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FACTORIZATION, FIBRATION AND TORSION

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(*communicated by George Janelidze*)

Abstract

A simple definition of torsion theory is presented, as a factorization system with both classes satisfying the 3–for–2 property. Comparisons with the traditional notion are given, as well as connections with the notions of fibration and of weak factorization system, as used in abstract homotopy theory.

1. Introduction

That full reflective subcategories may be characterized by certain factorization systems is well known, thanks to the works of Ringel [Ri] and Cassidy, Hébert and Kelly [CHK]. While the former paper treats the characterization in the context of the Galois correspondence that leads to the definition of weak factorization systems (as given in [AHRT]), the latter paper carefully analyzes construction methods for the factorizations in question. To be more specific, following [CHK], we call a factorization system $(\mathcal{E}, \mathcal{M})$ *reflective* if $\mathcal{E}$ satisfies the cancellation property that g and gf in $\mathcal{E}$ force f to be in $\mathcal{E}$; actually, $\mathcal{E}$ must then have what homotopy theorists call the 3-for-2 property. When there is a certain one–step procedure for constructing such factorizations from a given reflective subcategory, the system is called *simple*. Following a pointer given to the second author by André Joyal, in this paper we characterize simple reflective factorization systems of a category $\mathcal{C}$ in terms of generalized fibrations $P : \mathcal{C} \to \mathcal{B}$: they are all of the form $\mathcal{E} = \{\text{morphisms inverted by } P\}$, $\mathcal{M} = \{P - \text{cartesian morphisms}\}$ (see Theorem 3.9). In preparation for the theorem, we not only carefully review some needed facts on factorization systems, but characterize them also within the realm of weak factorization systems (Prop. 2.3), using a somewhat hidden result of [Ri], and we frequently allude to the use of weak factorization systems in the context of Quillen model categories. Furthermore, we have included a new result for many types of categories, including extensive categories as well as additive categories, namely that ({coproduct injections}, {split epimorphisms}) form always a weak factoriza-

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tion system (Theorem 2.7), which is somewhat surprising since both classes appear to be small.

The main point of the paper, however, is to present an easy definition of torsion theory that simplifies the definition given by Cassidy, Hébert and Kelly [CHK]. Hence, here a torsion theory in any category is simply a factorization system $(\mathcal{E}, \mathcal{M})$ that is both reflective and coreflective, so that both $\mathcal{E}$ and $\mathcal{M}$ have the 3–for–2 property. At least in pointed categories with kernels and cokernels, such that every morphism factors into a cokernel followed by a morphism with trivial kernel, and dually, our torsion theories determine a pair of subcategories with the properties typically expected from a pair of subcategories of "torsion" objects and of "torsion–free" objects, at least when the system $(\mathcal{E}, \mathcal{M})$ is simple (Theorem 4.10). We present a precise characterization of "standard" torsion theories (given by pairs of full subcategories) in terms of our more general notion in Theorem 5.2, under the hypothesis that the ambient category is homological (in the sense of [BB]), such that every morphism factors into a kernel preceded by a morphism with trivial cokernel. At least all additive categories which are both regular and coregular (in the sense of Barr [Ba]) have that property.

We have dedicated this paper to the memory of Saunders Mac Lane, whose pioneering papers entitled "Groups, categories and duality" (Bulletin of the National Academy of Sciences USA 34(1948) 263-267) and "Duality for groups" (Bulletin of the American Mathematical Society 56 (1950) 485-516) were the first to not only introduce fundamental constructions like direct products and coproducts in terms of their universal mapping properties, but to also present a forerunner to the modern notion of factorization system, an equivalent version of which made its first appearance in John Isbell's paper "Some remarks concerning categories and subspaces" (Canadian Journal of Mathematics 9 (1957) 563-577), but which became widely popularized only through Peter Freyd's and Max Kelly's paper on "Categories of continuous functors, I" (Journal of Pure and Applied Algebra 2 (1972) 169-191).

Some of the results contained in this paper were presented by the second author at a special commemorative session on the works of Samuel Eilenberg and Saunders Mac Lane during the International Conference on Category Theory, held at White Point (Nova Scotia, Canada) in June 2006.

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2. Weak factorization systems and factorization systems

2.1

For morphisms e and m in a category $\mathcal{C}$ one writes

$$e \,\square\, m \quad (e \perp m)$$

if, for every commutative solid-arrow diagram

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "e"'] & \cdot \arrow[d, "m"] \\ \cdot \arrow[r] \arrow[ur, dashed, "d"] & \cdot \end{tikzcd}$$

one finds a (unique) arrow d making both emerging triangles commutative. For classes $\mathcal{E}$ and $\mathcal{M}$ of morphisms in $\mathcal{C}$ one writes

$$\mathcal{E}^{\Box} = \{m \mid \forall e \in \mathcal{E} : e \Box m\}, \quad {}^{\Box}\mathcal{M} = \{e \mid \forall m \in \mathcal{M} : e \Box m\},$$
$$\mathcal{E}^{\perp} = \{m \mid \forall e \in \mathcal{E} : e \perp m\}, \quad {}^{\perp}\mathcal{M} = \{e \mid \forall m \in \mathcal{M} : e \perp m\}.$$

Recall that $(\mathcal{E}, \mathcal{M})$ is a *weak factorization system (wfs)* if

1. $\mathcal{C} = \mathcal{M} \cdot \mathcal{E}$
2. $\mathcal{E} = {}^{\Box}\mathcal{M}$ *and* $\mathcal{M} = \mathcal{E}^{\Box}$,

and it is a *factorization system (fs)* if (1) holds and

(2*) $\mathcal{E} = {}^{\perp}\mathcal{M}$ *and* $\mathcal{M} = \mathcal{E}^{\perp}$

It is well known that, in the presence of (1) , condition (2) may be replaced by

(2a) $\mathcal{E} \Box \mathcal{M}$ (that is: $e \Box m$ for all $e \in \mathcal{E}$ and $m \in \mathcal{M}$), and

(2b) $\mathcal{E}$ *and* $\mathcal{M}$ *are closed under retracts in* $\mathcal{C}^2 (= \mathcal{C}^{\{\cdot \to \cdot\}})$,

and (2b) may be formally weakened even further to

(2b1) *if* $gf \in \mathcal{E}$ *with* g *split mono, then* $f \in \mathcal{E}$, and

(2b2) *if* $gf \in \mathcal{M}$ *with* f *split epi, then* $g \in \mathcal{M}$ (see [AHRT]).

Likewise, in the presence of (1), condition (2*) may be replaced by

(2*a) $\mathcal{E} \perp \mathcal{M}$, and

(2*b) $\mathcal{E}$ *and* $\mathcal{M}$ *are closed under isomorphisms in* $\mathcal{C}^2$.

2.2

Every factorization system is a wfs (see [AHS], [AHRT]), and for every wfs $(\mathcal{E}, \mathcal{M})$ one has $\mathcal{E} \cap \mathcal{M} = \mathrm{Iso}\mathcal{C}$, $\mathcal{E}$ and $\mathcal{M}$ are closed under composition, $\mathcal{E}$ is stable under pushout and closed under coproducts, and $\mathcal{M}$ has the dual properties. For a factorization system $(\mathcal{E}, \mathcal{M})$, the class $\mathcal{E}$ is actually closed under every type of colimit and satisfies the cancellation property

(3) *if* $gf \in \mathcal{E}$ *and* $f \in \mathcal{E}$, *then* $g \in \mathcal{E}$.

Using an observation by Ringel [Ri] (see also [T, Lemma 7.1]) we show that each of these additional properties characterizes a wfs as an fs.

2.3 Proposition.

Let $(\mathcal{E}, \mathcal{M})$ *be a wfs of a category* $\mathcal{C}$ *with cokernelpairs of morphisms in* $\mathcal{E}$. *Then the following conditions are equivalent:*

(i) $(\mathcal{E}, \mathcal{M})$ *is a factorization system;*

(ii) $\mathcal{E}$ *is closed under any type of colimit (in the morphism category of* $\mathcal{C}$*);*

(iii) *for every* $e : A \to B$ *in* $\mathcal{E}$ *the canonical morphism* $e' : B +_A B \to B$ *lies also in* $\mathcal{E}$ *(where* $B +_A B$ *is the codomain of the cokernelpair of* e*);*

(iv) $\mathcal{E}$ *satisfies condition* (3)*;*

(v) *if* $gf = 1$ *with* $f \in \mathcal{E}$*, then* $g \in \mathcal{E}$*.*

Proof. $(i) \implies (ii)$ and $(i) \implies (iv)$ are well known (see 2.2), and $(iv) \implies (v)$ is trivial. For $(ii) \implies (iii)$ consider the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "1"] & B +_A B \arrow[d, "e'"] \\ B \arrow[r, "1"] & B \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & B \end{tikzcd}$$

where both rows represent cokernelpairs. Since the connecting vertical arrows e and 1 lie in $\mathcal{E}$, e' lies also in $\mathcal{E}$, by hypothesis. For $(v) \implies (iii)$ observe that, since $\mathcal{E}$ is stable under pushout, one has $e'p_1 = 1$ with $p_1 \in \mathcal{E}$, so that $e' \in \mathcal{E}$ follows. Finally, for $(iii) \implies (i)$, consider the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "e"'] & C \arrow[d, "m"] \\ B \arrow[r, "v"'] \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] & D \end{tikzcd}$$

with $e \in \mathcal{E}$, $m \in \mathcal{M}$, $se = te = u$ and $ms = mt = v$. The morphism $r : B +_A B \to C$ with $rp_1 = s$ and $rp_2 = t$ makes

$$\begin{tikzcd} B +_A B \arrow[r, "r"] \arrow[d, "e'"'] & C \arrow[d, "m"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

commute. Hence, by hypothesis, one obtains $w : B \to C$ with $we' = r$, and

$$s = rp_1 = we'p_1 = w = we'p_2 = rp_2 = t$$

follows, as desired. □

Dualizing (part of) the Theorem we obtain:

2.4 Corollary

In a category with kernelpairs, $(\mathcal{E}, \mathcal{M})$ *is an fs if, and only if, it is a wfs and satisfies the condition*:

(v^{op}) *if* $gf = 1$ *with* $g \in \mathcal{M}$*, then* $f \in \mathcal{M}$*.*

2.5

If (Epi, Mono) in Set is the prototype of fs, then (Mono, Epi) in Set is the prototype of wfs. But the latter claim actually disguises a simple general fact which does not seem to have been stated clearly in the literature yet. In conjunction with two very special features of Set, namely that 1. every monomorphism is a coproduct injection and 2. every epimorphism splits (=Axiom of Choice), the following Proposition and Theorem give, *inter alia*, the (Mono, Epi) system:

2.6 Proposition.
In a category with binary coproducts, $({}^{\square}\mathrm{SplitEpi}, \mathrm{SplitEpi})$ *is a wfs, and a morphism* $f : A \to B$ *lies in* ${}^{\square}\mathrm{SplitEpi}$ *if, and only if, there is some* $k : B \to A + B$ *with* $kf = i : A \to A + B$ *the first coproduct injection, and with* $< f, 1_B > k = 1_B$*; in particular, every coproduct injection lies in* ${}^{\square}\mathrm{SplitEpi}$.

Proof. Every morphism $f : A \to B$ factors as $pi = f$, and the co-graph $p :=< f, 1_B >: A + B \to B$ is a split epimorphism; moreover, split epimorphisms satisfy condition (2b2) trivially. It now suffices to prove the given characterization of morphisms in ${}^{\square}\mathrm{SplitEpi}$, since it shows in particular that coproduct injections are in ${}^{\square}\mathrm{SplitEpi}$ (simply take k to be a coproduct injection), and since ${}^{\square}\mathrm{SplitEpi}$ (like any class ${}^{\square}\mathcal{M}$) satisfies (2b1). Given $f \in {}^{\square}\mathrm{SplitEpi}$ one obtains k from $f \square p$:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & A+B \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed, "k"] & B \end{tikzcd}$$

Conversely, having k with $kf = i$ and $pk = 1_B$, consider the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, shift left, "r"] \\ B \arrow[r, "v"] & Y \arrow[u, shift left, "t"] \end{tikzcd}$$

with $ru = vf$ and $rt = 1_Y$. Then $s :=< u, tv >: A + B \to X$ satisfies

$$rsi = ru = vf = vpi, \quad rsj = rtv = v = vpj,$$

with j the second coproduct injection, so that $rs = vp$. Hence, $d := sk : B \to X$ satisfies

$$df = skf = si = u, \quad rd = rsk = vpk = v,$$

as desired. □

In many important types of categories, the class ${}^{\square}\mathrm{SplitEpi}$ is remarkably small:

2.7 Theorem
Let $\mathcal{C}$ *be a category with binary coproducts, and* Sum *be the class of all coproduct injections. If* Sum *is stable under pullback in* $\mathcal{C}$*, or if* $\mathcal{C}$ *is pointed and* Sum *contains all split monomorphisms, then* (Sum, SplitEpi) *is a wfs in* $\mathcal{C}$*. The hypotheses on* $\mathcal{C}$ *are particularly satisfied when* $\mathcal{C}$ *is extensive (in the sense of* [CLW]*) or just Boolean (in the sense of* [M]*), or when* $\mathcal{C}$ *is an additive category with finite coproducts.*

Proof. It suffices to prove that $f : A \to B$ in ${}^{\square}$SplitEpi is a coproduct injection. With the (split) monomorphism k as in 1.6, consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "1_B"] \arrow[d, "1_B"] & B \arrow[d, "k"] \\ A \arrow[r, "f"'] & B \arrow[r, "k"'] & A+B \end{tikzcd}$$

which is composed of two trivial pullback diagrams. By hypothesis, since kf is a coproduct injection, its pullback f is also one.

If $\mathcal{C}$ is pointed, the morphism $f : A \to B$ in ${}^{\square}$SplitEpi is a split monomorphism (since $< 1_A, 0 > kf =< 1_A, 0 > i = 1_A$), and as such it is a coproduct injection, by hypothesis. $\square$

For the sake of completeness we mention another well-known general reason for (Mono, Epi) being a wfs in Set:

2.8 Proposition

[AHRT] *In every category with binary products and enough injectives,* (Mono, Mono$^{\square}$) *is a wfs.*

$\square$

2.9

In an extensive (or just Boolean)category, one has Sum $\subseteq$ Mono, hence Mono$^{\square}$ $\subseteq$ Sum$^{\square}$ = SplitEpi. But in the presence of enough injectives, Mono$^{\square}$ = SplitEpi only if Sum = Mono, a condition that rarely holds even in a presheaf category: $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ satisfies Sum = Mono if, and only if, $\mathcal{C}$ is an equivalence relation. For $\mathcal{C} = \{ \cdot \rightrightarrows \cdot \}$, so that $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the category of (directed multi-)graphs, with the Axiom of Choice granted, Mono$^{\square}$ contais precisely the full morphisms that are surjective on vertices; here a morphism $f : G \to H$ of graphs is *full* if every edge $f(a) \to f(b)$ in H is the f-image of an edge $a \to b$ in G.

2.10

For a wfs $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ with terminal object 1, the full subcategory

$$\mathcal{F}(\mathcal{M}) := \{B \in \mathrm{ob}\mathcal{C} \mid (B \to 1) \in \mathcal{M}\}$$

is weakly reflective in $\mathcal{C}$, in fact weakly $\mathcal{E}-$reflective, with a weak reflection $\rho_A \in \mathcal{E}$ of an object A being obtained by an $(\mathcal{E}, \mathcal{M})$-factorization of $A \to 1$:

$$A \xrightarrow{\rho_A} RA \xrightarrow{\mathcal{M}} 1.$$

If $(\mathcal{E}, \mathcal{M})$ is an fs, $\mathcal{F}(\mathcal{M})$ is $\mathcal{E}$–reflective in $\mathcal{C}$.

2.11 Remark

Weak factorization systems are abundant in homotopy theory. In fact, a *Quillen model category* $\mathcal{C}$ is defined as a complete and cocomplete category together with three classes of morphisms $\mathcal{E}$ (*cofibrations*), $\mathcal{M}$ (*fibrations*) and $\mathcal{W}$ (*weak equivalences*) such that $\mathcal{W}$ has the 3–for–2 property, is closed under retracts in $\mathcal{C}^2$ and $(\mathcal{E}, \mathcal{M}_0)$, $(\mathcal{E}_0, \mathcal{M})$ are weak factorization systems where

$$\mathcal{M}_0 = \mathcal{M} \cap \mathcal{W}, \quad \mathcal{E}_0 = \mathcal{E} \cap \mathcal{W}$$

denote the classes of trivial fibrations and cofibrations, respectively. The *3–for–2 property* means that whenever two of the morphisms gf, f and g lie in $\mathcal{W}$, the third one lies also in $\mathcal{W}$.

Objects of the weakly reflective subcategory $\mathcal{F}(\mathcal{M})$ are called *fibrant*. Dually, when $\mathcal{C}$ has an initial object 0, there is a weakly coreflective subcategory

$$\mathcal{T}(\mathcal{E}) = \{A \in \mathrm{ob}\mathcal{C} \mid (0 \to A) \in \mathcal{E}\}$$

of *cofibrant* objects.

3. Reflective factorization systems and prefibrations

3.1

For a factorization system $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ with terminal object 1, the $\mathcal{E}$–reflective full subcategory $\mathcal{F}(\mathcal{M})$ of 2.10 is even *firmly $\mathcal{E}$-reflective*, in the sense that any morphism $A \to B$ in $\mathcal{E}$ with $\mathcal{B} \in \mathcal{F}(\mathcal{M})$ serves as a reflection of the object A into $\mathcal{F}(\mathcal{M})$. Such reflective subcategories are easily characterized:

3.2 Proposition

For a factorization system $(\mathcal{E}, \mathcal{M})$ and an $\mathcal{E}$-reflective subcategory $\mathcal{F}$ of $\mathcal{C}$, the following conditions are equivalent:

(i) $\mathcal{F} = \mathcal{F}(\mathcal{M})$,

(ii) *$\mathcal{F}$ is firmly $\mathcal{E}$-reflective in $\mathcal{C}$,*

(iii) $\mathcal{E} \subseteq R^{-1}(\mathrm{Iso}\mathcal{C})$.

If these conditions hold, one has $\mathcal{E} = R^{-1}(\mathrm{Iso}\mathcal{C})$ if, and only if, $\mathcal{E}$ satisfies (in addition to (3) *of* 2.2 *) the cancellation property*

(4) *if $gf \in \mathcal{E}$ and $g \in \mathcal{E}$, then $f \in \mathcal{E}$.*

Proof. (i) $\Longrightarrow$ (ii): see 3.1. (ii) $\Longrightarrow$ (iii): Considering the ρ-naturality diagram for $e : A \to B$ in $\mathcal{E}$,

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "\rho_A"'] & B \arrow[d, "\rho_B"] \\ RA \arrow[r, "Re"] & RB \end{tikzcd}$$

we see that $\rho_B e$ serves as a reflection for A, by hypothesis, so that Re must be an isomorphism. (iii) $\Longrightarrow$ (i): For $B \in \mathcal{F}$, consider the $(\mathcal{E}, \mathcal{M})$-factorization

$$B \xrightarrow{e} C \xrightarrow{m} 1.$$

Since $1 \in \mathcal{F}$ and $m \in \mathcal{M}$, also $\mathcal{C}$ lies in the $\mathcal{E}$-reflective subcategory $\mathcal{F}$. Hence $e \cong Re$ is an isomorphism, by hypothesis, and $(B \to 1) \in \mathcal{M}$ follows. Conversely, having

$(B \to 1) \in \mathcal{M}$, $1 \in \mathcal{F}$ implies $B \in \mathcal{F}$, as above. $\mathcal{E} = R^{-1}(\mathrm{Iso}\mathcal{C})$ trivially implies (4), and (4) implies $R^{-1}(\mathrm{Iso}\mathcal{C}) \subseteq \mathcal{E}$, by inspection of the ρ-naturality diagram above. □

We adopt the terminology of [CHK] and call an fs $(\mathcal{E}, \mathcal{M})$ in any category $\mathcal{C}$ *reflective* if (4) holds. Since $\mathcal{E}$ is always closed under composition and satisfies (3) of Section 2, we see that an fs $(\mathcal{E}, \mathcal{M})$ is a reflective fs if, and only if, $\mathcal{E}$ satisfies the 3–for–2 property, granted the existence of cokernelpairs in $\mathcal{C}$ (see 1.3).

A reflective fs $(\mathcal{E}_0, \mathcal{M})$ makes $\mathcal{C}$ a Quillen model category, with $\mathcal{W} = \mathcal{E}_0$, $\mathcal{E} = \mathcal{C}$ and $\mathcal{M}_0 = \mathrm{Iso}\mathcal{C}$. The corresponding homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ is $\mathcal{F}$.

3.3

A reflective fs $(\mathcal{E}, \mathcal{M})$ in a category with terminal object depends only on the reflective subcategory $\mathcal{F}(\mathcal{M})$, since $\mathcal{E} = R^{-1}(\mathrm{Iso}\mathcal{C})$ and $\mathcal{M} = \mathcal{E}^{\perp}$. Conversely, given any reflective subcategory $\mathcal{F}$ of $\mathcal{C}$ with reflector R and reflection morphism $\rho : 1 \to R$, one may ask when is $\mathcal{E} := R^{-1}(\mathrm{Iso}\mathcal{C})$ part of (a necessarily reflective) fs. This question is discussed in general in [CHK], [JT1]. Here we are primarily interested in the case when, moreover, $\mathcal{E}$'s factorization partner $\mathcal{M} = \mathcal{E}^{\perp}$ can be presented as

$$\mathcal{M} = \mathrm{Cart}(R, \rho),$$

where $\mathrm{Cart}(R, \rho)$ is the class of ρ-*cartesian morphisms*, i.e., of those morphisms whose ρ-naturality diagram is a pullback.

3.4 Proposition

For a reflective subcategory $\mathcal{F}$ of the finitely complete category $\mathcal{C}$ with reflection $\rho : 1 \to R$, $(\mathcal{E}, \mathcal{M}) = \big(R^{-1}(\mathrm{Iso}\mathcal{C}), \mathrm{Cart}(R, \rho)\big)$ is a factorization system of $\mathcal{C}$ if, and only if, for every morphism $f : A \to B$, the induced morphism $e = (f, \rho_A) : A \to B \times_{RB} RA$ into the pullback of Rf along ρ_B lies in $\mathcal{E}$. In this case, $\mathcal{F} = \mathcal{F}(\mathcal{M})$.

Proof. See Theorem 4.1 of [CHK] or Theorem 2.7 of [JT1]. □

Adopting again the terminology used in [CHK], we call a reflective factorization system $(\mathcal{E}, \mathcal{M})$ *simple* if $\mathcal{M} = \mathrm{Cart}(R, \rho)$, that is: if the reflective subcategory $\mathcal{F} = \mathcal{F}(\mathcal{M})$ satisfies the equivalent conditions of Proposition 3.4. We also make use of Theorem 4.3 of [CHK]:

3.5 Proposition

For a reflective fs $(\mathcal{E}, \mathcal{M})$ of a finitely complete category $\mathcal{C}$, in the notation of 3.3 *the following conditions are equivalent, and they imply simplicity of $(\mathcal{E}, \mathcal{M})$:*

(i) $\mathcal{E}$ is stable under pullback along morphisms in $\mathcal{M}$;

(ii) R preserves pullbacks of morphisms in $\mathcal{M}$ along any other morphisms;

(iii) the pullback of a reflection $\rho_A : A \to RA$ along a morphism in $\mathcal{F}$ is a reflection morphism.

Reflective factorization systems $(\mathcal{E}, \mathcal{M})$ satisfying these equivalent conditions are called *semi–left exact*. The reflective subcategory $\mathcal{F}$ is a *semilocalization* of $\mathcal{C}$ if property (iii) holds; equivalently, if the associated reflective fs is semi–left exact. A reflective fs need not be simple, and a simple fs need not be semi–left exact(see [CHK]).

3.6

A Quillen model category $\mathcal{C}$ is called *right proper* if every pullback of a weak equivalence along a fibration is a weak equivalence (see [H]). Since each weak equivalence w has a factorization $w = w_2w_1$ where w_1 is a trivial cofibration and w_2 a trivial fibration, and since (trivial) fibrations are stable under pullback, $\mathcal{C}$ *is right proper if, and only if, trivial cofibrations are stable under pullback along fibrations, that is: if the wfs* $(\mathcal{E}_0, \mathcal{M})$ *of* 2.10 *has property* 3.5(i). Hence, a semi–left exact reflective fs $(\mathcal{E}_0, \mathcal{M})$ makes $\mathcal{C}$ a right proper Quillen model category with $(\mathcal{E}, \mathcal{M}_0) = (\mathcal{C}, \mathrm{Iso}\mathcal{C})$ and $\mathcal{W} = \mathcal{E}_0$.

3.7

Simple and semi-left exact reflective factorization systems occur most naturally in the context of fibrations. Hence, recall that a functor $P : \mathcal{C} \to \mathcal{B}$ is a *(quasi-)fibration* if the induced functors

$$P_C : \mathcal{C}/C \to \mathcal{B}/PC$$

have full and faithful right adjoints, for all $\mathcal{C} \in \mathrm{ob}\mathcal{C}$. Let us call P a *prefibration* if, for all C, there is an adjunction

$$P_C \underset{\varepsilon}{\overset{\eta}{\dashv}} I_C \quad .$$

whose induced monad is idempotent. (Janelidze's notion of admissible reflective subcategory $\mathcal{B}$ of $\mathcal{C}$ asks the right adjoints I_C to be full and faithful, so that each P_C is a fibration, in particular a prefibration; see [J], [CJKP].) With the notation

$$I_C : (g : B \to PC) \mapsto (v_g : g^*C \to C)$$

we can state right adjointness of P_C more explicitly, as follows: for every morphism $g : B \to PC$ in $\mathcal{B}$ one has a commutative diagram

$$\begin{tikzcd} P(g^*C) \arrow[r, "Pv_g"] \arrow[d, "\varepsilon_g"'] & PC \arrow[d, "1"] \\ B \arrow[r, "g"] & PC \end{tikzcd}$$

in $\mathcal{B}$, and whenever

$$\begin{tikzcd} PA \arrow[r, "Pf"] \arrow[d, "u"'] & PC \arrow[d, "1"] \\ B \arrow[r, "g"] & PC \end{tikzcd}$$

commutes in $\mathcal{B}$ (with $f : A \to C$ in $\mathcal{C}$), then there is a unique morphism $t : A \to g^*C$ in $\mathcal{C}$ with $v_gt = f$ and $\varepsilon_g \cdot Pt = u$.

If $u = 1$, then $t = \eta_f$, and we obtain the factorization

$$\begin{tikzcd} & (Pf)^*C \arrow[dr, "v_{Pf}"] & \\ A \arrow[ur, "\eta_f"] \arrow[rr, "f"] & & C \end{tikzcd}$$

and the idempotency condition amounts to the requirement that $P\eta_f = \varepsilon_{Pf}^{-1}$ be an isomorphism. One then has $v_{Pf} \in \mathrm{Cart}P$, with

$$\mathrm{Cart}P = \{f \mid \eta_f \text{ iso}\}.$$

(As we will see shortly, there is no clash with the notation used in 3.3.) In fact, $\big(P^{-1}(\mathrm{Iso}\mathcal{B}), \mathrm{Cart}P\big)$ is a factorization system of $\mathcal{C}$, and it is trivially reflective.

Let us now *assume that P preserves the terminal object* 1 *of* $\mathcal{C}$. Then

$$\mathcal{F}(\mathrm{Cart}P) = \{A \mid A \to 1 \ P\text{–cartesian}\}$$

contains precisely the P*–indiscrete* objects of $\mathcal{C}$, e.g. those $A \in \mathrm{ob}\mathcal{C}$ for which every $h : PD \to PA$ in $\mathcal{B}$ (with $D \in \mathrm{ob}\mathcal{C}$) can be written uniquely as $h = Pd$, with $d : D \to A$ in $\mathcal{C}$. If we denote the adjunction

$$\mathcal{C} \simeq \mathcal{C}/1 \underset{I_1}{\overset{P_1}{\rightleftarrows}} \mathcal{B}/P1 \simeq \mathcal{B}$$

simply by $P \underset{\varepsilon}{\overset{\eta}{\dashv}} I : \mathcal{B} \to \mathcal{C}$, then

$$\mathcal{F}(\mathrm{Cart}P) = \{A \mid \eta_A \ \text{ iso}\}$$

is the reflective subcategory of $\mathcal{C}$ fixed by the adjunction $P \dashv I$. Hence its reflector R (as an endofunctor of $\mathcal{C}$) is IP, with reflection morphism η.

A routine exercise shows

$$P^{-1}(\mathrm{Iso}B) = R^{-1}(\mathrm{Iso}\mathcal{C}), \mathrm{Cart}P = \mathrm{Cart}(R, \eta).$$

In particular, *the fs* $\big(P^{-1}(\mathrm{Iso}\mathcal{B}), \mathrm{Cart}P\big)$ *given by a prefibration P with* $P1 \cong 1$ *is simple.* An easy calculation shows also that $P^{-1}(\mathrm{Iso}\mathcal{B})$ is stable under pullback along morphisms in $\mathrm{Cart}P$ when P preserves such pullbacks. Consequently, *for $\mathcal{C}$ finitely complete and with the prefibration P preserving pullbacks of* $\mathrm{Cart}P$*–morphisms and the terminal object, the fs is actually semi-left exact.*

3.8

Conversely to 3.7, let us show that any simple reflective fs $(\mathcal{E}, \mathcal{M})$ of a finitely complete category $\mathcal{C}$ is induced by a prefibration P with $P1 \cong 1$. More precisely, we show that the restriction $\mathcal{C} \to \mathcal{F}(\mathcal{M})$ of the reflector R (notation as in 3.3) is a prefibration. To this end, for $g : B \to RC$ with $B \in \mathcal{F}(\mathcal{M})$ we form the (outer) pullback diagram

$$\begin{tikzcd}
B \times_{RC} C \arrow[rr, "v_g"] \arrow[dd, "p"'] \arrow[dr, "\rho_{B \times_{RC} C}"] & & C \arrow[d, "\rho_C"] \\
& R(B \times_{RC} C) \arrow[r, "Rv_g"'] \arrow[dl, "\varepsilon_g"] & RC \arrow[d, "1"] \\
B \arrow[rr, "g"'] & & RC
\end{tikzcd}$$

The pullback projection p factors through $R(B \times_{RC} C)$ by a unique morphism ε_g since $B \in \mathcal{F}(\mathcal{M})$. To verify the required universal property, consider $f : A \to C$ and $u : RA \to B$ with $Rf = gu$. Since

$$gu\rho_A = Rf \cdot \rho_A = \rho_C f,$$

there is a unique morphism $t : A \to B \times_{RC} C$ with $pt = u\rho_A$, $v_g t = f$. From

$$\varepsilon_g \cdot Rt \cdot \rho_A = \varepsilon \rho_{B \times_{RC} C} t = pt = u\rho_A$$

one obtains $\varepsilon_g \cdot Rt = u$, as required. Since, conversely, $\varepsilon_g \cdot Rt = u$ implies $pt = u\rho_A$, we have shown right adjointness of R_C. Furthermore, when $u = 1$, the pullback diagram above can simply be taken to be the ρ–naturality diagram of f, by simplicity of $(\mathcal{E}, \mathcal{M})$. Hence, $A \cong B \times_{RC} C$ and $p \cong \rho_A$, so that ε_{Pf} is an isomorphism, and this shows the required idempotency. Consequently, the reflector of $\mathcal{F}(\mathcal{M})$ is a prefibration, and since $\mathcal{E} = R^{-1}(\mathrm{Iso}\mathcal{C})$, the induced factorization system must be the given fs $(\mathcal{E}, \mathcal{M})$. By 3.5, the system is semi-left exact precisely when the reflector preserves pullbacks of morphisms in $\mathcal{M}$. Hence, with 3.7 we proved here:

3.9 Theorem

In a finitely complete category $\mathcal{C}$, $(\mathcal{E}, \mathcal{M})$ is a simple reflective factorization system of $\mathcal{C}$ if, and only if, there exists a prefibration $P : \mathcal{C} \to \mathcal{B}$ preserving the terminal object with

$$\mathcal{E} = P^{-1}(\mathrm{Iso}B), \ \mathcal{M} = \mathrm{Cart}P.$$

$(\mathcal{E}, \mathcal{M})$ is semi–left exact precisely when P can be chosen to preserve every pullback along a P–cartesian morphism.

□

4. Torsion Theories

4.1

Let $(\mathcal{E}, \mathcal{M})$ be a reflective fs in a category $\mathcal{C}$ with zero object $0 = 1$. (There is no further assumption on $\mathcal{C}$ until 4.6.) Then we have not only the $\mathcal{E}$-reflective subcategory $\mathcal{F} = \mathcal{F}(\mathcal{M})$ with reflection $\rho : 1 \to R$, but also the $\mathcal{M}$-coreflective subcategory $\mathcal{T} = \mathcal{T}(\mathcal{E})$ (see 2.11), whose coreflections $\sigma_B : S_B \to B$ are obtained by $(\mathcal{E}, \mathcal{M})$-factoring $0 \to B$, for all B in $\mathcal{C}$. Let us first clarify how $\mathcal{T}$ and $\mathcal{F}$ are related.

4.2 Proposition

In the setting of 4.1, *the following assertions are equivalent for an object A in $\mathcal{C}$:*

(i) $A \in \mathcal{T}$;

(ii) $\mathcal{C}(A, B) = \{0\}$, *for all* $B \in \mathcal{F}$;

(iii) $RA \cong 0$.

Proof. (i) $\Longrightarrow$ (ii) follows from $(0 \to A)\perp(B \to 0)$. (ii) $\Longrightarrow$ (iii): Since $RA \in \mathcal{F}$, one has $\rho_A = 0$ and obtains $1_{RA} = 0$ from $\rho_A \perp (RA \to 0)$. (iii) $\Longrightarrow$ (i): Since $RA \cong 0$, one has $(A \to 0) \in \mathcal{E}$, and this implies $(0 \to A) \in \mathcal{E}$ by (4) of 3.2, hence $A \in \mathcal{T}$. $\square$

Dualizing Propositions 3.2 and 4.2 we obtain:

4.3 Corollary

In the setting of 4.1, $\mathcal{M} = S^{-1}(\mathrm{Iso}\mathcal{C})$ *if, and only if,* $\mathcal{M}$ *satisfies the cancellation property*:

(4^{op}) *if* $gf \in \mathcal{M}$ *and* $f \in \mathcal{M}$, *then* $g \in \mathcal{M}$.

In this case,

$$\mathcal{F} = \{B \in \mathrm{ob}\mathcal{C} \mid SB \cong 0\} = \{B \mid \mathcal{C}(A, B) = \{0\} \text{ for all } A \in \mathcal{T}\}.$$

Factorization systems $(\mathcal{E}, \mathcal{M})$ satisfying (4^{op}) are called *coreflective.*

4.4 Definitions and Summary

A *torsion theory* in a category $\mathcal{C}$ is a reflective and coreflective factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathcal{C}$, i.e., a fs of $\mathcal{C}$ in which both classes satisfy the 3–for–2 property. If $\mathcal{C}$ has kernelpairs or cokernelpairs, it actually suffices to assume that $(\mathcal{E}, \mathcal{M})$ be a wfs in this definition (see 2.7, 2.8). If $\mathcal{C}$ has a zero object, then $\mathcal{T} = \mathcal{T}(\mathcal{E})$ is the *torsion subcategory* and $\mathcal{F} = \mathcal{F}(\mathcal{M})$ the *torsion-free subcategory* associated with the theory. For an object C, the coreflection σ_C into $\mathcal{T}$ and the reflection ρ_C into $\mathcal{F}$ are obtained by $(\mathcal{E}, \mathcal{M})$-factoring $0 \to C$ and $C \to 0$, respectively as in

$$0 \longrightarrow SC \xrightarrow{\sigma_C} C \xrightarrow{\rho_C} RC \longrightarrow 0.$$

R and S determine all $\mathcal{E}, \mathcal{M}, \mathcal{T}, \mathcal{F}$, via

$$\begin{aligned} \mathcal{E} = R^{-1}(\mathrm{Iso}\mathcal{C}) = {}^{\perp}\mathcal{M}, \quad & \mathcal{M} = S^{-1}(\mathrm{Iso}\mathcal{C}) = \mathcal{E}^{\perp}, \\ \mathcal{T} = R^{-1}(\{0\}) = \mathcal{F}^{\leftarrow}, \quad & \mathcal{F} = S^{-1}(\{0\}) = \mathcal{T}^{\rightarrow}, \end{aligned}$$

with $\mathcal{F}^{\leftarrow} := \big\{A \mid \forall B \in \mathcal{F}\big(\mathcal{C}(A, B) = \{0\}\big)\big\}, \mathcal{T}^{\rightarrow} := \big\{B \mid \forall A \in \mathcal{T}\big(\mathcal{C}(A, B) = \{0\}\big)\big\}$. Furthermore, if $\mathcal{C}$ has pullbacks and $\mathcal{E}$ is stable under pullback along morphisms in $\mathcal{M}$, i.e., if the torsion theory is *semi-left-exact* and, hence, *simple*, then an $(\mathcal{E}, \mathcal{M})$-factorization of $f : A \to B$ can be presented as

$$\begin{tikzcd} & RA \times_{RB} B \arrow[dr, "\pi_2"] & \\ A \arrow[ur, "(\rho_A, f)"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

where π_2 is the pullback of Rf along ρ_B. In this case, $\mathcal{M} = \mathrm{Cart}(R, \rho)$. We note that without the hypothesis of semi-left-exactness or simplicity, one still has:

$$f \in \mathcal{E} \iff \pi_2 \text{ iso}, \; f \in \mathcal{M} \iff (\rho_A, f) \in \mathcal{M}.$$

The condition dual to semi-left-exactness is called *semi-right-exactness*, and it yields $\mathcal{E} = \mathrm{Cocart}(S, \sigma)$, along with an alternative presentation of the $(\mathcal{E}, \mathcal{M})$-factorization of f:

$$\begin{tikzcd}
& A +_{SA} SB \arrow[dr, "{(f,\sigma_B)}"] & \\
A \arrow[ur, "\kappa_1"] \arrow[rr, "f"'] & & B
\end{tikzcd}$$

where κ_1 is the pushout of Sf along σ_A.

In a category $\mathcal{C}$ with zero object, let 0Ker be the class of morphisms whose kernel is 0, and 0Coker the class of morphisms with zero cokernel. Note that Mono $\subseteq$ 0Ker and Epi $\subseteq$ 0Coker.

4.5 Proposition

In a category $\mathcal{C}$ with 0*, any pair of full subcategories $\mathcal{T} = \mathcal{F}^{\leftarrow}$ and $\mathcal{F} = \mathcal{T}^{\rightarrow}$ satisfies the following properties, for any morphisms $k : A \to B, p : B \to C$ in $\mathcal{C}$.*

(1) *for $k \in \mathrm{0Ker}, B \in \mathcal{F}$ implies $A \in \mathcal{F}$;*

(2) *for $p \in \mathrm{0Coker}, B \in \mathcal{T}$ implies $C \in \mathcal{T}$;*

(3) *for k the kernel of p, $A, C \in \mathcal{F}$ imply $B \in \mathcal{F}$;*

(4) *for p the cokernel of k, $A, C \in \mathcal{T}$ imply $B \in \mathcal{T}$.*

Proof. (3) implies (1), and (2), (4) are dual to (1), (3), respectively. Hence, if suffices to prove (3): any morphism $f : T \to B$ with $T \in \mathcal{T}$ satisfies $pf = 0$. Hence, it factors through k, by a morphism $T \to A$, which must be 0, so that also $f = 0$. □

4.6

We call a full subcategory $\mathcal{F}$ *closed under left-extensions* in $\mathcal{C}$ if it satisfies (3) of 4.5. If $\mathcal{C}$ has (NormEpi, 0Ker)-factorizations, with NormEpi the class of normal epimorphisms (i.e. of morphisms that appear as cokernels), and if $\mathcal{F}$ satisfies property (1) of 4.5, then the morphism p in (3) may be taken to be the cokernel of k, so that closure under left-extensions amounts to the selfdual property of being *closed under extensions.* Note that $\mathcal{C}$ has (NormEpi, 0Ker)-factorization if $\mathcal{C}$ has kernels and cokernels (of kernels), and if pullbacks of normal epimorphisms along normal monomorphisms have cokernel 0 (see Prop. 2.1 of [CDT]). From 4.5 (1), (2) one obtains:

4.7 Corollary

The reflection morphisms of the torsion-free subcategory of a torsion theory in a pointed category with (NormEpi, 0Ker)*-factorization are normal epimorphisms. Dually, if there are* (0Coker, NormMono)*–facto-rizations, then the coreflection morphisms of the torsion subcategory are normal monomorphisms.*

4.8

In a pointed category with kernels and cokernels, let $(\mathcal{E},\mathcal{M})$ be a torsion theory. With the notation of 4.4, let $\kappa_C = \mathrm{ker}\rho_C$ and $\pi_C = \mathrm{coker}\sigma_C$. If, as in 4.7, ρ_C is a normal epimorphism and σ_C a normal monomorphism, so that $\rho_C = \mathrm{coker}\kappa_C$ and $\sigma_C = \mathrm{ker}\pi_C$, we obtain induced morphisms α_C and β_C that, in the next diagram, make squares 1, 2, 3 pullbacks and squares 2, 3, 4 pushouts:

$$\begin{tikzcd}
SC \arrow[r, "1"] \arrow[d, "\alpha_C"'] \arrow[dr, phantom, "\boxed{1}"] & SC \arrow[r] \arrow[d, "\sigma_C"] \arrow[dr, phantom, "\boxed{2}"] & 0 \arrow[d] \\
KC \arrow[r, "\kappa_C"] \arrow[d] \arrow[dr, phantom, "\boxed{3}"] & C \arrow[r, "\pi_C"] \arrow[d, "\rho_C"] \arrow[dr, phantom, "\boxed{4}"] & QC \arrow[d, "\beta_C"] \\
0 \arrow[r] & RC \arrow[r, "1"'] & RC
\end{tikzcd}$$

Since $\rho_C \in \mathcal{E}$, also $\beta_C \in \mathcal{E}$ (since $\mathcal{E}$ is pushout stable), whence $\pi_C \in \mathcal{E}$ (by the 3-for-2 property) and $R\pi_C$ iso. But since ρ_{RC} is iso, this means that β_C may be replaced by ρ_{QC}. Likewise, replacing α_C by σ_{QC}, we can redraw the above diagram as:

$$\begin{tikzcd}
SKC \arrow[r, "S\kappa_C", "\sim"'] \arrow[d, "\sigma_{KC}"'] \arrow[dr, phantom, "\boxed{1}"] & SC \arrow[r] \arrow[d, "\sigma_C"] \arrow[dr, phantom, "\boxed{2}"] & 0 \arrow[d] \\
KC \arrow[r, "\kappa_C"] \arrow[d] \arrow[dr, phantom, "\boxed{3}"] & C \arrow[r, "\pi_C"] \arrow[d, "\rho_C"] \arrow[dr, phantom, "\boxed{4}"] & QC \arrow[d, "\rho_{QC}"] \\
0 \arrow[r] & RC \arrow[r, "R\pi_C"'] & RQC
\end{tikzcd}$$

The endofunctors K and Q behave just like S and R when we want to describe the subcategories $\mathcal{T}$ and $\mathcal{F}$:

4.9 Proposition

Under the hypothesis of 4.8, *for every object* C *one has the following equivalences*:

$$C \in \mathcal{F}(\mathcal{M}) \iff KC \in \mathcal{F}(\mathcal{M}) \iff KC = 0,$$
$$C \in \mathcal{T}(\mathcal{E}) \iff QC \in \mathcal{T}(\mathcal{E}) \iff QC = 0.$$

Proof. Since $\kappa_C = \mathrm{ker}\rho_C$ and $RC \in \mathcal{F}$, one has $(C \in \mathcal{F} \iff KC \in \mathcal{F})$ by Prop. 4.5. Furthermore, $(C \in \mathcal{F} \iff \rho_C$ iso $\iff \kappa_C = 0 \iff KC = 0)$. The rest follows dually. □

The normal monomorphism $\alpha_C \cong \sigma_{KC}$ and the normal epimorphism $\beta_C \cong \rho_{QC}$ measure the "distance" from κ_C to the coreflection σ_C and from π_C to the reflection π_C, respectively. The following Theorem indicates when that "distance" is zero:

4.10 Theorem

Under the hypothesis of 4.8, *the following conditions are equivalent for every object C*:

(i) $\pi_C \cdot \kappa_C = 0$;

(ii) $\ker \rho_{QC} = 0$;

(iii) π_{QC} *is an isomorphism*;

(iv) $QC \in \mathcal{F}(\mathcal{M})$;

(v) $\mathrm{coker}\sigma_{KC} = 0$;

(vi) κ_{KC} *is an isomorphism*;

(vii) $KC \in \mathcal{T}(\mathcal{E})$;

(viii) $(0 \to QC) \in \mathcal{M}$;

(ix) $(KC \to 0) \in \mathcal{E}$.

All conditions are satisfied when $(\mathcal{E}, \mathcal{M})$ *is simple* (see 3.4).

Proof. Since $\rho_{QC} \cdot \pi_C \cdot \kappa_C = 0$, (i) $\iff$ (ii) is obvious. (iv) implies ρ_{QC} iso, hence (ii), and also (iii), since

$$\rho_{QQC} \cdot \pi_{QC} = R\pi_{QC} \cdot \rho_{QC},$$

with $R\pi_{QC}$ iso. Conversely, (ii) $\implies$ (iv) holds since ρ_{QC} is a normal epimorphism, and (iii) $\implies$ (iv) holds since $\pi_{QC} = \mathrm{coker}\sigma_{QC}$ iso means $SQC = 0$, hence $QC \in \mathcal{F}(\mathcal{M})$. Consequently, we have (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv), and (i) $\iff$ (v) $\iff$ (vi) $\iff$ (vii) follows dually. Since

$$KC \xrightarrow{\rho_{KC}} RKC \longrightarrow 0$$

is the $(\mathcal{E}, \mathcal{M})$-factorization system of $KC \to 0$, one has $RKC \to 0$ iso, if, and only if, $KC \to 0$ lies in $\mathcal{E}$. This shows (vii) $\iff$ (ix), and (iv) $\iff$ (viii) follows dually. Finally, assume $(\mathcal{E}, \mathcal{M})$ to be simple and consider the commutative diagram

$$\begin{tikzcd} KC \arrow[r, "1"] \arrow[d, "\rho_{KC}"'] & KC \arrow[r, "\kappa_C"] \arrow[d] \arrow[dr, phantom, "\boxed{3}"] & C \arrow[d, "\rho_C"] \\ RKC \arrow[r] & 0 \arrow[r] & RC \end{tikzcd}$$

Since $0 = \rho_C \kappa_C = R\kappa_C \cdot \rho_{KC}$ with ρ_{KC} epi, the buttom row is $R\kappa_C$, and diagram 1 of 4.9 shows that κ_C lies in $\mathcal{M}$, since $(\mathcal{E}, \mathcal{M})$ is coreflective. Hence, the whole diagram is a pullback, by simplicity of $(\mathcal{E}, \mathcal{M})$, and therefore also its left square: $KC \cong KC \times RKC$. Now the morphism $t = \langle 0 : RKC \to KC, 1_{RKC} \rangle$ shows that ρ_{KC} must be 0, which means $RKC = 0$ and, hence, $KC \in \mathcal{T}$. □

4.11 Remarks

(1) Following the terminology of [CHK] we call a torsion theory *normal* if the equivalent conditions of 4.10 hold. Hence *every simple torsion theory is normal*, provided that $\mathcal{C}$ satisfies the hypothesis of 3.8. Moreover, square 3 of 4.8 and condition (ix) of 4.10 show that $(\mathcal{E}, \mathcal{M})$ is normal if, and only if, $\mathcal{E}$ satisfies a very particular

pullback-stability condition. No failure of this condition is known since the following open problem of [CHK] remains unsolved: *is there a non-normal torsion theory?*

(2) The advantage of our definition of torsion theory is that we do not need to assume the existence of kernels and cokernels in $\mathcal{C}$. It applies, for example, to a triangulated category $\mathcal{C}$. Such a category has only weak kernels and weak cokernels and our definition precisely corresponds to torsion theories considered there as pairs $\mathcal{F}$ and $\mathcal{T}$ of colocalizing and localizing subcategories (see [HPS]).

It is also easy to express torsion theories in terms of prefibrations, since Theorem 3.9 gives immediately:

4.12 Corollary

In a finitely complete category $\mathcal{C}$, the class $\mathcal{M}$ belongs to a torsion theory $(\mathcal{E},\mathcal{M})$ if, and only if, there is a prefibration $P : \mathcal{C} \to \mathcal{B}$ with $P1 \cong 1$ such that $\mathcal{M} = \mathrm{Cart}P$ has the 3–for–2 property. Dually, in a finitely cocomplete category $\mathcal{C}$, the class $\mathcal{E}$ belongs to a torsion theory $(\mathcal{E},\mathcal{M})$ if, and only if, there is a precofibration $Q : \mathcal{C} \to \mathcal{A}$ with $Q0 \cong 0$ such that $\mathcal{E} = \mathrm{Cocart}Q$ has the 3–for–2 property.

□

5. Characterization of normal torsion theories

5.1

In a finitely complete category $\mathcal{C}$ with a zero object and cokernels (of normal monomorphisms), we wish to compare the notion of normal torsion theory (as presented in 4.4, 4.11) with concepts considered previously, specifically with the more classical notion used in [BG] and [CDT]. Hence here let us refer to a pair $(\mathcal{T},\mathcal{F})$ of full replete subcategories of $\mathcal{C}$ satisfying

1. $\mathcal{C}(A,B) = \{0\}$ for all $A \in \mathcal{T}$ and $B \in \mathcal{F}$,
2. for every object C of $\mathcal{C}$ there exists $A \xrightarrow{k} C \xrightarrow{q} B$ with $A \in \mathcal{T},\ B \in \mathcal{F},\ k = \mathrm{ker}q,\ q = \mathrm{coker}k$.

as a *standard torsion theory* of $\mathcal{C}$; its torsion-free part is necessarily normal-epireflective in $\mathcal{C}$. The main result of [JT2] states that, when normal epimorphisms are stable under pullback in $\mathcal{C}$, *a normal-epireflective subcategory $\mathcal{F}$ is part of a standard torsion theory if, and only if, $\mathcal{F}$ satisfies the following equivalent conditions:*

(i) *$\mathcal{F}$ is a semilocalization of $\mathcal{C}$ (see* 3.5*);*

(ii) *the reflector $\mathcal{C} \to \mathcal{F}$ is a (quasi)fibration (see* 3.7 *);*

(iii) *$\mathcal{F}$ is closed under extensions, and the pushout of the kernel $A \xrightarrow{k} C$ of ρ_C along ρ_A is a normal monomorphism, for every $C \in \mathrm{ob}\mathcal{C}$ (with ρ_C the $\mathcal{F}$-reflection of C).*

Recall that $\mathcal{C}$ is *homological* [BB] if it is regular [Ba] and protomodular [Bo]; here the latter property amounts to: if in the commutative diagram

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "p"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

with regular epimorphism p the left and the whole rectangles are pullbacks, so is the right one. In such categories one has (NormEpi, 0Ker) = (RegEpi, Mono).

We are now ready to prove:

5.2 Theorem

Every standard torsion theory of $\mathcal{C}$ determines a simple reflective factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathcal{C}$ with $\mathcal{F}(\mathcal{M})$ normal-epireflective and $\mathcal{T}(\mathcal{E})$ normal-monocoreflective. When $\mathcal{C}$ is homological, such factorization systems are normal torsion theories. When both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are homological, then normal torsion theories correspond bijectively to standard torsion theories.

Proof. Since a standard torsion theory $(\mathcal{T}, \mathcal{F})$ is given by the semilocalization $\mathcal{F}$, its reflective factorization system $(\mathcal{E}, \mathcal{M})$ is simple (see 3.4, 3.5), and one has $\mathcal{T} = \mathcal{T}(\mathcal{E})$ (see 4.2). This proves the first statement. For the second, let $(\mathcal{E}, \mathcal{M})$ be a simple reflective factorization system such that the reflections of $\mathcal{F}(\mathcal{M})$ are normal epimorphisms and the coreflections of $\mathcal{T}(\mathcal{M})$ are normal monomorphisms. Simplicity means $\mathcal{M} = \mathrm{Cart}(R, \rho)$ by 3.4, and since the reflections ρ_C ($C \in \mathrm{ob}\mathcal{C}$) are regular epimorphisms, protomodularity of $\mathcal{C}$ gives immediately that $\mathcal{M}$ satisfies the 3–for–2 property. Hence $(\mathcal{E}, \mathcal{M})$ is a torsion theory, and its normality follows from 4.10, which is applicable since the assumptions of 4.8 are fulfilled, by hypothesis. When both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are homological, because of 4.7 we can apply 4.10 and obtain the last statement. □

5.3 Remarks

(1) As the proof of 5.2 shows, for the bijective correspondence between normal torsion theories and standard torsion theories, it suffices to have $\mathcal{C}$ homological with (0Coker, NormMono)-factorizations. The latter condition is, of course, still quite restrictive: even standard semi-abelian categories (like the categories of groups or of commutative rings) do not satisfy it. However, the type of categories that are both homological and co-homological is very well studied. As George Janelidze observed, these are precisely the "Raikov semi-abelian" [Ra], [K] or "almost-abelian" [Ru] categories. In fact, in a pointed protomodular category, the canonical morphism $A + B \to A \times B$ is an extremal epimorphism, hence it is an isomorphism when the category is also co-protomodular. Since protomodular categories are Mal'cev, co-protomodularity makes such categries additive. Hence, *the following conditions are equivalent for a category $\mathcal{C}$:*

(i) *$\mathcal{C}$ is regular, coregular and additive;*

(ii) *$\mathcal{C}$ is homological and co-homological;*

(iii) *$\mathcal{C}$ is Raikov semi-abelian (= almost-abelian).*

Clearly, these conditions imply that $\mathcal{C}$ is homological with (0Coker, NormMono)-factorizations, but we don't know whether these properties are equivalent to (i)-(iii).

(2) Consider the additive homological category $\mathcal{C}$ of abelian groups satisfying the implication ($4x = 0 \implies 2x = 0$). As shown in [JT2], the subcategory $\mathcal{F}$ of groups satisfying $2x = 0$ is closed under extensions and normal epireflective, but is not part of a standard torsion theory. Its reflective factorization system is not simple (likewise when one considers it not in $\mathcal{C}$ but in the abelian category of all abelian groups, see [CHK]), and it is not a normal torsion theory of $\mathcal{C}$. In fact, for $C = \mathbb{Z}$, the diagram of 4.8 is as follows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\sigma"] & 0 \arrow[r] \arrow[d, "\sigma"] & 0 \arrow[d] \\
\mathbb{Z} \cong 2\mathbb{Z} \arrow[r, hook, "\kappa"] \arrow[d] & \mathbb{Z} \arrow[r, "\pi=1"] \arrow[d, "\rho"'] & \mathbb{Z} \arrow[d, "\rho"] \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "1"] & \mathbb{Z}_2
\end{tikzcd}$$

But we do not know whether $(\mathcal{E}, \mathcal{M})$ is a torsion theory.

5.4

A standard torsion theory is called *hereditary* if $\mathcal{T}$ is closed under normal subobjects, and it is *cohereditary* if $\mathcal{F}$ is closed under normal quotients. While hereditary standard torsion theories are of principal importance, coheredity is a very restrictive property, as we show in the next proposition, which is well–known in the case of groups (see [N]).

5.5 Proposition

Let $\mathcal{C}$ be a pointed variety of universal algebras where free algebras are closed under normal subobjects. Then each standard cohereditary torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ is trivial, i.e., $\mathcal{T} = \mathcal{C}$ or $\mathcal{F} = \mathcal{C}$.

Proof. Assume $\mathcal{F} \neq \mathcal{C}$. Since $\mathcal{F}$ is closed under normal quotients, there is a free algebra V not belonging to $\mathcal{F}$. Hence, the $\mathcal{T}$–coreflection of V satisfies

$$0 \neq KV \in \mathcal{T},$$

and KV is free (as a normal subobject of a free algebra)and belongs to $\mathcal{T}$. Since $\mathcal{T}$ is closed under coproducts and quotients, $\mathcal{T} = \mathcal{C}$ follows. □

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