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NORMALISATION FOR THE FUNDAMENTAL CROSSED COMPLEX OF A SIMPLICIAL SET

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Abstract

Crossed complexes are shown to have an algebra sufficiently rich to model the geometric inductive definition of simplices, and so to give a purely algebraic proof of the Homotopy Addition Lemma (HAL) for the boundary of a simplex. This leads to the *fundamental crossed complex* of a simplicial set. The main result is a normalisation theorem for this fundamental crossed complex, analogous to the usual theorem for simplicial abelian groups, but more complicated to set up and prove, because of the complications of the HAL and of the notion of homotopies for crossed complexes. We start with some historical background, and give a survey of the required basic facts on crossed complexes.

Introduction

Crossed complexes are analogues of chain complexes but with nonabelian features in dimensions 1 and 2. So one aim of the use of crossed complexes is to increase the power of methods analogous to those of chain complexes.

Crossed complexes can incorporate information on presentations of groups, or groupoids. Thus another aim is to bring features of the fundamental group nearer to the centre of the toolkit of algebraic topology. From the early 20th century, the fundamental group $\pi_1(X, p)$ has played an anomalous rôle in algebraic topology. This invariant of a pointed space has the properties:

- it is nonabelian;
- it can under some circumstances be calculated precisely by a van Kampen theorem;

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- presentations of it are important;
- it models all pointed, connected, weak homotopy 1-types.

Yet higher dimensional tools (homology groups, homotopy groups) were generally abelian. An exception was the use of crossed modules, as developed by J.H.C. Whitehead in [**48**], and independently by Peiffer, [**42**], and Reidemeister, [**43**]. They were shown in [**38**] to model all pointed, connected, weak homotopy 2-types (there called 3-types).

A possible resolution of this anomaly – nonabelian invariant in dimension 1, abelian invariants in dimensions greater than 1 – has appeared with the transition from groups to groupoids. The fundamental groupoid $\pi_1(X, A)$ on a set A of base points was shown to have computational and conceptual utility, [**7, 9**]. Groupoids were found to have 2-dimensional nonabelian generalisations, for example crossed modules and forms of double groupoids, again with computational and conceptual utility, [**25, 24**]. Whitehead's crossed modules derived from homotopy theory could in some cases be *calculated precisely as nonabelian structures* by a 2-dimensional van Kampen type theorem, whose proof used a relative homotopy double groupoid, [**14, 26**].

These results were generalised in [**16**] to all dimensions, using crossed complexes, whose definition and basic theorems we recall in section 2. It is important that these results are proved by working directly with homotopically defined functors and without the use of traditional tools such as homology and simplicial approximation. Surveys of the use of crossed complexes are in [**10, 11**]. We also mention the work of Huebschmann, [**34**], on the representation of group cohomology by crossed n-fold extensions, and the use in [**44**] of Whitehead's methods from CHII to model algebraically filtrations of a manifold given by Morse functions.

We can also see from the use of the Homotopy Addition Lemma in proofs of the Hurewicz and relative Hurewicz Theorems, [**46**], that the category Crs lies at the transition between homology and homotopy. For further work on crossed complexes, see for example [**1, 3, 4, 41, 45**]. The works with Baues refer to crossed complexes as 'crossed chain complexes', and, as with Huebschmann, are in the one vertex case. It can be argued that the category Crs gives a linear approximation to homotopy theory: that is, crossed complexes can incorporate presentations of the fundamental group(oid) and its actions. They do not incorporate, say, higher dimensional Whitehead products, or composition operators. The tensor product of crossed complexes (see later in 2.2) allows, analogously to work on chain complexes, for corresponding notions of an 'algebra' and so for the modelling of more structure, as in [**12, 3, 4**].

The main result of this paper, Theorem 11.7, extends the theory by proving for crossed complexes an analogue of a basic normalisation theorem for the traditional chain complex associated with a simplicial set, due originally to Eilenberg and Mac Lane, [**31**]. We use the Homotopy Addition Lemma to construct two crossed complexes associated to a simplicial set K, the unnormalised $\Pi^{\Upsilon}K$ and the normalised ΠK, of which the first is free on all simplices of K and the second is free on the nondegenerate simplices. It is important to have both crossed complexes for applications of acyclic model theory; this is analogous to the application of acyclic models to the usual singular chain complex by Eilenberg and Mac Lane in [**29, 30**]. Indeed it was this relation with acyclic model theory which motivated this investigation, and which we plan to deal with elsewhere. As an example of a corollary from our normalisation theorem and properties of crossed complexes we obtain the well known fact that the projection $\|K\| \to |K|$, from the thick to the standard geometric realisation of a simplicial set K, is a homotopy equivalence (Corollary 1.3).

To make this paper self-contained, we give a fairly full account of the necessary properties of crossed complexes, so that this paper can form an introduction to their use. The structure of this paper is as follows: Section 1 gives an introduction to the Homotopy Addition Lemma: this Lemma is essential for describing the fundamental crossed complex of a simplicial set. Section 2 gives an account of the basic results on crossed complexes that are needed. Section 3 states the main relation with chain complexes with a groupoid of operators; this is useful for understanding many constructions on crossed complexes. Sections 4, 5, 6 give brief accounts of generating crossed complexes, normal subcrossed complexes and free crossed complexes; the first two topics are not easily available in the literature. Sections 7, 8 give specific rules for the crossed complex constructions of cylinders and homotopies, and then cones, and so allow the algebraic deduction of the Homotopy Addition Lemma. Section 9 defines the (non normalised) fundamental crossed complex of a simplicial set. Section 10 normalises this crossed complex at ε_0. Section 11 gives the full normalisation theorem.

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1. The Homotopy Addition Lemma

The normalisation theorem for simplicial abelian groups (see, for example, Eilenberg-Mac Lane [**31**, Theorem 4.1], Mac Lane [**37**, §VIII.6]), is of importance in homological algebra and in geometric applications of simplicial theory. It is based on the formula, fundamental in much of simplicial based algebraic topology and homological algebra, that if x has dimension n then

$$\partial x = \sum_{i=0}^{n} (-1)^i \partial_i x, \tag{1}$$

which can be interpreted intuitively as: 'the boundary of a simplex is the alternating sum of its faces'. The setting for this formula is the theory of chain complexes: these are sequences of morphisms of abelian groups (or R-modules) $\partial : A_n \to A_{n-1}$ such that $\partial\partial = 0$. For simplicial abelian groups, each ∂_i is a morphism of abelian groups and the formula (1) is just the alternating sum of morphisms. Thus we have a chain complex (A, ∂). Further, if $(DA)_n$ is for $n \geqslant 0$ the subgroup of A_n generated by degenerate elements, then DA is a contractible subchain complex of (A, ∂). This is the normalisation theorem.

In *homotopy*, rather than homology, theory, there is another and more complicated basic formula, known as the *Homotopy Addition Lemma* (HAL) (or theorem) [**5**, §12], [**33**], [**46**, Theorem IV-6.1, p174 ff.]. The intuition of 'the boundary of a simplex' given in (1) is strengthened in the HAL by taking account also of:

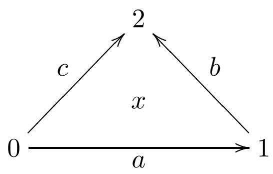
- a *set* of base points (the vertices of the simplex);
- nonabelian structures in dimensions 1 and 2;
- operators of dimension 1 on dimensions $\geqslant 2$.

From our standpoint, the set of base points is taken into account through the use of groupoids in dimension 1, while the boundary from dimension 3 to dimension 2 uses crossed modules of groupoids. This leads to basic formulae, which with our conventions are as follows:

In dimension 2 we have a groupoid rule:

$$\delta_2 x = -\partial_1 x + \partial_2 x + \partial_0 x, \tag{HAL2}$$

which is represented by the diagram



(HAL2-diagram)

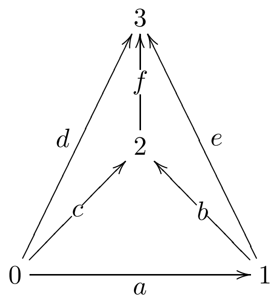
and the easy to understand formula (HAL2) says that $\delta_2 x = -c + a + b$. Note that we use additive notation throughout for group or groupoid composition. Our convention is that the *base point* of an ordered n-simplex x is the final vertex $\partial_0^n x$.

In dimension 3 we have the nonabelian rule:

$$\delta_3 x = (\partial_3 x)^{\partial_0^2 x} - \partial_0 x - \partial_2 x + \partial_1 x. \tag{HAL3}$$

Understanding of this is helped by considering the diagram

$x :=$



(HAL3-diagram)

in which $f = \partial_0^2 x$. The base point of the above 3-simplex x is 3, while the base point of $\partial_3 x$ is 2. The exponent f relocates $\partial_3 x$ to have base point at 3, and so yields a well defined formula. Given the labelling in (HAL3-diagram) we have the groupoid formula

$$-f + (-c + a + b) + f - (-e + b + f) - (-d + a + e) + (-d + c + f) = 0.$$

This is a translation of the rule $\delta_2\delta_3 = 0$, provided we assume $\delta_2(y^f) = -f+\delta_2 y+f$, which is the first rule for a crossed module.

In dimension $n \geqslant 4$ we have the abelian rule with operators:

$$\delta_n x = (\partial_n x)^{u_n x} + \sum_{i=0}^{n-1}(-1)^{n-i}\partial_i x \quad \text{for } n \geqslant 4, \qquad \text{(HAL}\geqslant 4)$$

where $u_n x = \partial_0^{n-1} x$. We have some difficulty in drawing a diagram for this! These, or analogous, formulae underly much nonabelian cohomology theory.

The rule $\delta_{n-1}\delta_n = 0$ is straightforward to verify for $n > 4$, through working in abelian groups; for $n = 4$ we require the second crossed module rule, that for x, y of dimension 2

$$-y + x + y = x^{\delta_2 y}.$$

A consequence is that $\operatorname{Ker}\delta_2$, is central. Hence so also is $\operatorname{Im}\delta_3$, since we have verified $\delta_2\delta_3 = 0$. The type of argument that is used for the $n = 4$ case, [**46**], and which we shall use later, is the simple:

Lemma 1.1. *If $\gamma = \alpha + \beta$ is a central element in a group, then $\gamma = \beta + \alpha$.*

Proof.

$$\gamma - \alpha - \beta = -\alpha + \gamma - \beta = 0. \qquad \text{(centrality)}$$

$\square$

The above formulae are not exactly as will be found generally in the literature; they follow from our conventions given in section 7 for crossed complexes and their 'cylinder object' $\mathcal{I}\otimes C$ and cone object $\operatorname{Cone}(C)$. Thus the chain complex and homological boundary formula (1) becomes a much more complicated result in homotopy theory, as the formulae in the Homotopy Addition Lemma. Formulae of this type occur frequently in mathematics, for example in the cohomology of groups, [**23**], in differential geometry, [**36**], and in the cohomology of stacks, [**6**].

The formal structure required for this HAL is known as a *crossed complex of groupoids*, and such structures form the objects of a category which we write Crs.

We give more details of this category in section 2. It is complete and cocomplete. It contains 'free' objects, satisfying the universal property that a morphism $f : F \to C$ from a free crossed complex is defined by its values on a free basis, subject to certain geometric conditions. Note that in the formulae for the HAL, the ∂_i are *not morphisms*, but x and all $\partial_i x$ are elements of a free basis.

Our first aim here is to show how the HAL for a simplex fits neatly into an algebraic pattern in crossed complexes, using a cone construction $\operatorname{Cone}(B)$ for a crossed complex B. We define algebraically and inductively an 'algebraic' or 'crossed complex simplex' $a\Delta^n$ by

$$a\Delta^0 = \{0\}, \quad a\Delta^n = \operatorname{Cone}(a\Delta^{n-1}). \qquad (2)$$

Our conventions for the tensor product, [**17**], ensure that this definition yields algebraically precisely the HAL given above.

Our main result is an application to the (non normalised) fundamental crossed complex, written here $\Pi^{\Upsilon}K$, of a simplicial set K. This is defined to be the free crossed complex on the elements of $K_n, n \geqslant 0$, with boundary given by the homotopy addition lemma HAL. Thus $\Pi^{\Upsilon}K$ contains basis elements which are degenerate simplices, of the form $\varepsilon_i y$ for some y. Full details are given in section 9.

We may also construct $\Pi^{\Upsilon}K$ as a coend as follows. Let $\mathbf{\Delta}$ be the simplicial operator category, so that a simplicial set K is a functor $\mathbf{\Delta}^{op} \to \mathsf{Sets}$. Let $\mathbf{\Upsilon}$ be the subcategory of $\mathbf{\Delta}$ generated by the injective maps, i.e. those which correspond to simplicial face operators. Then we can see the unnormalised fundamental crossed complex of K as the coend in the category of crossed complexes

$$\Pi^{\Upsilon}K = \int^{\mathbf{\Upsilon},n} K_n \times a\Delta^n.$$

We are also interested in the *normalised* crossed complex, defined as the coend

$$\Pi K = \int^{\mathbf{\Delta},n} K_n \times a\Delta^n.$$

In Section 11 we complete the proof of:

Theorem 1.2 (Normalisation theorem). *For a simplicial set K, the quotient morphism $p : \Pi^{\Upsilon}K \to \Pi K$ is a homotopy equivalence with section $q : \Pi K \to \Pi^{\Upsilon}K$, and the quotient crossed complex ΠK is free.*

This has application to the usual thick and standard geometric realisations of a simplicial set K, defined respectively as coends:

$$\|K\| = \int^{\mathbf{\Upsilon},n} K_n \times \Delta^n,$$

$$|K| = \int^{\mathbf{\Delta},n} K_n \times \Delta^n,$$

where Δ^n is the geometric simplex. Then the normalisation theorem together with standard properties of crossed complexes, implies:

Corollary 1.3. *For a simplicial set K, the projection $\|K\| \to |K|$ from the thick to the standard geometric realisation is a homotopy equivalence.*

Proof. We use the Higher Homotopy van Kampen Theorem (HHvKT) of [**16**] to give natural isomorphisms

$$\Pi^{\Upsilon}K \cong \Pi(\|K\|_*), \Pi K \cong \Pi(|K|_*).$$

It is immediate that the projection induces an isomorphism of fundamental groupoids and of the homologies of the universal covers at all base points. □

We assume work on groupoids as in [**9, 32**].

2. Basics on crossed complexes

Crossed complexes, first called *group systems*, were first defined, in the one vertex case, in 1946 by Blakers in [**5**], following a suggestion of Eilenberg. He combined into a single structure the fundamental group $\pi_1(X, p)$ and the relative homotopy groups $\pi_n(X_n, X_{n-1}, p), n \geqslant 2$ associated to a reduced filtered space X_*, i.e. when X_0 is a singleton $\{p\}$. We now call this structure the *fundamental crossed complex* $\Pi(X_*)$ of the filtered space (see below).

Blakers' concept was taken up in J.H.C. Whitehead's deep paper 'Combinatorial homotopy theory II' (CHII) [**48**], in the reduced and free case, under the term 'homotopy system'; this paper is much less read than the previous paper 'Combinatorial homotopy I' (CHI) [**47**], which introduced the basic concept of CW-complex. We give below a full definition of the category Crs of crossed complexes: our viewpoint, following that of CHII, is that Crs should be seen as a basic category for applications in algebraic topology, with better realisability properties, [**19, 22**], than the more usual chain complexes with a group of operators, [**18**].

We use relative homotopy theory to construct the functor

$$\Pi : \mathsf{FTop} \to \mathsf{Crs} \tag{3}$$

where FTop is the category of *filtered spaces*, whose objects

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty$$

consist of a compactly generated topological space X_∞ and an increasing sequence of subspaces $X_n, n \geqslant 0$. The morphisms $f : X_* \to Y_*$ of FTop are maps $f : X_\infty \to Y_\infty$ such that for all $n \geqslant 0$ $f(X_n) \subseteq Y_n$.

The functor Π is given on a filtered space X_* by

$$(\Pi X_*)_n = \begin{cases} X_0 & \text{if } n = 0, \\ \pi_1(X_1, X_0) & \text{if } n = 1, \\ \pi_n(X_n, X_{n-1}, X_0) & \text{if } n \geqslant 2. \end{cases} \tag{4}$$

Here $\pi_1(X_1, X_0)$ is the fundamental groupoid of X_1 on the set X_0 of base points, and $\pi_n(X_n, X_{n-1}, X_0)$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, p)$ for all $p \in X_0$.

If we write $C_n = (\Pi X_*)_n$, then we find that there is a structure of a family of groupoids over C_0 with source and target maps s, t:

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "t"] & C_{n-1} \arrow[r] \arrow[d, "t"] & \cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ & C_0 & C_0 & & C_0 & C_0 \end{tikzcd} \tag{5}$$

in which: for $n \geqslant 2$, C_n is totally disconnected, i.e. $s = t$; C_1 operates (on the right) on $C_n, n \geqslant 2$, and on the family of vertex groups of C_1 by conjugation; and the axioms are, in addition to the usual operation rules, that:

CC1) $s\delta_2 = t\delta_2$, $\delta_{n-1}\delta_n = 0$;

CC2) δ_n is an operator morphism;

CC3) $\delta_2 : C_2 \to C_1$ is a crossed module;

CC4) for $n \geqslant 3$, C_n is abelian and $\delta_2 C_2$ operates trivially on C_n.

It will be convenient to write all group and groupoid compositions additively, and the operations as x^a: if $a : p \to q$ in dimension 1, then $p = sa, q = ta$, and if further $b : q \to r$ then $a + b : p \to r$; if $n \geqslant 2$ and $x \in C_n(p)$, then $tx = p$ and $x^a \in C_n(q)$.

These laws CC1)-CC4) for $C = \Pi X_*$ reflect basic facts in relative homotopy theory. Indeed, for any crossed complex C there is a filtered space X_* such that $C \cong \Pi X_*$, [**16**]. These laws also define the objects of the category Crs of crossed complexes. The morphisms $f : C \to D$ of crossed complexes consist of groupoid morphisms $f : C_n \to D_n, n \geqslant 1$, preserving all the structure.

A crossed complex C has a *fundamental groupoid* $\pi_1 C$ defined to be $C_1/(\delta_2 C_2)$, whose set of components is written $\pi_0 C$, and also called the *set of components* of C. It also has a family of *homology groups* given for $n \geqslant 2$ by

$$H_n(C, p) = \operatorname{Ker}(\delta_n : C_n(p) \to C_{n-1}(p))/\operatorname{Im}(\delta_{n+1}(C_{n+1}(p) \to C_n(p)),$$

which can be seen to be a module over $\pi_1 C$. A morphism $f : C \to D$ of crossed complexes induces a morphism of fundamental groupoids and homology groupoids, and is called a *weak equivalence* if it induces an equivalence of fundamental groupoids and isomorphisms $H_*(C, p) \to H_*(D, fp)$ for all $p \in C_0$.

If X_* is the skeletal filtration of a CW-complex X, then (see [**48**]) there are natural isomorphisms

$$\pi_1(\Pi X_*) \cong \pi_1(X, X_0), \qquad H_n(\Pi X_*, p) \cong H_n(\widetilde{X}_p),$$

where $\widetilde{X}_p$ is the universal cover of X based at p. It follows from this and Whitehead's theorem from [**47**] that if $f : X \to Y$ is a cellular map of CW-complexes X, Y which induces a weak equivalence $\Pi f : \Pi X_* \to \Pi Y_*$, then f is a homotopy equivalence.

The following additional facts on crossed complexes were found in a sequence of papers by Brown and Higgins:

2.1 ([16]). *The functor* $\Pi : \mathsf{FTop} \to \mathsf{Crs}$ *from the category of filtered spaces to crossed complexes preserves certain colimits.*

2.2 ([17]). *The category* Crs *is monoidal closed, with an exponential law of the form*

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C)). \qquad \text{(exponential law)}$$

2.3 ([13]). *The category* Crs *has a unit interval object written* $\{0\} \rightrightarrows \mathcal{I}$*, which is essentially just the indiscrete groupoid on two objects* $0, 1$*, and so has in dimension*

1 *only one element* $\iota : 0 \to 1$. *For a crossed complex* B, *this gives rise to a cylinder object*

$$\mathrm{Cyl}(B) = (B \rightrightarrows \mathcal{I} \otimes B),$$

and so a homotopy theory for crossed complexes.

The result 2.1 is a kind of Higher Homotopy van Kampen Theorem (HHvKT). Among its consequences are the relative Hurewicz Theorem, seen from this viewpoint as a relation between $\pi_n(X, A)$ and $\pi_n(X \cup CA)$. It also implies nonabelian results in dimension 2 not obtained by other means, [**8, 26**]. The proof of the HHvKT, [**16**], uses cubical higher homotopy groupoids and is independent of standard methods in algebraic topology, such as homology or simplicial approximation.

The monoidal closed structure for crossed complexes allows us to define homotopies for morphisms $B \to C$ of crossed complexes as morphisms $\mathcal{I} \otimes B \to C$, or, equivalently, as morphisms $\mathcal{I} \to \mathsf{CRS}(B, C)$. The detailed structure of this *cylinder object* $\mathcal{I} \otimes C$, [**35**], will be given in section 7. The model structure of this homotopy theory is developed in [**13**].

The full structure of the internal hom $\mathsf{CRS}(B, C)$ is quite complicated. This complexity is also reflected in the structure of the tensor product $A \otimes B$ of crossed complexes A, B: it is generated in dimension n by elements $a \otimes b$ where $a \in A_l, b \in B_k, l + k = n$; the full list of structure and laws is again quite complex (see [**17**]).

We need that the category FTop of filtered spaces is monoidal closed with an exponential law

$$\mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)). \tag{6}$$

Here $(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q$. A standard example of a filtered space is a CW-complex with its skeletal filtration, and among the CW-complexes we have the n-ball E^n with its cell structure

$$E^n = \begin{cases} e^0 & \text{if } n = 0, \\ e^0_\pm \cup e^1 & \text{if } n = 1, \\ e^0 \cup e^{n-1} \cup e^n & \text{if } n > 1. \end{cases} \tag{7}$$

The complications of the cell structure of $E^m \times E^n$ are modelled in the tensor product of crossed complexes, as shown by:

2.4 ([19]). *There is for filtered spaces* X_*, Y_* *a natural transformation*

$$\eta : \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*),$$

which is an isomorphism if X_*, Y_* *are the skeletal filtrations of* CW*-complexes,* [**19**], *and more widely,* [**2**].

Remark 2.5. From the early days, basic results of relative homotopy theory have been proved by relating the geometries of cells and cubes. This geometric relation was translated into a relation between algebraic theories in several papers, particularly [**15, 17**], which give an equivalence of monoidal closed categories between a category of 'cubical ω-groupoids' and the category Crs. While many constructions and proofs are clearer in the former category, both categories are required for some results. For example the natural transformation η of 2.4 is easy to see in the cubical category, [**19**]. For a survey on crossed complexes and their uses, see [**11**]. A book in preparation, [**20**], is planned to give a full account of all these main properties, and make the theory more accessible and hopefully more usable.

An important result on crossed complexes is the following:

2.6 ([19]). *There is a classifying space functor $B : \mathsf{Crs} \to \mathsf{Top}$ and a homotopy classification theorem*

$$[X, BC] \cong [\Pi X_*, C]$$

for a CW-complex X with its skeletal filtration, and crossed complex C.

This result is a homotopy classification theorem in the non simply connected case, and includes many classical results. It relates to Whitehead's comment in [**48**] that crossed complexes have better realisation properties than chain complexes with a group of operators. It is also relevant to nonabelian cohomology, and cohomology with local coefficients, as discussed in [**19**]. See also [**27, 39, 40**].

3. Crossed complexes and chain complexes

As is clear from the definition, crossed complexes differ from chain complexes of modules over groupoids only in dimensions 1 and 2. It is useful to make this relationship more precise. We therefore define a category Chn of such chain complexes which is monoidal closed, give a functor $\nabla : \mathsf{Crs} \to \mathsf{Chn}$, and state that this functor is monoidal and has a right adjoint. The results of this section are taken from [**18**], which develops results from [**48**]. See also [**21**] for the low dimensional and reduced case.

We first define the category Mod of *modules over groupoids*. We will often write G_0 for $\mathrm{Ob}\, G$ for a groupoid G. The objects of the category Mod are pairs (G, M) where G is a groupoid and M is a family $M(p)$, $p \in G_0$, of disjoint abelian groups on which G operates. The notation for this will be the same as for the operations of C_1 on C_n for $n \geqslant 3$ in the definition of a crossed complex. The morphisms of Mod are pairs $(\theta, \phi) : (G, M) \to (H, N)$ where $\theta : G \to H$ is a morphism of groupoids and $\phi : M \to N$ is a family $\phi(p) : M(p) \to N(\theta p)$ of morphisms of abelian groups preserving the operations. Instead of writing (G, M) we often say M is a G-module. For a morphism $\phi : M \to N$ of G-modules it is assumed that $\theta = 1_G$.

A *chain complex C over a groupoid G* is a sequence of morphisms of G-modules $\partial : C_n \to C_{n-1},\ n \geqslant 1$, such that $\partial\partial = 0$. A *morphism of chain complexes* consists of a morphism $\theta : G \to H$ of groupoids and a family $\phi_n : C_n \to D_n$ such that (θ, ϕ_n) is a morphism of modules and also $\partial\phi_n = \phi_{n-1}\partial, n \geqslant 0$. This defines the category Chn.

The category Mod is monoidal closed. We define here only the tensor product: for modules (G, M), (H, N) we set

$$(G, M) \otimes (H, N) = (G \times H, M \otimes N)$$

using the product of groupoids and with

$$(M \otimes N)(p, q) = M(p) \otimes N(q),$$

the usual tensor product of abelian groups, and action the product action $(m \otimes n)^{(g,h)} = m^g \otimes n^h$.

The category Chn is also monoidal closed with the usual tensor product $(G, C) \otimes (H, D) = (G \times H, C \otimes D)$ where $(C \otimes D)_n = \bigotimes_{p+q=n} C_p \otimes D_q$. For details of the internal hom, see [**18**].

A particular module we need is $(G, \mathbb{Z}^{\to}G)$, also written $\mathbb{Z}^{\to}G$, for a groupoid G. If $p \in G_0$, then $\mathbb{Z}^{\to}G(p)$ is the free abelian group on the elements of G with final point p, and with operations induced by the right action of G. Note that in contrast to the single object case, i.e. of groups, we obtain a module and not an analogue of a ring. A set J defines a discrete groupoid on J also written J and so a module $(J, \mathbb{Z}^{\to}J)$: when the set J is understood, we abbreviate this module to $\mathbb{Z}^{\to}$. The *augmentation map* in this context is given as usual by the sum of the coefficients. It is a module morphism $\varepsilon : (G, \mathbb{Z}^{\to}G) \to (G_0, \mathbb{Z}^{\to})$ and its kernel is the *augmentation module* $(G, I^{\to}G)$ which we abbreviate to $I^{\to}G$.

Let $\psi : G \to H$ be a groupoid morphism which is bijective on objects, and let (H, N) be a module. We need the generalisation to groupoids of the universal derivations of Crowell [**28**]. A ψ-derivation $d : G \to N$ assigns to each $g \in G$ with final point p an element $d(g) \in N(\psi p)$ satisfying the rule that $d(g' + g) = d(g')^{\psi g} + d(g)$ whenever $g' + g$ is defined in G. The ψ-derivation d is *universal* if given any other ψ-derivation $d' : G \to L$ where L is an H-module, there is a unique H-module morphism $f : L \to N$ such that $fd' = d$. The construction of the universal ψ-derivation is straightforward, and is written $\alpha : G \to D_\psi$.

3.1. *Let C be a crossed complex, and let $\phi : C_1 \to G$ be a cokernel of δ_2 of C. Then there are G-morphisms*

$$C_2^{\mathrm{ab}} \xrightarrow{\partial_2} D_\phi \xrightarrow{\partial_1} \mathbb{Z}^{\to}G \tag{8}$$

such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "="'] & C_{n-1} \arrow[r] \arrow[d, "="'] & \cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "="'] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\alpha_2"] & C_1 \arrow[r, "\phi"] \arrow[d, "\alpha_1"] & G \arrow[d, "\alpha_0"] \\
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"'] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"'] & C_2^{\mathrm{ab}} \arrow[r, "\partial_2"'] & D_\phi \arrow[r, "\partial_1"'] & \mathbb{Z}^{\to}G
\end{tikzcd} \tag{9}$$

commutes and the lower line is a chain complex over G. Here α_2 is abelianisation, α_1 is the universal ϕ-derivation, α_0 is the G-derivation $x \mapsto x - 1_q$ for $x \in G(p, q)$, as a composition $G \to I^{\to}G \to \mathbb{Z}^{\to}G$, and $\partial_n = \delta_n$ for $n \geqslant 4$. Further

(i) *the sequence* (8) *is exact at* D_ϕ *and the image of* ∂_1 *is the augmentation module* $I^{\to}G$*;*

(ii) *if* C_1 *is a free groupoid on a generating graph* X_1 *and* ψ *is surjective, then* D_ϕ *is the free* H*-module on the basis at* $p \in H_0$ *of elements* x *of* X_1 *such that* $\psi tx = p$*;*

(iii) *if* C_1 *is free, then* α_2 *is injective on* $\operatorname{Ker}\delta_2$*.*

3.2. (i) *The bottom row of diagram* (9) *defines a functor* $\nabla : \mathsf{Crs} \to \mathsf{Chn}$*, which has a right adjoint. Hence* ∇ *preserves colimits.*

(ii) *The functor* ∇ *preserves tensor products: there is a natural equivalence for crossed complexes* A, B

$$\nabla(A) \otimes \nabla(B) \cong \nabla(A \otimes B).$$

This last result shows that the major unusual complications of the tensor product of crossed complexes occur in dimensions 1 and 2. These cases are analysed in [**17**].

Remark 3.3. These results show the close relation of crossed complexes and these chain complexes. The functor ∇ loses information. Whitehead remarks in [CHII] that (using our terminology) these chain complexes have less good realisation properties even than free crossed complexes. Indeed, the problem of which 2-dimensional free chain complexes are realisable by a crossed complex is known to be hard.

4. Generating structures

Let C be a crossed complex, and let R_* be a family of subsets $R_n \subseteq C_n$ for all $n \geqslant 0$. We have to explain what is meant by the subcrossed complex $\langle R_* \rangle$ of C *generated* by R_*.

A formal definition of $B = \langle R_* \rangle$ is easy: it is the smallest sub-crossed complex D of C such that $R_n \subseteq D_n$ for all $n \geqslant 0$, and so is also the intersection of all such D. A direct construction is as follows. We set

$$B_0 = R_0 \cup sR_1 \cup \bigcup_{n \geqslant 1} tR_n.$$

Let B_1 be the subgroupoid of C_1 generated by $R_1 \cup \delta_2(R_2)$ and the identities at B_0. Let B_2 be the subcrossed B_1-module of C_2 generated by R_2. For $n \geqslant 3$, let B_n be the sub-B_1-module of C_n generated by $R_n \cup \delta(R_{n+1})$ and the identities at elements of B_0.

Note that this definition is inductive. The usual property of a generating structure holds: thus if R_* generates C, i.e. $\langle R_* \rangle = C$, and $f, g : C \to D$ are two crossed complex morphisms which agree on R_*, then $f = g$. This is proved by induction.

We say the family R_* is a *generating structure* for a subcrossed complex B of C if for each $n > 0$ the boundaries in C of elements of R_n lie in the subcrossed complex generated by the R_i for $i < n$, and R_* generates B.

5. Normal subcrossed complexes

We assume work on normal subgroupoids, as in [**9, 32**].

Definition 5.1. A subcrossed complex A of a crossed complex C is called *normal* in C if:

N1) A is wide in C, i.e. $A_0 = C_0$;

N2) A_1 is a totally disconnected normal subgroupoid of C_1;

N3) for $n \geqslant 2$, A_n is C_1-invariant, i.e. $a \in A_n$, $x \in C_1$ and a^x is defined implies $a^x \in A_n$;

N4) for $n \geqslant 2$, if $a \in A_1$, $x \in C_n$, and x^a is defined then $x - x^a \in A_n$. □

Note that N3) implies that A_2 is a normal subgroupoid of C_2 since $-x+a+x = a^{\delta_2 x}$.

The above are necessary and sufficient conditions for A to be the kernel of a morphism $C \to D$ of crossed complexes which is injective on objects for some D, as the following proposition shows.

Proposition 5.2. If A is a normal subcrossed complex of the crossed complex C, then the family of quotients $C_n/A_n,\ n \geqslant 1$ inherits the structure of crossed complex, which we call the quotient crossed complex C/A.

We leave the proof to the reader.

Let R_* be a family of subsets of the crossed complex C as in the previous section, and such that R_1 is totally disconnected, i.e. just a family of subsets of vertex groups of the groupoid C_1. We say that R_* *normally generates* a subcrossed complex A of C if A is the smallest wide normal subcrossed complex of C containing R_*, and then we say A is the *normal closure* of R_* in C, and write $A = \langle\langle R_* \rangle\rangle$. We also say R_* is a *normal structure in C* if for each $n > 0$ the boundaries of elements of R_n are in the normal closure of the R_i for $i < n$.

We consider how to construct $A = \langle\langle R_* \rangle\rangle$. In dimension 1, this is the normal closure of $R_1 \cup \delta_2(R_2)$ as extended to the groupoid case in [**9, 32**], i.e we take the 'consequences' of $R_1 \cup \delta_2(R_2)$ in C_1. Suppose this A_1 has been constructed.

Proposition 5.3. For $n \geqslant 2$, $A_n = \langle\langle R_* \rangle\rangle_n$ is generated as a group (abelian if $n \geqslant 3$) by the elements

$$r^c,\ x - x^a \ \text{ for all } r \in R_n \cup \delta_{n+1}(R_{n+1}),\ c \in C_1,\ x \in C_n,\ a \in A_1.$$

Proof. Clearly these elements belong to A_n. We now prove the set of these is C_1-invariant.

This is clear for the set of elements of the form r^c as above. Suppose then x, c, a are as above. Then

$$\begin{aligned}(x - x^a)^c &= x^c - x^{a+c} \\ &= x^c - (x^c)^{-c+a+c},\end{aligned}$$

which implies what we want since $-c+a+c \in A_1$ by normality.

It follows that the group generated by these elements is C_1-invariant. In dimension 2, this implies normality, by the crossed module rules. □

6. Free crossed complexes

Write $\mathbb{F}(n)$ for the crossed complex freely generated by one generator c_n in dimension n. So $\mathbb{F}(0)$ is a singleton in all dimensions; $\mathbb{F}(1)$ is essentially the groupoid $\mathcal{I}$; and for $n \geqslant 2$, $\mathbb{F}(n)$ is in dimensions n and $n-1$ an infinite cyclic group with generators c_n and δc_n respectively, and otherwise trivial. Let $\mathbb{S}(n-1)$ be the subcrossed complex of $\mathbb{F}(n)$ generated by δc_n. Thus $\mathbb{S}(-1)$ is empty.

If E^n_* and S^{n-1}_* denote the skeletal filtrations of the standard n-ball and $(n-1)$-sphere respectively, then a basic result in algebraic topology is that

$$\Pi E^n_* \cong \mathbb{F}(n), \quad \Pi S^{n-1}_* \cong \mathbb{S}(n-1).$$

This is also a consequence of the Higher Homotopy van Kampen Theorem indicated in 2.1, see [**16**].

We now define a particular kind of morphism $j : A \to F$ of crossed complexes which we call a *morphism of relative free type*. Let A be any crossed complex. A sequence of morphisms $j_n : F^{n-1} \to F^n$ may be constructed inductively as follows. Set $F^{-1} = A$. Supposing F^{n-1} is given, choose any family of morphisms as in the top row of the diagram

$$\begin{tikzcd}
\bigsqcup_{\lambda\in\Lambda_n} \mathbb{S}(n-1) \arrow[r, "(f^\lambda)"] \arrow[d] & F^{n-1} \arrow[d, "j_n"] \\
\bigsqcup_{\lambda\in\Lambda_n} \mathbb{F}(n) \arrow[r] & F^n
\end{tikzcd}$$

and form the pushout in Crs to obtain $j_n : F^{n-1} \to F^n$. Let $F = \operatorname{colim}_n F^n$, and let $j : A \to F$ be the canonical morphism. The image x^n_λ of the element c_n in the summand indexed by λ is called a *basis element of F relative to A*, and we may conveniently write

$$F = A \cup \{x^n_\lambda\}_{\lambda\in\Lambda_n, n\geqslant 0}.$$

We now give some useful results on this notion, see [**19**].

Proposition 6.1. Given two morphism of relative free type, so is their composite.

Proposition 6.2. If in a pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A' \arrow[d] \\
F \arrow[r] & F'
\end{tikzcd}$$

the morphism $A \to F$ is of relative free type, so is the morphism $A' \to F'$.

Proposition 6.3. If in a commutative diagram

$$\begin{tikzcd} A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^n \arrow[r] \arrow[d] & \cdots \\ F^0 \arrow[r] & F^1 \arrow[r] & \cdots \arrow[r] & F^n \arrow[r] & \cdots \end{tikzcd}$$

each vertical morphism is of relative free type, so is the induced morphism $\operatorname{colim}_n A^n \to \operatorname{colim}_n F^n$.

In particular:

Corollary 6.4. *If in a sequence of morphisms of crossed complexes*

$$F^0 \to F^1 \to \cdots \to F^n \to \cdots$$

each morphism is of relative free type, so are the composites $F^0 \to F^n$ and the induced morphism $F^0 \to \operatorname{colim}_n F^n$.

A crossed complex F is *free on* R_* if in the first place R_* generates F, and secondly morphisms on F to any crossed complex can be defined inductively by their values on R_*.

So in the first instance we have $R_0 = F_0$, and F_1 is the free groupoid on the graph (R_1, R_0, s, t). We assume this concept as known; it is fully treated in [**9, 32**]

Secondly, R_2 comes with a function $w : R_2 \to F_1$ given by the restriction of δ_2. We require that the inclusion $R_2 \to F_2$ makes F_2 the free crossed F_1-module on R_2.

By this stage, the fundamental groupoid $\pi_1 F$ is defined; we require that for $n \geqslant 3$, F_n is the free $\pi_1 F$-module on R_n.

A standard fact, due in the reduced case to Whitehead in [CHII], is that if X_* is the skeletal filtration of a CW-complex, then ΠX_* is the free crossed complex on the characteristic maps of the cell structure of X_*. This may be proved using the relative Hurewicz theorem, and is also a consequence of the Higher Homotopy van Kampen Theorem*Higher Homotopy van Kampen Theorem, (HHvKT)*[1] of [**16**].

Proposition 6.5. If C is a free crossed complex on R_*, then a morphism $f : C \to D$ is specified by the values $fx \in D_n, x \in R_n, n \geqslant 0$ provided only that the following geometric conditions hold:

$$sfx = fsx, x \in R_1, tfx = ftx, x \in R_n, n \geqslant 1, \delta fx = f\delta x, x \in R_n, n \geqslant 2. \qquad (10)$$

We refer also to [**13, 19**] for more details on free crossed complexes. It is proved in [**13**] that a weak equivalence of free crossed complexes is a homotopy equivalence.

We now illustrate some of the difficulties of working with free crossed modules by giving a proposition and a counterexample due essentially to Whitehead, [**49**].

[1] Jim Stasheff has recently suggested this term to Brown should replace the previous 'Generalised van Kampen Theorem' in order to emphasise the nature of the theorem.

Theorem 6.6. *Let C be the free crossed complex on R_*, and suppose $S_* \subseteq R_*$ generates a subcrossed complex B of C. Let F be the free crossed complex on S_*. Then the induced morphism $F \to C$ is injective if the induced morphism $\pi_1 B \to \pi_1 C$ is injective.*

Proof. First of all, we know that a subgroupoid of a free groupoid is free. Also in dimensions > 2 C_n is the free $\pi_1 C$-module on the basis R_n. So injectivity, under the given condition, is clear in this case.

Thus the only problem is in dimension 2, and here we generalise an argument of Whitehead, [**49**]. We use the functor $\nabla : \mathsf{Crs} \to \mathsf{Chn}$ given in section 3.

The abelianised groupoids $F_2^{\mathrm{ab}}, C_2^{\mathrm{ab}}$ are respectively free $\pi_1 F, \pi_1 C$-modules on the bases S_2, R_2. Since the induced morphism on π_1 is injective, so also is the induced morphism $F_2^{\mathrm{ab}} \to C_2^{\mathrm{ab}}$. The morphism $C_2 \to C_2^{\mathrm{ab}}$ is injective on $\operatorname{Ker} \delta_2 : C_2 \to C_1$, since C_1 is a free groupoid, by (ii) of 3.1. So $F \to C$ is injective in dimension 2. □

Example 6.7. Let $X = Y = \{x\}, R = \{a, b\}, S = \{b\}$ where $a = x, b = 1$. The group presentations $\langle Y \mid S\rangle, \langle X \mid R\rangle$ determine free crossed modules $\delta_S : C(S) \to F(X), \delta_R : C(R) \to F(X)$. The inclusion $i : S \to R$ determines $C(i) : C(S) \to C(R)$. Now $F(X) = F(Y) = \mathsf{C}$, the infinite cyclic group, while $C(S)$ is abelian and is the free C-module on the generator b. Also in $C(R)$, $ab = ba$ since $\delta_R b = 1$. Hence

$$C(i)(b^x) = (C(i)(b))^{\delta_R a} = a^{-1}ba = b = C(i)(b),$$

and so $C(i)$ is not injective.

Of course the geometry of this example is the cell complex $K = E^2 \vee S^2$ and the subcomplex $S^1 \vee S^2$. □

7. Cylinder and homotopies

It is useful to write out first all the rules for the cylinder $\operatorname{Cyl}(C) = \mathcal{I} \otimes C$, as a reference. For full details of the tensor product, see [**17, 11**].

Let C be a crossed complex. The cylinder $\mathcal{I} \otimes C$ is generated by elements $0 \otimes x, 1 \otimes x$ of dimension n and $\iota \otimes x, (-\iota) \otimes x$ of dimension $(n+1)$ for all $n \geqslant 0$ and $x \in C_n$, with the following defining relations for $a \in \mathcal{I}$:

Source and target

$$\begin{array}{ll} t(a \otimes x) = ta \otimes tx & \text{for all } \ a \in \mathcal{I}, \in C \\ s(a \otimes x) = a \otimes sx & \text{if } \ a = 0, 1, n = 1\ , \\ s(a \otimes x) = sa \otimes x & \text{if } a = \iota, -\iota, n = 0\ . \end{array}$$

Relations with operations

$$a \otimes x^c = (a \otimes x)^{ta \otimes c} \qquad \text{if } \ n \geqslant 2, \ c \in C_1.$$

Relations with additions

$$a \otimes (x+y) = \begin{cases} (a \otimes x)^{ta \otimes y} + a \otimes y, & \text{if } a = \iota, -\iota, n = 1, \\ a \otimes x + a \otimes y, & \text{if } a = 0, 1, n \geqslant 1 \text{ or if } a = \iota, -\iota, \ n \geqslant 2, \end{cases}$$

$$(-\iota) \otimes x = \begin{cases} -(\iota \otimes x) & \text{if } n = 0, \\ -(\iota \otimes x)^{(-\iota) \otimes tx} & \text{if } n \geqslant 1. \end{cases}$$

Boundaries

$$\delta(a \otimes x) = \begin{cases} a \otimes \delta x & \text{if } a = 0, 1, \ n \geqslant 2; \\ -ta \otimes x - a \otimes sx + sa \otimes x + a \otimes tx & \text{if } a = \iota, -\iota, \ n = 1; \\ -(a \otimes \delta x) - (ta \otimes x) + (sa \otimes x)^{a \otimes tx} & \text{if } a = \iota, -\iota, \ n \geqslant 2. \end{cases}$$

□

Now we can translate the rules for a cylinder into rules for a homotopy. Thus a *homotopy* $f^0 \simeq f$ of morphisms $f^0, f : C \to D$ of crossed complexes is a pair (h, f) where h is a family of functions $h_n : C_n \to D_{n+1}$ with the following properties:

$$th_n(x) = tf(x) \qquad \text{for all } x \in C; \tag{11}$$

$$h_1(x+y) = h_1(x)^{fy} + h_1(y) \qquad \text{if } x, y \in C_1 \text{ and } x+y \text{ is defined;} \tag{12}$$

$$h_n(x+y) = h_n(x) + h_n(y) \qquad \text{if } x, y \in C_n, \ n \geqslant 2 \text{ and } x+y \text{ is defined;} \tag{13}$$

$$h_n(x^c) = (h_n x)^{fc} \qquad \text{if } x \in C_n, n \geqslant 2, \ c \in C_1, \text{ and } x^c \text{ is defined.} \tag{14}$$

Then f^0, f are related by

$$f^0(x) = \begin{cases} sh_0 x & \text{if } x \in C_0, \\ (h_0 sx) + (fx) + (\delta_2 h_1 x) - (h_0 tx) & \text{if } x \in C_1, \\ \{fx + h_{n-1}\delta_n x + \delta_{n+1} h_n x\}^{-(h_0 tx)} & \text{if } x \in C_n, \ n \geqslant 2. \end{cases} \tag{15}$$

Remark 7.1. Part of the force of this statement is that if (h, f) satisfy properties (11-14), then f^0 defined by (15) is a morphism of crossed complexes.

The following is a substantial result:

Proposition 7.2 ([19]). If F, F' are free crossed complexes, on bases R_*, R'_*, then $F \otimes F'$ is the free crossed complex on the basis $R \otimes R'$.

The proof in [**19**] uses the inductive construction of free complexes as successive pushouts given in section 6; the exponential law and the symmetry of $\otimes$ show that $\otimes$ preserves colimits on either side, and this gives an inductive proof, analogous to a corresponding result for CW-complexes.

A consequence, which may also be proved directly, is:

Proposition 7.3. If $f : C \to D$ is a morphism of crossed complexes and C is a free crossed complex on R_*, then a homotopy $(h, f) : f^0 \simeq f : C \to D$ is specified by the values $hx \in D_{n+1}, x \in R_n, n \geqslant 0$ provided only that the following geometric conditions hold:

$$thx = tfx, x \in R_n, n \geqslant 0. \tag{16}$$

Proof. The main special fact we need here is that an f-derivation on a free groupoid is uniquely defined by its values on a free basis. This follows easily from the fact that an f-derivation $h_1 : C_1 \to D_2$ corresponds exactly to a section of a semidirect product construction $F_1 \ltimes C_2 \to F_1$. □

8. Cones and the HAL

Definition 8.1. Let C be a crossed complex. The *cone* $\mathrm{Cone}(C)$ is defined by the pushout

$$\begin{tikzcd} \{1\} \otimes C \arrow[r] \arrow[d] & \{v\} \arrow[d] \\ \mathcal{I} \otimes C \arrow[r] & \mathrm{Cone}\,(C). \end{tikzcd}$$

We call v the *vertex* of the cone. □

Because the cone is formed from the cylinder by shrinking the end at 1 to a point, the rules for the cylinder now simplify nicely.

Proposition 8.2. If C is a crossed complex, then the cone $\mathrm{Cone}\,(C)$ on C is generated by elements $0 \otimes x, \iota \otimes x, x \in C_n$ of dimensions $n, n+1$ respectively, and v of dimension 0 with the following rules, for all $a \in \mathcal{I}$:

Source and target

$$t(a \otimes x) = \begin{cases} 0 \otimes tx, & \text{if } a = 0, \\ v & \text{otherwise.} \end{cases}$$

Relations with operations

$$a \otimes x^c = a \otimes x \quad \text{if} \quad n \geqslant 2, \ c \in C_1.$$

Relations with additions

$$a \otimes (x + y) = a \otimes x + a \otimes y.$$

and

$$(-\iota) \otimes x = \begin{cases} -(\iota \otimes x) & \text{if } n = 0, \\ -(\iota \otimes x)^{-\iota \otimes tx} & \text{if } n \geqslant 1. \end{cases}$$

Boundaries

$$\delta_n(0 \otimes x) = 0 \otimes \delta_n x \qquad \text{if } n \geqslant 2.$$

$$\delta_{n+1}(\iota \otimes x) = \begin{cases} -\iota \otimes sx + 0 \otimes x + \iota \otimes tx & \text{if } n = 1, \\ -(\iota \otimes \delta_n x) + (0 \otimes x)^{\iota \otimes tx} & \text{if } n \geqslant 2. \end{cases}$$

Proposition 8.3. Let F be a free crossed complex on a basis R_*. Then $\mathrm{Cone}(F)$ is the free crossed complex on v in dimension 0, and elements $0 \otimes r, \iota \otimes r$ for all $r \in R_*$, with boundaries given by proposition 8.2.

Proof. This follows from proposition 6.2. □

We use the above to work out the fundamental crossed complex of the simplex $a\Delta^n$ in an algebraic fashion. We define $a\Delta^0 = \{0\}$, $a\Delta^n$ inductively by

$$a\Delta^n = \mathrm{Cone}(a\Delta^{n-1}).$$

The vertices of $a\Delta^1 = \mathcal{I}$ are ordered as $0 < 1$. Inductively, we get vertices $v_0, \dots, v_n$ of $a\Delta^n$ with $v_n = v$ being the last introduced in the cone construction, the other vertices v_i being $(0, v_i)$. The fact that our algebraic formula corresponds to the topological one follows from facts stated earlier on the tensor product and on the GvKT.

We now define inductively top dimensional generators of the crossed complex $a\Delta^n$ by, in the cone complex:

$$\sigma^0 = v,\ \sigma^1 = \iota,\ \sigma^n = (\iota \otimes \sigma^{n-1}),\ n \geqslant 2,$$

with σ^0 being the vertex of $a\Delta^0$.

Next we need conventions for the faces of σ^n. We define inductively

$$\partial_i \sigma^n = \begin{cases} \iota \otimes \partial_i \sigma^{n-1} & \text{if } i < n, \\ 0 \otimes \sigma^{n-1} & \text{if } i = n. \end{cases}$$

Theorem 8.4 (Homotopy Addition Lemma). *The following formulae hold, where $u_n = \iota \otimes v_{n-1}$:*

$$\delta_2 \sigma^2 = -\partial_1 \sigma^2 + \partial_2 \sigma^2 + \partial_0 \sigma^2, \tag{17}$$

$$\delta_3 \sigma^3 = (\partial_3 \sigma^3)^{u_3} - \partial_0 \sigma^3 - \partial_2 \sigma^3 + \partial_1 \sigma^3, \tag{18}$$

while for $n \geqslant 4$

$$\delta_n \sigma^n = (\partial_n \sigma^n)^{u_n} + \sum_{i=0}^{n-1} (-1)^{n-i} \partial_i \sigma^n. \tag{19}$$

Proof. For the case $n = 2$ we have

$$\begin{aligned} \delta_2 \sigma^2 &= \delta_2(\iota \otimes \iota) \\ &= -\iota \otimes 0 + 0 \otimes \iota + \iota \otimes 1 \\ &= -\partial_1 \sigma^2 + \partial_2 \sigma^2 + \partial_0 \sigma^2. \end{aligned}$$

For $n = 3$ we have:

$$\begin{aligned}\delta_3\sigma^3 &= \delta_3(\iota \otimes \sigma^2)\\ &= (0 \otimes \sigma^2)^{\iota\otimes v_2} - \iota \otimes \delta_2\sigma^2\\ &= (0 \otimes \sigma^2)^{u_3} - \iota \otimes (-\partial_1\sigma^2 + \partial_2\sigma^2 + \partial_0\sigma^2)\\ &= (\partial_3\sigma^3)^{u_3} - \partial_0\sigma^3 - \partial_2\sigma^3 + \partial_1\sigma^3.\end{aligned}$$

We leave the general case to the reader, using the inductive formula

$$\delta_{n+1}\sigma^{n+1} = (0 \otimes \sigma^n)^{\iota\otimes v_n} - \iota \otimes \delta_n\sigma^n.$$

The key points that make it easy are the rules on operations and additions of Proposition 8.2. □

Corollary 8.5. *The formula for the boundary of a simplex is as given by the HAL in section 1.*

Proof. We use the fact that for $n \geqslant 2$, the geometric n-simplex Δ^n may be regarded as the cone $\mathrm{Cone}(\Delta^{n-1})$. Our previous results thus give an isomorphism

$$\Pi\Delta^n_* \cong \mathrm{Cone}(\Pi\Delta^{n-1}_*).$$

Since $\Delta^1_* = E^1_*$, the HAL now follows from theorem 8.4. □

9. The unnormalised fundamental crossed complex of a simplicial set

We now give full details of the definition of the (unnormalised) fundamental crossed complex of a simplicial set, which we referred to in section 1.

Definition 9.1. We define $\Pi^\Upsilon K$ the *(unnormalised) fundamental crossed complex of the simplicial set* K to be the free crossed complex having the elements of K_n as generators in dimension n and boundary maps given by the Homotopy Addition Lemma. In detail this gives the crossed complex $\Pi^\Upsilon K$ as follows:

1. The objects are the vertices of K: $(\Pi^\Upsilon K)_0 = K_0$;
2. The groupoid $(\Pi^\Upsilon K)_1$ is the free groupoid associated to the directed graph K_1. So it has a free generator $x : \partial_1 x \to \partial_0 x$ for each $x \in K_1$;
3. The crossed module $(\Pi^\Upsilon K)_2 \to (\Pi^\Upsilon K)_1$ is the free $(\Pi^\Upsilon K)_1$-crossed module generated by the map $\delta_2 : K_2 \to (\Pi^\Upsilon K)_1$ given by
$$\delta_2 x = -\partial_1 x + \partial_2 x + \partial_0 x$$
for all $x \in K_2$. We set $\pi_1 K = \mathrm{Coker}\,\delta_2$, the *fundamental groupoid* of K.
4. For all $n \geqslant 3$, $(\Pi^\Upsilon K)_n$ is the free $\pi_1 K$-module with generators K_n and boundary given by
$$\delta_n x = \begin{cases} (\partial_3 x)^{u_3 x} - \partial_0 x - \partial_2 x + \partial_1 x & \text{if } n = 3,\\ (\partial_n x)^{u_n x} + \sum_{i=0}^{n-1} (-1)^{n-i}\partial_i x & \text{if } n \geqslant 4,\end{cases}$$

where $u_n = \partial_0^{n-1}$.

This construction is natural, giving a fundamental crossed complex functor of simplicial sets
$$\Pi^{\Upsilon} : \mathsf{Simp} \to \mathsf{Crs}. \qquad \square$$

Remark 9.2. There are two notions of realisation of a simplicial set K, usually written $\|K\|$, and $|K|$. In the first the only identifications are along faces, and in the second the degenerate simplices are also factored out. Each realisation is a CW-complex with its skeletal filtration, and the Higher Homotopy van Kampen Theorem of 2.1, [**16**], shows that there is a canonical isomorphism $\Pi^{\Upsilon} K \cong \Pi(\|K\|_*)$.

10. 0-normalisation

We first contrast with the usual case of a simplicial abelian group A, where the simplicial operators $\partial_i, \varepsilon_i$ are morphisms of abelian groups. The *associated chain complex* (A, ∂) is then A_n in dimension $n \geqslant 0$ with boundary
$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i.$$
Let $(DA)_n$ for $n \geqslant 0$ be the subgroup of A_n generated by the degenerate elements. It is an easy calculation from the rules for simplicial operators that $\partial(DA)_n \subseteq (DA)_{n-1}$ and so (DA, ∂) is a subchain complex of (A, ∂).

In the nonabelian case, the formulae cope well with the increased technicalities. For the rest of this section, K is a simplicial set.

Proposition 10.1. Let E_* be the set of degenerate elements in K, together with the elements of E_0. Then E_* is a normal structure in $\Pi^{\Upsilon} K$.

Proof. By the rules $\partial_i \varepsilon_i = \partial_{i+1}\varepsilon_i = 1$, and the Homotopy Addition Lemma, we get immediate cancellation in $\delta_n \varepsilon_i y$ for $0 < i < n-1$ but not necessarily for $i = 0, n-1$, because of the operators, and the nonabelian structures in dimensions 1,2. Thus terms of concern are:
$$\begin{aligned}
\delta_2 \varepsilon_0 y &= -y + \varepsilon_0 \partial_1 y + y,\\
\delta_3 \varepsilon_0 y &= (\varepsilon_0 \partial_2 y)^{\partial_0 y} + (-y - \varepsilon_0 \partial_1 y + y),\\
\delta_3 \varepsilon_2 y &= (y)^{\varepsilon_0 \partial_0^2 y} - \varepsilon_1 \partial_0 y - y + \varepsilon_1 \partial_0 y,\\
&= (y^{\varepsilon_0 \partial_0^2 y} - y) + (y - \varepsilon_1 \partial_0 y - y) + \varepsilon_1 \partial_0 y,
\end{aligned}$$
and for $n \geqslant 4$
$$\delta_n \varepsilon_{n-1} y = y^{\varepsilon_0 \partial_0^{n-1} y} - y + \text{terms involving } \varepsilon_{n-2}.$$
This proves the result in view of the definitions in section 5. $\square$

Definition 10.2. We define a normal subcrossed complex E_0K of $\Pi^\Upsilon K$ to be K_0 in dimension 0 and in higher dimensions to be normally generated by degenerate elements of the form $\varepsilon_0 y$. □

Definition 10.3. We define the 0-normalised crossed complex of K to be

$$\Pi^{0N}K = (\Pi^\Upsilon K)/E_0K.$$

Our first result is:

Theorem 10.4. *The projection $p^0 : \Pi^\Upsilon K \to \Pi^{0N}K$ has a section q such that $qp^0 \simeq 1$.*

The proof will occupy the rest of this section.

We first need a lemma, which will be used later as well.

Lemma 10.5. *Let $h_1 : (\Pi^\Upsilon K)_1 \to (\Pi^\Upsilon K)_2$ be a derivation. Then for $x \in K_2$ we have*

$$h_1\delta_2 x = -(h_1\partial_1 x)^{\delta_2 x} + (h_1\partial_2 x)^{\partial_0 x} + h_1\partial_0 x.$$

Proof

$$\begin{aligned} h_1\delta_2 x &= h_1(-\partial_1 x + \partial_2 x + \partial_0 x) \\ &= (h_1(-\partial_1 x + \partial_2 x))^{\partial_0 x} + h_1\partial_0 x \\ &= ((h_1(-\partial_1 x))^{\partial_2 x})^{\partial_0 x} + (h_1\partial_2 x)^{\partial_0 x} + h_1\partial_0 x \\ &= -(h_1\partial_1 x)^{\delta_2 x} + (h_1\partial_2 x)^{\partial_0 x} + h_1\partial_0 x. \end{aligned}$$ □

Lemma 10.6. *If $h : \psi \simeq 1 : \Pi^\Upsilon K \to \Pi^\Upsilon K$ is given by $h_0 = \varepsilon_0$ in dimension 0, and in dimension 1 by h_1 is ε_0 or ε_1 on the free basis given by K_1, then ψ is given in dimensions 0, 1 by*

$$\psi x = \begin{cases} x & \text{if } \dim x = 0, \\ x - \varepsilon_0\partial_0 x & \text{if } \dim x = 1, \end{cases}$$

and hence $\psi\varepsilon_0 y = 0_y$ for all $y \in K_0$.

Proof The case $\dim x = 0$ is clear. For the case $\dim x = 1$ and for $h_1 = \varepsilon_0$ we have

$$\begin{aligned} \psi x &= \varepsilon_0 sx + x + \delta_2(-\varepsilon_0 x) - \varepsilon_0 tx \\ &= \varepsilon_0\partial_1 x + x - (-x + \varepsilon_0\partial_1 x + x) - \varepsilon_0 tx \\ &= x - \varepsilon_0\partial_0 x. \end{aligned}$$

and for $h_1 = \varepsilon_1$ we have

$$\begin{aligned} \psi x &= 0_{sx} + x + \delta_2(-\varepsilon_1 x) - 0_{tx} \\ &= x + (x - x - \varepsilon_0\partial_0 x) \\ &= x - \varepsilon_0\partial_0 x. \end{aligned}$$ □

Now we define simultaneously a morphism $\psi : \Pi^{\Upsilon} K \to \Pi^{\Upsilon} K$ and a homotopy $h : \psi \simeq 1$ such that $\psi(E_0 K)$ is trivial.

Proposition 10.7 (0-normalisation). Let K be a simplicial set. Then a homotopy $(h, 1)$ on $\Pi^{\Upsilon} K$ may be defined on generators from K by $h_n = (-1)^n \varepsilon_0$, yielding $h : \psi \simeq 1$ where ψ is given on generators by

$$\psi(x) = \begin{cases} x & \text{if } \dim x = 0, \\ x - \varepsilon_0 \partial_0 x & \text{if } \dim x = 1, \\ (x - \varepsilon_0 \partial_0 x)^{-\varepsilon_0 t x} & \text{if } \dim x > 1. \end{cases}$$

This ψ satisfies

1.- $\psi(\varepsilon_0 x) = 0_{tx}$ for all $x \in K$.

2.- The induced morphism $\bar{\psi} : \Pi^{0N} K \to \Pi^{\Upsilon} K$ satisfies $p_0 \bar{\psi} = 1$ and $\psi = \bar{\psi} p_0 \simeq 1$. Thus p_0 is a homotopy equivalence.

Proof. To verify the formula for ψ requires working out a formula for $\overline{\varepsilon}_0 \delta_n x - \delta_{n+1} \varepsilon_0 x$, where $\overline{\varepsilon}_0$ is the derivation or operator morphism defined by ε_0 on generators, and we also have to use the crossed module rules.

Thus for $x \in K_2$, we have by Lemma 10.5:

$$\overline{\varepsilon}_0 \delta_2 x = -(\varepsilon_0 \partial_1 x)^{\delta_2 x} + (\varepsilon_0 \partial_2 x)^{\partial_0 x} + \varepsilon_0 \partial_0 x$$

while

$$\begin{aligned} \delta_3 \varepsilon_0 x &= (\partial_3 \varepsilon_0 x)^{\partial_0^2 \varepsilon_0 x} - x - \varepsilon_0 \partial_1 x + x \\ &= (\varepsilon_0 \partial_2 x)^{\partial_0 x} + (-\varepsilon_0 \partial_1 x)^{\delta_2 x} \\ &= (-\varepsilon_0 \partial_1 x)^{\delta_2 x} + (\varepsilon_0 \partial_2 x)^{\partial_0 x}, \quad \text{by centrality of } \delta_3 \varepsilon_0 x \end{aligned}$$

From this we get

$$-\delta_3 \varepsilon_0 x + \overline{\varepsilon}_0 \delta_2 x = \varepsilon_0 \partial_0 x.$$

More easily, we have for $n \geqslant 3$ and $x \in K_n$

$$\delta_{n+1} \varepsilon_0 x = (\varepsilon_0 \partial_n x)^{\partial_0^{n-1} x} + \sum_{i=2}^{n} (-1)^{n+1-i} \partial_i \varepsilon_0 x$$

and

$$\overline{\varepsilon}_0 \delta_n x = (\varepsilon_0 \partial_n x)^{\partial_0^{n-1} x} + \sum_{i=0}^{n-1} (-1)^{n-i} \varepsilon_0 \partial_i x$$

so that

$$\overline{\varepsilon}_0 \delta_n x - \delta_{n+1} \varepsilon_0 x = (-1)^n \varepsilon_0 \partial_0 x.$$

With these computations we get $h : \psi \simeq 1$ where ψ is the morphism given in the statement. Hence $\psi(\varepsilon_0^n v) = 0_v$ for all $n \geqslant 1$, and in fact $\psi\varepsilon_0 x = 0_{tx}$ for all $x \in K$. From this we easily deduce that $\psi(\Pi^0 K)$ is the trivial subcomplex on K_0. The morphism ψ then defines a morphism $\bar{\psi} : {\Pi^{\Upsilon}}^{0N} K \to \Pi^{\Upsilon} K$ satisfying $\bar{\psi} p_0 = 1$.

The homotopy $\overline{\varepsilon}_0$ gives also $p_0\bar{\psi} \simeq 1$. Thus $\bar{\psi}$ is a homotopy equivalence (actually a deformation retract). □

Remark 10.8. Let v be a vertex of the simplicial set K. Then in $\Pi^{\Upsilon} K$ we have

$$\delta_2(\varepsilon_0^2 v) = \varepsilon_0 v$$

and so $\varepsilon_0 v$ acts trivially on $(\Pi^{\Upsilon} K)_n$ for $n \geqslant 3$. Further for $n \geqslant 3$

$$\delta_n(\varepsilon_0^n v) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \varepsilon_0^{n-1} v & \text{if } n \text{ is even.} \end{cases} \qquad \Box$$

Proposition 10.9. *The crossed complex $\Pi^{0N} K$ is isomorphic by $\bar{\psi}$ to the (free) subcrossed complex of $\Pi^{\Upsilon} K$ on the elements of K not of the form $\varepsilon_0 y$ for $y \in K_{n-1},\ n \geqslant 1$.*

Proof. This follows from theorem 6.6. □

Remark 10.10. An advantage of working in the 0-normalised complex is that certain awkward exponents, which would vanish or not appear in the usual abelian case, now disappear in the 0-normalised complex. For example if $y \in K_1$ we have

$$\begin{aligned} \delta_2\varepsilon_1 y &= -\partial_1\varepsilon_1 y + \partial_2\varepsilon_1 y + \partial_0\varepsilon_1 y \qquad & \delta_2\varepsilon_0 y &= -\partial_1\varepsilon_0 y + \partial_2\varepsilon_0 y + \partial_0\varepsilon_y \\ &= -y + y + \varepsilon_0\partial_0 y & &= -y + \varepsilon_0\partial_1 y + y \\ &= 0_{ty} \qquad \operatorname{mod} \varepsilon_0. & &= 0_{sy} \qquad \operatorname{mod} \varepsilon_0. \end{aligned} \qquad \Box$$

Remark 10.11. There is another way of proceeding, by first reducing in $\Pi^{\Upsilon} K$ all degeneracies of the vertices. Let K_0 denote also the simplicial set on the vertices of K, and also the discrete crossed complex on the object set K_0. Then the inclusion $K_0 \to \Pi^{\Upsilon} K_0$ is a strong deformation retract, as is easily seen from Remark 10.8, with retraction $r_0 : \Pi^{\Upsilon} K \to K_0$, say. So we may form the pushout

$$\begin{tikzcd} \Pi^{\Upsilon} K_0 \arrow[r, "r_0"] \arrow[d] & K_0 \arrow[d] \\ \Pi^{\Upsilon} K \arrow[r, "r"'] & \Pi^0 K \end{tikzcd}$$

Then r is also a strong deformation retract, by the methods of the homotopy theory of crossed complexes, [**13**]. We can then apply the previous methods to $\Pi^0 K$ to factor out the 0-degeneracies. We leave details and comparisons to the reader.

11. Normalisation

Now we are able to define, in analogy with Mac Lane [**37**, §VIII.6], some further homotopies on $\Pi^{0N}(K)$ to obtain the normalisation theorem. We can model more closely the classical case on this 0-normalised crossed complex. Note that if $x \in K_n$ we write also x for the corresponding elements of both $\Pi^{\Upsilon} K$ and $\Pi^{0N} K$.

Definition 11.1. For any $k \geqslant 0$ we define a subcrossed complex $D_k K \subseteq \Pi^{0N} K$ as follows:

- $(D_k K)_0 = (\Pi^{0N} K)_0 = K_0$.
- $(D_k K)_1$ is trivial, i.e. consists only of identities.
- $(D_k K)_n$ is normally generated by $\varepsilon_i y$ for $y \in K_{n-1}$, $i \leqslant k$ and $i \leqslant n-1$.

Also, we define the *degeneracy subcomplex* $DK = \bigcup_k D_k K$, i.e. $(DK)_n = \bigcup_k (D_k K)_n$ for all $n \in \mathbb{N}$. □

Now we define a sequence of homotopies from the identity to morphisms of crossed complexes sending $D_k K$ into $D_{k-1} K$ and leaving fixed the elements up to dimension $k-1$. Then, the composition of these morphisms is well defined and kills all the degeneracy subcomplex. Let us formalise this sketch.

Definition 11.2. For any $k \geqslant 0$ we define a homotopy $(\tau^k, 1) : \Pi^{0N} K \to \Pi^{0N} K$ given on the free basis $x \in K_n$ by

$$\tau^k x = \begin{cases} 0_{tx} & \text{if } n < k, \\ (-1)^{n+k} \varepsilon_k x & \text{if } n \geqslant k. \end{cases}$$ □

Therefore, for any $k \geqslant 0$ the homotopy τ^k defines a morphism of crossed complex, $\phi^k : \Pi^{0N} K \to \Pi^{0N} K$ such that $\tau^k : \phi^k \simeq 1$. Clearly $\phi^0 = \psi$. For $n \geqslant 1$ this map is given when $x \in K_n$ by

$$\phi^k x = \begin{cases} x & \text{if } n < k, \\ x + (-1)^{k+n-1} \overline{\varepsilon}_k \delta_n x + (-1)^{k+n} \delta_{n+1} \varepsilon_k x & \text{if } k \leqslant n. \end{cases}$$

where $\overline{\varepsilon}_i$ is the extension of ε_i on the basis to a derivation or operator morphism as appropriate.

Proposition 11.3. $\phi^k : \Pi^{0N} K \to \Pi^{0N} K$ satisfies

(i) $\phi^k D_j K \subseteq D_j K$ when $j < k$, and
(ii) $\phi^k D_k K \subseteq D_{k-1} K$.

Proof. (i) By the definition of ϕ^k we have to prove the inclusion only in the case $k \leqslant n$. In this case the generators of $(D_j K)_n$ are elements $\varepsilon_i x$ for $i \leqslant \min\{j, n-1\}$, so the definition of ϕ^k is

$$\phi^k \varepsilon_i x = \varepsilon_i x + (-1)^{k+n-1} \varepsilon_k \delta_n \varepsilon_i x + (-1)^{k+n} \delta_{n+1} \varepsilon_k \varepsilon_i x.$$

Therefore, since $\varepsilon_i x \in D_j K$, which is a subcrossed complex, we have that $\delta_n \varepsilon_i x \in D_j K$. So $\delta_n \varepsilon_i x$ can be written as a combination of $\varepsilon_p y$ with $y \in K_{n-2}$, $p \leqslant \min\{j, n-2\}$. Therefore, since we have

$$\varepsilon_k \varepsilon_p = \varepsilon_p \varepsilon_{k-1} \qquad \text{if } k > p$$

we have that $\varepsilon_k \delta_n \varepsilon_i x \in D_j K$.

On the other hand, for the same reason we have $\delta_{n+1} \varepsilon_k \varepsilon_i \in D_j K$. Therefore, $\phi^k \varepsilon_i x \in D_j K$.

(ii) Now let us prove $\phi^k D_k K \subseteq D_{k-1} K$. Since $(D_k K)_1$ is trivial we have to prove this inclusion only for generators of dimension $n \geqslant 2$.

We first deal with the case $n = 2$. Suppose then $x \in K_2$. Then

$$\begin{aligned} \delta_3 \varepsilon_1 x &= (\partial_3 \varepsilon_1 x)^{\partial_0 x} - \varepsilon_0 \partial_0 x - x + x \\ &= (\partial_3 \varepsilon_1 x)^{\partial_0 x} \qquad \mod \varepsilon_0. \\ \overline{\varepsilon}_1 \delta_2 x &= (-\varepsilon_1 \partial_1 x)^{\delta_2 x} + (\varepsilon_1 \partial_2 x)^{\partial_0 x} + \varepsilon_1 \partial_0 x \end{aligned}$$

so that $\mod \varepsilon_0$ and by centrality

$$\begin{aligned} \phi^1 x &= x + \overline{\varepsilon}_1 \delta_2 x - \delta_3 \varepsilon_1 x \\ &= x - (\varepsilon_1 \partial_1 x)^{\delta_2 x} + \varepsilon_1 \partial_0 x. \end{aligned}$$

Now it is clear that, $\mod \varepsilon_0$, $x = \varepsilon_1 y$ implies $\phi^1 x = 0$.

Let $\varepsilon_i y \in (D_k K)_n$, where $i \leqslant \min\{k, n-1\}$. If $i < k$ then $\varepsilon_i y \in D_{k-1} K$ and so $\phi^k \varepsilon_i y \in D_{k-1} K$ by (i). It only remains to prove $\phi^k \varepsilon_k y \in D_{k-1} K$ for $y \in K_{n-1}$.

We have already done the case of $n \leqslant 2$. In general

$$\phi^k \varepsilon_k y = \varepsilon_k y + (-1)^{k+n-1} \varepsilon_k \delta_n \varepsilon_k y + (-1)^{k+n} \delta_{n+1} \varepsilon_k \varepsilon_k y,$$

for $y \in K_{n-1}$ with $n > 2$, and, in this case, $(D_{k-1} K)_n$ is abelian. We can write,

$$\varepsilon_k \delta_n \varepsilon_k y = \varepsilon_k (\partial_n \varepsilon_k y)^{\partial_0^{n-1} \varepsilon_k y} + \sum_{j=0}^{n-1} (-1)^{n-j} \varepsilon_k \partial_j \varepsilon_k y$$

and

$$\delta_{n+1} \varepsilon_k \varepsilon_k y = (\partial_{n+1} \varepsilon_k \varepsilon_k y)^{\partial_0^n \varepsilon_k \varepsilon_k y} + \sum_{j=0}^{n} (-1)^{n+1-j} \partial_j \varepsilon_k \varepsilon_k y.$$

Therefore $\phi^k (D_k K) \subseteq D_{k-1} K$ follows from

$$\varepsilon_k \partial_j \varepsilon_k y = \begin{cases} \varepsilon_{k-1} \varepsilon_{k-1} \partial_j y & \text{if } j < k \\ \varepsilon_k y & \text{if } j = k, k+1 \\ \varepsilon_k \varepsilon_k \partial_{j-1} y & \text{if } j > k+1 \end{cases}$$

and on the other hand,

$$\partial_j \varepsilon_k \varepsilon_k y = \begin{cases} \varepsilon_{k-1}\varepsilon_{k-1}\partial_j y & \text{if } j < k \\ \varepsilon_k y & \text{if } j = k, k+1, k+2 \\ \varepsilon_k \varepsilon_k \partial_{j-1} y & \text{if } j > k+2. \end{cases}$$

□

Now we define $\phi = \phi_0 \phi^1 \cdots \phi^k \cdots : \Pi^{0N} K \to \Pi^{0N} K$.

Notice that since $\phi^k x = x$ for $k > \dim x$, this composite is finite in each dimension.

Proposition 11.4. $\phi DK = 0$.

Proof. We have $(DK)_0 = 0$ and for $n > 0$, $(DK)_n$ is generated by $\varepsilon_i y$ where $y \in K_{n-1}$ and $i \leqslant n-1$. Therefore,

$$\phi \varepsilon_i y = \phi^0 \phi^1 \cdots \phi^n \varepsilon_i y$$

If $i = n-1$ we have that $\varepsilon_i y \in (D_n K)_n$. So,

$$\phi^n \varepsilon_i y \in D_{n-1} K, \quad \phi^{n-1} \phi^n \varepsilon_i y \in D_{n-2} K, \quad \cdots, \quad \phi^0 \cdots \phi^n \varepsilon_i y \in D_0 K.$$

If $i < n-1$ we have that $\varepsilon_i y \in (D_i K)_n$. Therefore, since $\phi^j D_i K \subseteq D_i K$ for $i < j$ we have $\phi^{i+1} \cdots \phi^n \varepsilon_i y \in D_i K$. So, as above , $\phi^0 \cdots \phi^n \varepsilon_i y \in D_0 K$. □

Definition 11.5. We define the *normalised fundamental crossed complex of the simplicial set K* by

$$\Pi K = \frac{\Pi^{0N} K}{DK}.$$

Theorem 11.6. *The quotient morphism*

$$p : \Pi^{0N} K \to \Pi K$$

is a homotopy equivalence with a section q. Further, ΠK has free generators given by the images of the non degenerate elements of K.

This follows as for the 0-normalised case in the previous section. Putting the two results together gives:

Theorem 11.7. *The quotient morphism*

$$p : \Pi^{\Upsilon} K \to \Pi K$$

is a homotopy equivalence with a section q. Further, ΠK has free generators given by the images of the non degenerate elements of K.

The crossed complex ΠK homotopy equivalent to $\Pi^{\Upsilon} K$ can be described as freely generated by the non degenerate simplices of K, with boundary maps given by the HAL, forgetting the degenerate parts. In this sense, we have two alternative descriptions of ΠK, one as just given and another in terms of coends as described in section 1. The latter is used in classifying space results in [**19**].

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