.

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Stephen Lack
`s.lack@uws.edu.au`

School of Computing and Mathematics
University of Western Sydney
Locked Bag 1797
Penrith South DC NSW 1797
Australia