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REPARAMETRIZATIONS OF CONTINUOUS PATHS

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(*communicated by Ronald Brown*)

Abstract

A reparametrization (of a continuous path) is given by a surjective weakly increasing self-map of the unit interval. We show that the monoid of reparametrizations (with respect to compositions) can be understood via “stop-maps” that allow to investigate compositions and factorizations, and we compare it to the distributive lattice of countable subsets of the unit interval. The results obtained are used to analyse the space of traces in a topological space, i.e., the space of continuous paths up to reparametrization equivalence. This space is shown to be homeomorphic to the space of regular paths (without stops) up to increasing reparametrizations. Directed versions of the results are important in directed homotopy theory.

1. Introduction and Outline

1.1. Introduction

In elementary *differential geometry*, the most basic objects studied (after points perhaps) are *paths*, i.e., *differentiable* maps $p : I \to \mathbf{R}^n$ defined on the closed interval $I = [0, 1]$. Such a path is called *regular* if $p'(t) \neq \mathbf{0}$ for all $t \in]0, 1[$. A *reparametrization* of the unit interval I is a surjective differentiable map $\varphi : I \to I$ with $\varphi'(t) > 0$ for all $t \in]0, 1[$, i.e. a (strictly increasing) self-diffeomorphism of the unit interval.

Given a path $p : I \to \mathbf{R}^n$ and a reparametrization $\varphi : I \to I$, the paths p and $p \circ \varphi$ represent the same geometric object. In differential geometry one investigates equivalence classes (identifying p with $p \circ \varphi$ for any reparametrization φ) and their invariants, like curvature and torsion.

Motivated by applications in *concurrency theory*, a branch of theoretical Computer Science trying to model and to understand the coordination between many different processors working on a common task, we are interested in *continuous* paths $p : I \to X$ in more general topological spaces up to more general reparametrizations $\varphi : I \to I$. When the *state space* of a concurrent program is viewed as

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a topological space (typically a cubical complex; cf. [**6**]), "directed" paths in that space respecting certain "monotonicity" properties correspond to *executions*. A nice framework to handle *directed* topological spaces (with an eye to homotopy properties) is the concept of a d-space proposed and investigated by Marco Grandis in [**9**]. Essentially, a topological space comes equipped with a subset of preferred *d-paths* in the set of all paths in X, cf. Definition 4.1. Note in particular, that the reverse of a directed path in general is *not* directed; the slogan is "breaking symmetries".

We do not try to capture the *quantitative* behaviour of executions, corresponding to particular parametrizations of paths, but merely the *qualitative* behaviour, such as the order of shared resources used, or the result of a computation. Hence the objects of study are paths up to certain reparametrizations which

1. do not alter the *image* of a path, and
2. do not alter the *order of events*.

We are thus interested in general paths in topological spaces, up to *surjective* reparametrizations $\varphi : I \to I$ which are *increasing* (and thus continuous!—cf. Lemma 2.7), but not necessarily strictly increasing. Two paths are considered to have the same behaviour if they are *reparametrization equivalent*, cf. Definition 1.2.

To understand this equivalence relation, we have to investigate the space of all reparametrizations which includes strange (e.g. nowhere differentiable) elements. Nevertheless, it enjoys remarkable properties: It is a monoid, in which compositions and factorizations can be completely analysed through an investigation of *stop intervals* and of *stop values*. The quotient space after dividing out the self-homeomorphisms has nice algebraic lattice properties.

A path is called *regular* if it does not "stop"; and we are able to show that the space of general paths *modulo reparametrizations* is homeomorphic to the space of *regular* paths *modulo increasing auto-homeomorphisms* of the interval. Hence to investigate properties of the former, it suffices to consider the latter. This is one of the starting points in the homotopy theoretical and categorical investigation of invariants of d-spaces in [**15**]. Further possible areas of application of the results (and of higher-dimensionsal generalisations still to be investigated) include categorical homotopy theory as in [**8**], categorified gauge theory as in [**1**] and n-transport theory; cf. the blog "The n-category café" at `http://golem.ph.utexas.edu/category`.

This article does not build on any sophisticated machinery. Most of the concepts and proofs can be understood with an undergraduate mathematical background. There are certain parallels to the elementary theory of distribution functions in probability theory, cf. e.g. [**13**]. The flavour is nevertheless different, since continuity (no jumps, i.e., surjectivity) is essential for us. For the sake of completeness, we have chosen to include also elementary results and their proofs (some of which may be well-known).

Marco Grandis has studied piecewise linear reparametrizations in [**10**] for different purposes, but also in the framework of "directed algebraic topology".

1.2. Basic definitions

Let always X denote a Hausdorff topological space and $I = [0, 1]$ the unit interval. The set of all (nondegenerate) closed subintervals of I will be denoted by $\mathfrak{P}_{[\,]}(I) =$

$\{[a,b] \mid 0 \leqslant a < b \leqslant 1\}$. Let $p : I \to X$ denote a continuous map (a path), and remark that the pre-image $p^{-1}(x)$ of any element $x \in X$ is a closed set.

Definition 1.1.

1. An interval $J \in \mathfrak{P}_{[\,]}(I)$ is called a *p-stop interval* if the restriction $p|_J$ is constant and if J is a maximal interval with that property.
2. The set of all p-stop intervals will be denoted as $\Delta_p \subseteq \mathfrak{P}_{[\,]}(I)$. Remark that the intervals in Δ_p are disjoint and that Δ_p carries a natural total order. We let $D_p := \bigcup_{J\in\Delta_p} J \subset I$ denote the *stop set* of p.
3. A path $p : I \to X$ is called *regular* if $\Delta_p = \emptyset$ or if $\Delta_p = \{I\}$ (no stop or constant).
4. A continuous map $\varphi : I \to I$ is called a *reparametrization* if $\varphi(0) = 0, \varphi(1) = 1$ and if φ is *increasing*, i.e. if $s \leqslant t \in I$ implies $\varphi(s) \leqslant \varphi(t)$.

Remark that neither a regular path nor a reparametrization need be injective.

Definition 1.2. Two paths $p, q : I \to X$ are called *reparametrization equivalent* if there exist reparametrizations φ, ψ such that $p \circ \varphi = q \circ \psi$.

We will show later (Corollary 3.3) that reparametrization equivalence is indeed an equivalence relation. As in differential geometry, we are interested in equivalence classes of paths modulo reparametrization equivalence. We call these equivalence classes *traces*[1] in the space X. In particular, we would like to know *whether every trace can be represented by a regular path.* The (positive) answer to this question in Proposition 3.7 is based on a closer look at the space of reparametrizations of the unit interval.

1.3. Outline of the article

Section 2 contains a detailed study of reparametrizations (in their own right) and characterizes their behaviour essentially by an order-preserving bijection between the set of stop intervals and the set of stop values (Definition 1.1 and Proposition 2.13). This pattern analysis allows to study compositions, and in particular, factorizations in the monoid of reparametrizations from an algebraic point of view. In particular, Proposition 2.18 shows that the space of all reparametrizations "up to homeomorphisms" is a distributive lattice isomorphic to the lattice of countable subsets of the unit interval.

Section 3 investigates the *space* of all paths in a Hausdorff space up to reparametrization equivalence. The main result (Theorem 3.6) states that two quotient spaces are in fact homeomorphic: the orbit space arising from the action of the group of all oriented homeomorphisms of the unit interval on the space of *regular* paths (with given end points, cf. Definition 1.1) on the one side, and the space of *all* paths with given end points up to reparametrization equivalence (Definition 1.2); in particular, *every trace can be represented by a regular path.* It might be a bit surprising that the proof makes essential use of the results on factorizations of reparametrizations from Section 2.

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[1]with a geometric meaning; the notion has nothing to do with algebraic traces.

The final Section 4 deals with spaces of *directed* traces (directed paths up to reparametrization equivalence) on a d-space (cf. Section 1.1 and Definition 4.1). Corollary 4.5 confirms that the result of Theorem 3.6 has an analogue for directed paths in *saturated* (cf. Definition 4.3) d-spaces. This result is one of the starting points for the (categorical) investigations into invariants of directed spaces in [**15**]. Furthermore, it is shown how to relate reparametrization equivalence of directed paths to *thin dihomotopies*; this result is needed in the study [**5**] of directed squares ("two-dimensional paths").

2. Reparametrizations

2.1. Stop and move intervals, stop values, stop maps

The following definitions (extending Definition 1.1) and elementary results will mainly be used for reparametrizations. For the sake of generality, we will state and prove them for general paths $p : I \to X$ in a Hausdorff space X.

Definition 2.1.

1. An element $c \in X$ is called a *p-stop value* if there is a p-stop interval $J \in \Delta_p$ with $p(J) = \{c\}$. We let $C_p \subseteq X$ denote the set of all p-stop values.
2. The map p induces the *p-stop map* $F_p : \Delta_p \to C_p$ with $F_p(J) = c \Leftrightarrow p(J) = \{c\}$.
3. An interval $J \in \mathfrak{P}_{[]}(I)$ is called a *p-move interval* if it does not contain any p-stop interval and if it is maximal with that property.
4. The set of all p-move intervals will be denoted $\Gamma_p \subseteq \mathfrak{P}_{[]}(I)$, a collection of disjoint closed intervals. We let $O_p := \bigcup_{J \in \Gamma_p} \operatorname{int} J \subseteq I$ denote the *p-move set.*

Lemma 2.2. *For any path $p : I \to X$, the sets of p-stop intervals Δ_p, of p-move intervals Γ_P and of p-stop values C_p are at most countable.*

Proof. The set O_p and $\bigcup_{J \in \Delta_p} \operatorname{int} J$ of *interior points* in move, resp. stop intervals are open subsets of I and thus unions of at most *countably* many maximal open intervals. Their closures constitute Γ_p, resp. Δ_p. The stop value set C_p is at most countable as image of Δ_p under the p-stop map F_p. □

Remark 2.3. This result is similar in spirit to the assertion (relevant for distribution functions in probability theory) that a nondecreasing function to an interval has at most countably many discontinuity points, cf. e.g. [**13**, Sec. 11].

It is important to analyse the *boundary* ∂D_p of the p-stop set: It can be decomposed as $\partial D_p = \partial_1 D_p \cup \partial_2 D_p$ as follows:

- $\partial_1 D_p = \partial D_p \cap D_p$ – the set of all boundary points of intervals in D_p, an at most countable set;
- $\partial_2 D_p = \partial D_p \setminus D_p$ – the set of all (honest) accumulation points of these boundary points. $\partial_2 D_p$ can be uncountable; compare Ex. 2.11.

The move set O_p is the complement $O_p = I \setminus \overline{D_p} \subset I$ of the closure of D_p. It does occur that O_p is empty; compare Ex. 2.11.

The following elementary technical lemma concerning stop sets will be needed in the proof of Proposition 3.7.

Lemma 2.4. *Let $p : I \to X$ denote a path and $U \subseteq X$ an open subspace. Then $p^{-1}(U)$ is a union of (at most) countably many disjoint open intervals, and for any maximal open interval $]a, b[$ in $p^{-1}(U)$, $[a, c] \not\subseteq D_p$ and $[c, b] \not\subseteq D_p$ for every $c \in]a, b[$.*

Proof. As an open subset of I, $p^{-1}(U)$ is a union of (at most) countably many disjoint open intervals. Let $]a, b[$ be one of these, and let $c \in]a, b[$. Then $p(c) \in U, p(a) \notin U, p(b) \notin U$. In particular, p is neither constant on $[a, c]$ nor on $[c, b]$. □

2.2. Spaces of reparametrizations

Within the set of all self-maps of the unit interval I fixing its boundary points, we study the following subsets:

Definition 2.5.

- $\mathrm{Mon}_+(I) := \{\varphi : I \to I \mid \varphi \text{ increasing}, \varphi(0) = 0, \varphi(1) = 1\}$;
- $\mathrm{Rep}_+(I) := \{\varphi \in \mathrm{Mon}_+(I) \mid \varphi \text{ continuous }\}$ –
  the set of all increasing reparametrizations;
- $\mathrm{Homeo}_+(I) = \{\rho \in \mathrm{Rep}_+(I) \mid \rho \text{ strictly increasing }\}$ –
  the set of all increasing auto-homeomorphisms of the interval.

Note that $\mathrm{Homeo}_+(I) \subset \mathrm{Rep}_+(I) \subset \mathrm{Mon}_+(I)$. The compact-open topology on the space of *all* continuous maps $C(I, I)$ induces topologies on the latter two spaces $\mathrm{Rep}_+(I)$ and $\mathrm{Homeo}_+(I)$. Composition $\circ$ of maps turns $\mathrm{Mon}_+(I)$ into a monoid, $\mathrm{Rep}_+(I)$ into a topological monoid and $\mathrm{Homeo}_+(I)$ into a topological group (consisting of the units in $\mathrm{Rep}_+(I)$).

All three mapping sets come equipped with a natural partial order: $\varphi \leqslant \psi$ if and only if $\varphi(t) \leqslant \psi(t)$ for all $t \in I$, and they form complete lattices with respect to $\leqslant$. Least upper bounds, resp. greatest lower bounds are given by the max, resp. min of the functions involved:

$$(\varphi \vee \psi)(t) := \max\{\varphi(t), \psi(t)\} \qquad (\varphi \wedge \psi)(t) := \min\{\varphi(t), \psi(t)\}$$

Lemma 2.6.

1. *All three sets $\mathrm{Mon}_+(I), \mathrm{Rep}_+(I), \mathrm{Homeo}_+(I)$ are convex. In particular, the latter two spaces are contractible.*
2. *Any two reparametrizations $\varphi, \psi \in \mathrm{Rep}_+(I)$ are d-homotopic (cf. Definition 4.7 for the general definition), i.e. there exists a reparametrization $\varphi, \psi \leqslant \eta \in \mathrm{Rep}_+(I)$ and increasing paths $G, H : \vec{I} \to \mathrm{Rep}_+(I)$ with $G(0) = \varphi, H(0) = \psi, G(1) = H(1) = \eta$.*

Proof. 1. The sets are closed under convex combinations $(1 - s)\varphi + s\psi$.

2. For $\eta = \varphi \vee \psi$, define $G(s) = (1 - s)\varphi + s\eta$ and $H(s) = (1 - s)\psi + s\eta$.

□

A characterization of the elements of $\mathrm{Mon}_+(I)$, $\mathrm{Rep}_+(I)$, and $\mathrm{Homeo}_+(I)$ is achieved in the elementary

Lemma 2.7. *Let $\varphi \in \mathrm{Mon}_+(I)$.*

1. *For every interval $J \subseteq I$, the pre-image $\varphi^{-1}(J) \subseteq I$ is an interval, as well. In particular, $\varphi^{-1}(a)$ is an interval (possibly degenerate) for every $a \in I$.*
2. *$\varphi \in \mathrm{Rep}_+(I)$ if and only if φ is* surjective.
3. *$\varphi \in \mathrm{Homeo}_+(I)$ if and only if φ is* bijective*.*

Proof. The only non-obvious statement is that surjectivity of $\varphi \in \mathrm{Mon}_+(I)$ implies continuity; we show that the pre-image $\varphi^{-1}(J)$ of an open interval $J \subset I$ is open:

Let $d \in \varphi^{-1}(J)$ and $\varphi(d) = c \in J$. Then there exist $\varepsilon > 0$ such that $[c-\varepsilon, c+\varepsilon] \subseteq J$ and $d_1, d_2 \in I$ such that $\varphi(d_1) = c - \varepsilon, \varphi(d_2) = c + \varepsilon$. Monotonicity implies: $\varphi([d_1, d_2]) \subseteq [c - \varepsilon, c + \varepsilon]$ and $d' \notin [d_1, d_2] \Rightarrow \varphi(d') \notin]c - \varepsilon, c + \varepsilon[$. Surjectivity implies: $\varphi([d_1, d_2]) = [c - \varepsilon, c + \varepsilon] \subseteq J$; hence d has an open neighbourhood in $\varphi^{-1}(J)$. □

The following information about images of intervals under reparametrizations is needed in the proof of Proposition 3.7:

Lemma 2.8. *Let $a, b \in I$ and $\varphi \in \mathrm{Rep}_+(I)$. Then*

1. *$\varphi([a, b]) = [\varphi(a), \varphi(b)]$.*
2. *$]\varphi(a), \varphi(b)[\;\subseteq \varphi(]a, b[) \subseteq [\varphi(a), \varphi(b)]$.*
3. *$\varphi(]a, b[) \neq]\varphi(a), \varphi(b)[$ if and only if there is $c \in]a, b[$ such that $[a, c] \subseteq D_\varphi$ or $[c, b] \subseteq D_\varphi$.*

Proof. Only the last assertion requires proof. If $[a, c] \subseteq D_\varphi$ for some $c \in]a, b[$, then $\varphi(a) = \varphi(c) \in \varphi(]a, b[)$; similarly, if $[c, b] \subseteq D_\varphi$, then $\varphi(b) \in \varphi(]a, b[)$. For the reverse direction, assume $\varphi(a) \in \varphi(]a, b[)$, and let $c \in]a, b[$ such that $\varphi(a) = \varphi(c)$. Then $\varphi(a) \leqslant \varphi(t) \leqslant \varphi(c) = \varphi(a)$ for any $t \in [a, c]$, hence $[a, c] \subseteq D_\varphi$. The other implication is similar. □

The following result deals with the relative size of the homeomorphisms within the reparametrizations. It will be needed in the proof of the main result in Section 3.

Lemma 2.9. *In the topology induced from the compact-open topology, both* $\mathrm{Homeo}_+(I)$ *and its complement are dense in* $\mathrm{Rep}_+(I)$*.*

Proof. The compact-open topology is induced by the supremum metric on the space $C(I, I)$ of all self-maps of the interval. Hence, for a given $\varphi \in \mathrm{Rep}_+(I)$ and $n \in \mathbf{N}$, we need to construct $\rho \in \mathrm{Homeo}_+(I)$ such that $\|\varphi - \rho\| \leqslant \frac{1}{n}$: Choose $c_k, 0 \leqslant k \leqslant n$, such that $c_0 = 0, c_n = 1$ and $\varphi(c_k) = \frac{k}{n}$; clearly c_k is strictly increasing with k. Hence the piecewise linear map ρ given by $\rho(c_k) = \frac{k}{n}$ is contained in $\mathrm{Homeo}_+(I)$. Furthermore, for $x \in [c_k, c_{k+1}], k < n$, we have $\frac{k}{n} \leqslant \rho(x), \varphi(x) \leqslant \frac{k+1}{n}$, and thus $\|\varphi - \rho\| \leqslant \frac{1}{n}$.

For the same $\varphi \in \mathrm{Rep}_+(I)$ and the same definition for c_0 and c_1 as above, let $\psi \in \mathrm{Rep}_+(I) \setminus \mathrm{Homeo}_+(I)$ be given by $\psi(x) = \varphi(x), x \geqslant c_1$; on the interval $[0, c_1]$, we let ψ be the piecewise linear map with $\psi(0) = \psi(\frac{c_1}{2}) = 0$ and $\psi(c_1) = \varphi(c_1)$. See also Figure 1. □

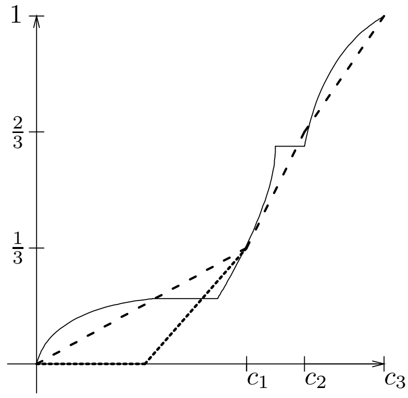

Figure 1: Reparametrization φ (full line), homeomorphism ρ (broken line), and reparametrization ψ (dotted line).

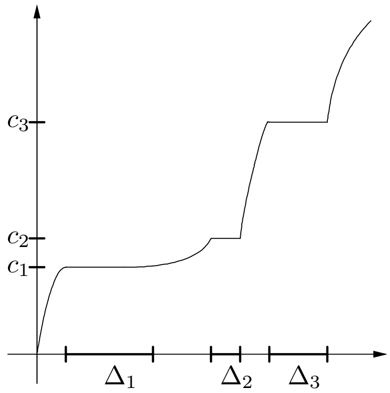

Figure 2: Stop intervals and stop values

2.3. Classification of reparametrizations

In the following, we are mainly interested in an investigation of the algebraic monoid structure on $\mathrm{Rep}_+(I)$ induced by *composition* $\circ$ of maps. Note that there is another structure on the sets (spaces) $\mathrm{Mon}_+(I)$, $\mathrm{Rep}_+(I)$, and $\mathrm{Homeo}_+(I)$, induced by *concatenation* of paths

$$(\varphi, \psi) \mapsto \varphi * \psi; \quad (\varphi * \psi)(t) = \begin{cases} \varphi(2t) & \text{for } t \leqslant \frac{1}{2} \\ \psi(2t-1) & \text{for } t > \frac{1}{2} \end{cases} \tag{2.1}$$

This composition does not induce a monoidal structure on these sets, as concatenation is not associative and does not have units "on the nose".

We wish to describe a reparametrization $\varphi \in \mathrm{Rep}_+(I)$ by its φ-stop map $F_\varphi : \Delta_\varphi \to C_\varphi$ illustrated in Figure 2 and by the restriction of φ to its φ-move set $O_\varphi \subseteq I$; cf. Definition 2.1.

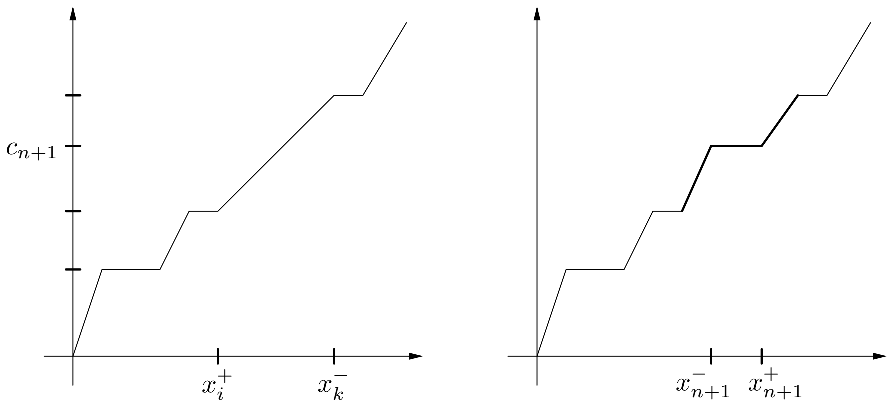

Figure 3: Inserting the stop value c_{n+1}

2.3.1. All countable sets in the interval are stop value sets

Lemma 2.2 tells us that the φ-stop value set $C_\varphi \subset I$ of a reparametrization φ is an at most countable ordered subset of I. Remark also that automorphisms $\rho \in \mathrm{Homeo}_+(I)$ are characterized by the properties $\Delta_\rho = C_\rho = \emptyset$, resp. $O_\rho = I$.

Which (countable) subsets of the unit interval can be realized as φ-stop sets of some $\varphi \in \mathrm{Rep}_+(I)$? It is easy to construct (piecewise linear) reparametrizations with a *finite* set of stop values. Rather surprisingly, this construction can be extended to arbitrary (at most) *countable* sets of stop values:

Lemma 2.10. *For every countable set $C \subset I$, there is a reparametrization $\varphi \in \mathrm{Rep}_+(I)$ with $C_\varphi = C$.*

Proof. Let $C = \{c_1, c_2, \dots\} \subset I$ denote an injective enumeration of the countable set C. We shall first construct a uniformly convergent sequence of piecewise linear maps $\varphi_n \in \mathrm{Rep}_+(I), n \geqslant 0$ with $C(\varphi_n) = \{c_1, \dots, c_n\}$ and thus $\Delta_{\varphi_n} = \{\varphi_n^{-1}(c_i) | 1 \leqslant i \leqslant n\}$. Let $[x_i^-, x_i^+] = \varphi_n^{-1}(c_i), 1 \leqslant i$; moreover, $x_0^- = 0, x_0^+ = 1$.

We start with $\varphi_0 = \mathrm{id}_I$. Inductively, assume φ_n given as above. Among the $x_j^\pm, 1 \leqslant j \leqslant n$, choose x_i^+, x_k^- such that $c_{n+1} \in \varphi(]x_i^+, x_k^-[)$ and such that the restriction of φ_n on that interval is strictly increasing (and linear). The map φ_{n+1} will differ from φ_n only on (the interior of) that subinterval $[c_i, c_k]$. The linear map on that interval is replaced by a piecewise linear map, which comes in three pieces. The middle one takes the constant value c_{n+1} on a subinterval $[x_{n+1}^-, x_{n+1}^+]$. On the left and right subinterval, we connect linearly to the values c_i, c_k on the boundaries. See also Figure 3.

The interval $[x_{n+1}^-, x_{n+1}^+]$ is chosen so small that $||\varphi_{n+1} - \varphi_n||_\infty < \frac{1}{2^n}$ ensuring uniform convergence of the maps φ_n to a continuous map $\varphi \in \mathrm{Rep}_+(I)$. For this map φ, we have $\Delta_\varphi = \{[x_n^-, x_n^+] | n > 0\}$ and $C_\varphi = C$. ☐

Example 2.11. If one chooses a *dense* countable subset $C \subset I$, e.g., $C = \mathbf{Q} \cap I$, then φ cannot be injective on *any* non-trivial interval; hence $O_\varphi = \emptyset$ and $\overline{D_\varphi} = I$.

The (uncountable) complement $I \setminus D_\varphi = \partial_2 D_\varphi$ does not contain any non-trivial interval.

2.3.2. Classification

What are the essential data to describe a reparametrization in terms of stop maps and move sets?

Proposition 2.12. *Let $\varphi \in \mathrm{Rep}_+(I)$ denote a reparametrization.*

1. *The reparametrization φ induces an* order-preserving bijection $F_\varphi : \Delta_\varphi \to C_\varphi$*. The restriction $\varphi|_J : J \to \varphi(J)$ to every move interval $J \in \Gamma_\varphi$ (Definition 2.1) is an (increasing) homeomorphism onto its image.*
2. *The restriction $\varphi|_{\overline{D_\varphi}} : \overline{D_\varphi} \to \overline{C_\varphi}$ of φ to $\overline{D_\varphi}$ is* onto.
3. *Two reparametrizations $\varphi, \psi \in \mathrm{Rep}_+(I)$ with $\Delta_\varphi = \Delta_\psi, C_\varphi = C_\psi, F_\varphi = F_\psi$ agree on $\overline{D_\varphi}$; if, moreover, $\varphi|_{O_\varphi} = \psi|_{O_\psi}$, then φ and ψ agree on all of I.*

Proof.

1. The first statement is obvious from the definitions. For the second, note that $J \cap D_\varphi = \partial J$ (or empty) for every such interval J.
2. Every element $b \in \overline{C_\varphi}$ is the limit of a monotone sequence of elements in C_φ which is the image of a monotone and bounded sequence of elements in D_φ; the limit of such a sequence exists and maps to b under φ.
3. By definition, φ and ψ agree on D_φ; by continuity, they have to agree on its closure $\overline{D_\varphi}$, as well. The last statement is obvious.

□

A reparametrization $\varphi \in \mathrm{Rep}_+(I)$ is thus *uniquely* characterized by its stop map $F_\varphi : \Delta_\varphi \to C_\varphi$ and by a (fitting) collection of homeomorphisms $\varphi|_{\overline{J}} : \overline{J} \to \varphi(\overline{J})$, $J \in \Gamma_\varphi$. Now we ask which conditions an "abstract" stop map has to satisfy in order to arise from a genuine reparametrization. We start with the following data:

- $\Delta \subseteq \mathfrak{P}_{[]}(I)$ denotes an (at most) countable subset of *disjoint closed* intervals – with a natural total order.
- $C \subseteq I$ denotes a subset with the same cardinality as Δ.
- $F : \Delta \to C$ denotes an order-preserving bijection.

Let $\Delta_-, \Delta_+ \subseteq I$ denote the set of lower, resp. upper boundaries of intervals in Δ. Define $D := \bigcup_{J \in \Delta} \Delta \subset I$.

Let $O = I \setminus \overline{D}$. Since O is open, it is a disjoint union $O = \bigcup_{J \in \Gamma} J$ of maximal open intervals indexed by an (at most) countable set Γ – possibly empty.

For every map $G : \Delta \to C$ we define a map $\varphi_G : D \to C$ by $\varphi_G(t) = G(J) \Leftrightarrow t \in J$. If F is order-preserving, then φ_F is increasing. Moreover:

Proposition 2.13.

1. *A reparametrization $\varphi \in \mathrm{Rep}_+(I)$ satisfies the following for every pair of strictly monotonely converging sequences $x_n \uparrow x$, $y_n \downarrow y$ for which $x, x_n \in (\Delta_\varphi)_+$ and $y, y_n \in (\Delta_\varphi)_-$:*

$$x = y \quad \Rightarrow \quad \lim \varphi(x_n) = \lim \varphi(y_n), \tag{2.2}$$
$$x < y \quad \Rightarrow \quad \lim \varphi(x_n) < \lim \varphi(y_n), \tag{2.3}$$
$$x = 1 \quad \Rightarrow \quad \lim \varphi(x_n) = 1, \tag{2.4}$$
$$y = 0 \quad \Rightarrow \quad \lim \varphi(y_n) = 0, \tag{2.5}$$

2. *For every order preserving bijection $F : \Delta \to C$ with φ_F satisfying* (1)–(4) *above for every pair of strictly monotonely converging sequences $x_n \uparrow x$, $y_n \downarrow y$ for which $x, x_n \in (\Delta_\varphi)_+$ and $y, y_n \in (\Delta_\varphi)_-$, there exists a reparametrization $\psi \in \mathrm{Rep}_+(I)$ with $\Delta_\psi = \Delta, C_\psi = C$ and $F_\psi = F$. The set of all such reparametrizations is in one-to-one correspondence with $\prod_\Gamma \mathrm{Homeo}_+(I)$.*

Proof.

1. By continuity, $\lim \varphi(x_n) = \varphi(x)$ and $\lim \varphi(y_n) = \varphi(y)$; this settles all but (2.3). Suppose $\varphi(x) = \varphi(y)$ in (2). Then $[x, y]$ is contained in a stop interval, hence $x \notin \Delta_+$ and $y \notin \Delta_-$.

2. First, we extend φ_F to $\overline{D}$: there is a *unique continuous* (and increasing!) extension of φ_F from D to $\overline{D}$: $\lim_{x_n \uparrow x} \varphi_F(x_n)$ exists and is independent of the sequence x_n by monotonicity and agrees with $\lim_{y_n \downarrow x} \varphi_F(y_n)$ by condition (2.2) of Proposition 2.13. Moreover, we let $\varphi_F(0) = 0$ and $\varphi_F(1) = 1$, in accordance with (3) and (4) above.
   Let $J =]a^J_-, a^J_+[\in \Gamma$ denote a maximal open interval. Its boundary points a^J_-, a^J_+ are contained in ∂D unless possibly if $a^J_- = 0$ and/or $a^J_+ = 1$, in which case we are covered by (2.4) and/or (2.5) above. In conclusion, φ_F is defined on ∂J. Moreover, $\varphi_F(a^J_-) < \varphi_F(a^J_+)$, since F is order preserving and injective and because of condition (2.3) above.
   Hence, every collection of strictly increasing homeomorphisms between $[a^J_-, a^J_+]$ and $[\varphi_F(a^J_-), \varphi_F(a^J_+)]$ – preserving endpoints – extends φ_F to a continuous increasing map $\psi : I \to I$ with $\Delta_\psi = \Delta, C_\psi = C$ and $F_\psi = F$. The set of all collections of such homeomorphisms is easily seen to be in one-to-one correspondence with $\prod_\Gamma \mathrm{Homeo}_+(I)$.

$\square$

2.4. Compositions and Factorizations

We shall now investigate the behaviour of $\mathrm{Rep}_+(I)$ under composition and factorization in view of the description and classification from Proposition 2.13 above. We need to introduce the following notation: For a (continuous) map $\psi : I \to I$, let $\psi_*, (\psi^{-1})^* : \mathfrak{P}_{[]}(I) \to \mathfrak{P}_{[]}(I)$ denote the maps induced on subintervals: $\psi_*(J) = \psi(J) \in \mathfrak{P}_{[]}(I), (\psi^{-1})^*(J) = \psi^{-1}(J)$.

2.4.1. Composition of reparametrizations

The results below follow easily from the definitions of stop-intervals, stop-values and stop-maps:

Lemma 2.14. *Let $\varphi, \psi \in \mathrm{Rep}_+(I)$ denote reparametrizations with associated stop maps $F_\varphi : \Delta_\varphi \to C_\varphi, F_\psi : \Delta_\psi \to C_\psi$. Then*

1. $\Delta_{\varphi\circ\psi} = \{J \in \Delta_\psi \mid F_\psi(J) \notin D_\varphi\} \cup (\psi^{-1})^*(\Delta_\varphi)$,
2. $C_{\varphi\circ\psi} = \varphi(C_\psi) \cup C_\varphi$,
3. $F_{\varphi\circ\psi} : \Delta_{\varphi\circ\psi} \to C_{\varphi\circ\psi}$ *is given by* $F_{\varphi\circ\psi}(J) = \begin{cases} \varphi(F_\psi(J)), & J \in \Delta_\psi \\ F_\varphi(\psi_*(J)), & J \in (\psi^{-1})^*(\Delta_\varphi). \end{cases}$

□

Corollary 2.15. *Let $\varphi, \psi \in \mathrm{Rep}_+(I)$ as in Lemma 2.14. If $\psi \in \mathrm{Homeo}_+(I)$, resp. $\varphi \in \mathrm{Homeo}_+(I)$, then*

1. $\Delta_{\varphi\circ\psi} = (\psi^{-1})^*(\Delta_\varphi)$, *resp.* $\Delta_{\varphi\circ\psi} = \Delta_\psi$,
2. $C_{\varphi\circ\psi} = C_\varphi$, *resp.* $C_{\varphi\circ\psi} = \varphi(C_\psi)$,
3. $F_{\varphi\circ\psi} = F_\varphi \circ \psi_* : \psi_*^{-1}(\Delta_\varphi) \to C_\varphi$, *resp.* $F_{\varphi\circ\psi} = \varphi \circ F_\psi : \Delta_\psi \to \varphi(C_\psi)$.

□

2.4.2. Factorizations of reparametrizations

Also factorizations can be studied effectively using stop-data of the reparametrizations involved. It turns out that the following result on factorizations on the *right* will be an essential tool in Section 3:

Proposition 2.16. *Let $\eta, \varphi \in \mathrm{Rep}_+(I)$ denote reparametrizations.*

1. *There exists a lift $\psi \in \mathrm{Rep}_+(I)$ in the diagram*

$$\begin{tikzcd} & I \arrow[d, "\varphi"] \\ I \arrow[ur, dashed, "\psi"] \arrow[r, "\eta"'] & I \end{tikzcd} \tag{2.6}$$

   if and only if $C_\varphi \subseteq C_\eta$.

2. *If $C_\varphi \subseteq C_\eta$ and C is any (at most) countable set with $\varphi^{-1}(C_\eta \setminus C_\varphi) \subseteq C \subseteq \varphi^{-1}(C_\eta \setminus C_\varphi) \cup D_\varphi$, then there exists such a lift $\psi \in \mathrm{Rep}_+(I)$ with $C_\psi = C$. In particular, if $C_\varphi = C_\eta$, there exist a lift $\psi \in \mathrm{Homeo}_+(I)$.*
3. *Assume $C_\varphi \subseteq C_\eta$ and let $\Delta_1 := F_\eta^{-1}(C_\varphi) \subseteq \Delta_\eta$. Then the space of all lifts $\{\psi \mid \eta = \varphi\circ\psi\} \subseteq \mathrm{Rep}_+(I)$ is in one-to-one-correspondence with $\prod_{L\in\Delta_1} \mathrm{Rep}_+(L) = \prod_{\Delta_1} \mathrm{Rep}_+(I)$.*

Proof. The "only if" part of 1. follows immediately from Lemma 2.14.2. For the "if" part, we analyse first the set-theoretic requirements to a lift ψ on relevant subintervals. To this end, decompose $\Delta_\eta =: \Delta_1 \sqcup \Delta_2$ with $\Delta_1 := F_\eta^{-1}(C_\varphi)$ and $\Delta_2 = F_\eta^{-1}(C_\eta \setminus C_\varphi)$, and $D_\eta = D_1 \sqcup D_2$ with $D_1 := \bigcup_{J\in\Delta_1}$ and $D_2 = \bigcup_{J\in\Delta_2}$. We construct a lift $\psi : I = D_1 \cup D_2 \cup O_\eta \to I$ by considering each of these three subsets of I:

Remark that Δ_2 necessarily has to be a subset of Δ_ψ and that for $J \in \Delta_2$, $F_\psi(J)$ has to be the unique element of $\varphi^{-1}(F_\eta(J))$.

On any move interval $K \in \Gamma_\eta$ (cf. Definition 2.1.4), the restriction $\eta\,|_K : K \to \eta(K)$ is an increasing homeomorphism; in particular, $\eta(K) \cap C_\eta$ consists at most of the two boundary points. Hence, the restriction $\varphi\,|_{\varphi^{-1}\eta(K)} : \varphi^{-1}\eta(K) \to \eta(K)$ is also an increasing homeomorphism, since $\eta(K) \cap C_\varphi \subseteq \eta(K) \cap C_\eta$ again consists at most of the two boundary points. The restriction of ψ to K has to be defined as $\psi\,|_K = (\varphi\,|_{\varphi^{-1}\eta K})^{-1} \circ \eta\,|_K$; it is onto $\varphi^{-1}\eta K$.

On any interval $L \in \Delta_1$, the restriction of ψ to L can be defined as *any* increasing continuous map $\psi\,|_L : L \to F_\varphi^{-1}(F_\eta(L)) \in \Delta_\varphi$ respecting the boundary points.

The map $\psi : \vec{I} \to \vec{I}$ thus defined altogether is by definition a lift, it is increasing and surjective. Lemma 2.7.2 settles 1.

The only freedom in the construction of a lift ψ is the choice of increasing continuous maps on the intervals $L \in \Delta_1$ (with given end points). As in Lemma 2.10, we can construct the set of stop values (on D_1) to be any countable subset of D_φ. To these one has of course to add the pre-images of the stop values in $C_\eta \setminus C_\varphi$. This settles 2. and 3. $\square$

We will also make use of the following result on factorizations on the *left*.

Proposition 2.17. *Let $\eta, \varphi \in \mathrm{Rep}_+(I)$ denote reparametrizations.*

1. *There exists a factorization with $\psi \in \mathrm{Rep}_+(I)$ in the diagram*

$$\begin{tikzcd} I \arrow[r, "\eta"] \arrow[d, "\varphi"'] & I \\ I \arrow[ur, dashed, "\psi"'] & \end{tikzcd} \tag{2.7}$$

   if and only if there exists a map $i_{\varphi\eta} : \Delta_\varphi \to \Delta_\eta$ such that $J \subseteq i_{\varphi\eta}(J)$ for every $J \in \Delta_\varphi$. (Δ_φ is a refinement *of Δ_η).*

2. *If it exists, the factor $\psi \in \mathrm{Rep}_+(I)$ is uniquely determined and satisfies*
   - $C_\psi = C_\eta \setminus \{F_\eta(J) \mid J \in \Delta_\eta \cap \Delta_\varphi\}$,
   - $\Delta_\psi = \{\varphi(K) \mid K \in \Delta_\eta \setminus \Delta_\varphi\}$,
   - *if $K \in \Delta_\psi$, then $F_\psi(K)$ is the unique element of $\eta(\varphi^{-1}(K))$, $K \in \Delta_\psi$.*

Proof. A lift ψ as in (2.7) has to satisfy $\psi(x) := \eta(\varphi^{-1}(x))$. It is well-defined (and then unique and increasing) if and only if the condition of Proposition 2.17 is satisfied. Since η is onto, ψ is onto as well, and thus continuous by Lemma 2.7.2. The description of the invariants of ψ follows by inspection. $\square$

2.5. The algebra of reparametrizations up to homeomorphisms

Consider the group action $\mathrm{Rep}_+(I) \times \mathrm{Homeo}_+(I) \to \mathrm{Rep}_+(I)$ given by composition on the right. An element in the quotient space $\mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)}$ preserves the set of stop values, whereas the exact distribution of stop intervals over the interval is factored out. Using the factorization tools from Section 2.4 above, this intuition will be made more formal in Propositions 2.18 and 2.22 below.

Consider the preorder on $\mathrm{Rep}_+(I)$ (different from the one considered in Section 2.2) given by $\varphi \leqslant \psi \Leftrightarrow \exists\ \eta \in \mathrm{Rep}_+(I) : \psi = \varphi \circ \eta$ ($\Leftrightarrow C_\varphi \subseteq C_\psi$ by Proposition 2.16.1). This preorder factors to yield a partial order on the quotient $\mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)}$ since $\psi = \varphi \circ \eta = (\varphi \circ \rho) \circ (\rho^{-1} \circ \eta)$ for $\rho \in \mathrm{Homeo}_+(I)$.

Moreover, let us consider the set $\mathfrak{P}_c(I)$ of *countable* subsets of I with the partial order given by inclusion.

Proposition 2.18. *The map* $C : \mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)} \to \mathfrak{P}_c(I)$ *given by* $C(\varphi) = C_\varphi$ *is an order-preserving bijection.*

Proof. By Corollary 2.15.2, the map C is well-defined; by Lemma 2.14, it is order-preserving, and by Lemma 2.10, it is surjective. Given two reparametrizations with the same set of stop values, Proposition 2.16 shows that one can construct a lift ψ from one into the other that is a homeomorphism ($C_\psi = \emptyset$); as a consequence, C is also injective. □

Proposition 2.19. *For every* $\varphi_1, \varphi_2 \in \mathrm{Rep}_+(I)$*, there exist* $\psi_1, \psi_2 \in \mathrm{Rep}_+(I)$ *completing the diagram*

$$\begin{tikzcd} I \arrow[r, dashed, "\psi_1"] \arrow[d, dashed, "\psi_2"'] & I \arrow[d, "\varphi_1"] \\ I \arrow[r, "\varphi_2"'] & I \end{tikzcd}$$

with $C_{\varphi_1 \circ \psi_1} = C_{\varphi_2 \circ \psi_2} = C_{\varphi_1} \cup C_{\varphi_2}$.

Proof. Using Lemma 2.10, construct $\psi_1 \in \mathrm{Rep}_+(I)$ with $C_{\psi_1} = \varphi_1^{-1}(C_{\varphi_2} \setminus C_{\varphi_1})$ and hence $C_{\varphi_1 \circ \psi_1} = C_{\varphi_1} \cup C_{\varphi_2}$ (cf. Lemma 2.14.2). Using Proposition 2.16, construct a lift ψ_2 in the diagram

$$\begin{tikzcd} & I \arrow[d, "\varphi_2"] \\ I \arrow[ur, dashed, "\psi_2"] \arrow[r, "\varphi_1 \circ \psi_1"'] & I \end{tikzcd}$$

with $C_{\psi_2} = \varphi_2^{-1}(C_{\varphi_1} \setminus C_{\varphi_2})$, i.e., without introducing superfluous extra stop values. □

Remark 2.20. In general, it is not possible to complete the dual diagram

$$\begin{tikzcd} I \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & I \arrow[d, dashed, "\psi_1"] \\ I \arrow[r, dashed, "\psi_2"'] & I \end{tikzcd}$$

since $D_{\psi_1 \circ \varphi_1} = D_{\psi_2 \circ \varphi_2} \supseteq D_{\varphi_1} \cup D_{\varphi_2}$. The latter set might be the entire interval I which is impossible for a reparametrization.

Is there a natural way to construct from two reparametrizations a third one (a common factor) with a set of stop values that is just the *intersection* of the sets of stop values of the given ones? In order to have the stop intervals appear in proper order, it is necessary to modify one of the reparametrizations by a homeomorphism first (which is not a problem if one works in the quotient $\mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)}$!)

Proposition 2.21. *For every $\varphi_1, \varphi_2 \in \mathrm{Rep}_+(I)$, there exist $\rho \in \mathrm{Homeo}_+(I)$, $\psi_1, \psi_2, \varphi \in \mathrm{Rep}_+(I)$ completing the diagram*

$$\begin{tikzcd}
I \arrow[r, "\varphi_1"] \arrow[d, dashed, "\rho"'] \arrow[ddr, dashed, "\varphi"] & I \\
I \arrow[d, "\varphi_2"'] & \\
I & I \arrow[l, dashed, "\psi_2"] \arrow[uu, dashed, "\psi_1"']
\end{tikzcd}$$

and such that $C_\varphi = C_{\varphi_1} \cap C_{\varphi_2}$.

Proof. Define $\Delta_\varphi := F_{\varphi_1}^{-1}(C_{\varphi_1} \cap C_{\varphi_2}) \subseteq \Delta_{\varphi_1}, C_\varphi := C_{\varphi_1} \cap C_{\varphi_2}$ and define $F_\varphi : \Delta_\varphi \to C_\varphi$ as the restriction of F_{φ_1}. By Proposition 2.13, there exists $\varphi \in \mathrm{Rep}_+(I)$ with F_φ as its stop map. By Proposition 2.17, there exists a lift $\psi_1 \in \mathrm{Rep}_+(I)$ in the right triangle of the diagram above.

In general, the reparametrization φ constructed above does not factor over φ_2 immediately, cf. Proposition 2.17. We need a "correction" homeomorphism $\rho \in \mathrm{Homeo}_+(I)$ whose restriction to D_φ fits into

$$\begin{tikzcd}
D_\varphi \arrow[r, dashed, "\rho"] \arrow[d, "F_\varphi"'] & D_{\varphi_2} \arrow[d, "F_{\varphi_2}"] \\
C_\varphi \arrow[r, "\subseteq"] & C_{\varphi_2}.
\end{tikzcd}$$

On intervals $J \in \Delta_\varphi$, ρ can be chosen as the increasing linear map sending J onto $F_{\varphi_2}^{-1}(F_\varphi(J))$ and then extended from D_φ to I as a homeomorphism as in the proof of Proposition 2.13. The condition of Proposition 2.17 is now satisfied to guarantee a lift of φ over $\varphi_2 \circ \rho$ in the left triangle of the diagram above. $\square$

Using the bijection from Proposition 2.18, one may introduce binary operations on the quotient $\mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)}$ in a purely algebraic manner, i.e., one may pull back the operations given by set union and intersection on $\mathfrak{P}_c(I)$. The results of this section allow us to give these operations an intrinsic meaning in terms of reparametrizations. Using the notation from Proposition 2.19, an operation $\vee$ ("least common multiple") is defined by $[\varphi_1] \vee [\varphi_2] := [\varphi_1 \circ \psi_1] = [\varphi_2 \circ \psi_2]$. Likewise, $[\varphi_1] \wedge [\varphi_2]$ can be represented by the reparametrization φ from Proposition 2.21. Altogether we obtain:

Proposition 2.22. *The operations*

$$\vee, \wedge : \mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)} \times \mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)} \to \mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)}$$

turn $\mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)}$ *into a distributive lattice with the class represented by* $\mathrm{Homeo}_+(I)$ *as a global minimum. The map*

$$C : (\mathrm{Rep}_+(I)/_{\mathrm{Homeo}_+(I)}, \vee, \wedge) \to (\mathfrak{P}_c(I), \cup, \cap)$$

from Proposition 2.18 is then an isomorphism of distributive lattices. □

3. Spaces of traces

3.1. The compact-open topology on path spaces

We now turn to spaces of equivalence classes of paths in a Hausdorff space X. To start with, we give a handy characterization of the compact-open topology of the space of all paths $P(X) = X^I$ in X. By definition, is has as a basis the sets $P(X)(\mathcal{C}, \mathcal{V}) := \{p \in P(X) \mid p(C_i) \subset V_i\}$, indexed by collections $\mathcal{C} = \{C_1, \dots, C_n\}$ of compact subsets $C_i \subseteq I$, resp. $\mathcal{V} = \{V_1, \dots, V_n\}$ of open subsets $V_i \subseteq X$, $n \in \mathbf{N}$.

A partition $A = \{0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1\}$ of the unit interval I gives rise to the collection $\mathcal{A} = \{[a_{j-1}, a_j]\}$ of closed intervals. We write $P(X)(A,\mathcal{U}) = P(X)(\mathcal{A},\mathcal{U}) \subseteq P(X)$.

Lemma 3.1. *The sets* $P(X)(A,\mathcal{U}) \subset P(X)$ *form a basis for the compact-open topology of* $P(X)$.

Proof. Compare [**4**, Ch. XII] for the first part of this proof. Every path q in a basis set $P(X)(\mathcal{C}, \mathcal{V})$ satisfies $C_i \subset q^{-1}(V_i)$; hence C_i is covered by the connected components of $q^{-1}(V_i)$, a set of open disjoint intervals. Finitely many of those, say the intervals J_{ij}, cover C_i; for each of those let $I_{ij} = [\min(J_{ij} \cap C_i), \max(J_{ij} \cap C_i)]$. Then $I_{ij} \subset J_{ij} \subseteq q^{-1}(V_i)$ whence $q(I_{ij}) \subseteq V_i$; moreover, $C_i \subseteq \bigcup_j I_{ij}$.

Let $A' = \bigcup_{i,j} \partial I_{ij} \subset I$ denote the finite subset of interval boundary points. Then, for two successive points $a, b \in A'$ we have: $q([a, b]) \subseteq \bigcap_{[a,b] \subseteq I_{ij}} V_i$; this intersection is to be interpreted as the set X if the index set is empty. We slightly alter the partition A' and arrive at a new partition A if one or several boundary points are both the upper and the lower boundary of some of the intervals I_{ij}: If a is such an upper and a lower boundary of some of the intervals I_{ij} and hence $q(a) \in \bigcap_{a \in I_{ij}} V_i$, we replace a with a pair a_-, a_+ satisfying $a' < a_- < a < a_+ < a''$ for all $a' < a < a''$ in A' and such that $q([a_-, a_+]) \subseteq \bigcap_{a \in I_{ij}} V_i$.

Let $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$ denote the elements of A in the induced order, and let $\mathcal{U}$ denote the collection of open sets $U_j := \bigcap_{q([a_{j-1}, a_j]) \subseteq V_i} V_i$. The open set $P(X)(\mathcal{C}, \mathcal{V})$ from above then contains the open neighbourhood $P(X)(A,\mathcal{U})$ of q. □

3.2. Regular traces versus traces

In this section we compare several spaces of paths in a Hausdorff space X *up to reparametrization.* Extending Definition 1.1, we get

Definition 3.2.

1. A path $p : I \to X$ in a topological space X is said to be *regular* if $\Delta_p = \emptyset$ or if $\Delta_p = \{I\}$.

2. The set of regular paths in X is denoted $R(X)$ and regarded as a subspace of $P(X) = X^I$.
3. The spaces of (regular) paths p starting in $x \in X$ and ending in $y \in X$ ($p(0) = x$, $p(1) = y$) are denoted by $R(X)(x, y) \subset P(X)(x, y)$; they are equipped with the induced topologies.

Composition on the right yields a group action of the topological group $\mathrm{Homeo}_+(I)$ on $R(X)$ and a monoid action of the topological monoid $\mathrm{Rep}_+(I)$ on $P(X)$. These actions respect the decompositions in subspaces $R(X)(x, y)$, resp. $P(X)(x, y)$.

In Definition 1.2, we called paths $p, q \in \mathrm{Rep}_+(I)$ reparametrization equivalent if there exist reparametrizations $\varphi, \psi \in \mathrm{Rep}_+(I)$ such that $p \circ \varphi = q \circ \psi$.

Corollary 3.3.

1. *Reparametrization equivalence of paths is an equivalence relation.*
2. *Two reparametrization equivalent paths are thinly homotopic.*

For a definition of thin homotopy see [**11**]; essentially, a homotopy $H : I \times I \to X$ fixing the endpoints is thin if it factors through a *tree* (the geometric realisation of an acyclic one-dimensional simplicial set), i.e. if $H : I \times I \to J \to X$ for a tree J. Remark that reparametrization equivalent paths have the same image: $p(I) = q(I) \subseteq X$. This is not necessarily true for thinly homotopic paths; e.g., the cancellation homotopy [**16**, p. 48] between the concatenation of a path with its inverse and the constant path is thin.

Proof.

1. Reparametrization equivalence is clearly a reflexive and symmetric relation. For transitivity, let $p, q, r \in P(X)$ denote three paths and assume that $p \circ \varphi = q \circ \psi$ and $q \circ \varphi' = r \circ \psi'$ for reparametrizations $\varphi, \varphi', \psi, \psi' \in \mathrm{Rep}_+(I)$. By Proposition 2.19, there are $\eta, \eta' \in \mathrm{Rep}_+(I)$ such that $\psi \circ \eta = \varphi' \circ \eta'$; hence $p \circ \varphi \circ \eta = r \circ \psi' \circ \eta'$.
2. It is enough to show that p and $p \circ \varphi$ are thinly homotopic for every $\varphi \in \mathrm{Rep}_+(I)$; consider the homotopy $H : I \times I \to I \xrightarrow{p} X,\ H(s, t) = p((1 - s)t + s\varphi(t))$, that even factors over I.

□

Factoring out the respective equivalence relations given by the actions above, we arrive at quotient spaces $T_R(X) = R(X)/_{\mathrm{Homeo}_+(I)}$, resp. $T(X) = P(X)/_{\mathrm{Rep}_+(I)}$ with subspaces $T_R(X)(x, y) = R(X)(x, y)/_{\mathrm{Homeo}_+(I)}$, resp. $T(X)(x, y) = P(X)(x, y)/_{\mathrm{Rep}_+(I)}$ for $x, y \in X$. They are considered as spaces of (regular) *traces* of paths in X and should be compared to the notions of curves or regular curves in elementary differential geometry.

These spaces can be organised in a topological category $T(X)$ (and likewise $T_R(X)$) with the elements of X as objects, with the topological spaces $T(X)(x, y)$ as morphism from x to y and with a composition $T(X)(x, y) \times T(X)(y, z) \to T(X)(x, z)$ induced by concatenation. Remark that one does *not* obtain a category structure on $P(X)$ since concatenation is not associative "on the nose". The categories $T(X)$ and their directed relatives are used as important tools in [**15**].

Lemma 3.4. *Let x, y be elements of a Hausdorff space X and let $p \in R(X)(x, y)$, $\varphi \in \mathrm{Rep}_+(I)$. If $p \circ \varphi = p$, then $\varphi = \mathrm{id}_I$ or p is constant. In particular, for $x \neq y$, the action of* $\mathrm{Homeo}_+(I)$ *on $R(X)(x, y)$ is* free.

Proof. If $\varphi \neq \mathrm{id}_I$, there exists an interval $J = [a, b] \subseteq I$ with $\varphi(a) = a, \varphi(b) = b$ and, without loss of generality, $\varphi(t) < t$ for all $a < t < b$. For all these t we conclude that $p(t) = p(\varphi(t)) = p(\varphi^n(t))$ for all $n > 0$, and hence that $p(t) = p(\lim_{n\to\infty}(\varphi^n(t))) = p(a)$. In particular, there is a non-trivial interval on which p is constant; this is not allowed for a regular path unless p is constant on the entire unit interval I (and thus $x = y$). ☐

Corollary 3.5. *Let $x \neq y$ be elements of a topological space X. The quotient map $R(X)(x, y) \to T_R(X)(x, y)$ is a weak homotopy equivalence.*

Proof. The free group action yields a fibration with contractible fiber $\mathrm{Homeo}_+(I)$. ☐

It is not clear to the authors whether one can sort out conditions under which the quotient map is a genuine homotopy equivalence.

3.3. Spaces of (regular) traces

For $x, y \in X$, the inclusion map $R(X)(x, y) \hookrightarrow P(X)(x, y)$ induces a natural map $i : T_R(X)(x, y) \to T(X)(x, y)$ between the corresponding quotient trace spaces. The main aim of this section is a proof of

Theorem 3.6. *For every two elements $x, y \in X$ of a Hausdorff space X, the map $i : T_R(X)(x, y) \to T(X)(x, y)$ is a* homeomorphism.

In particular, every trace can be represented by a regular trace (cf. Proposition 3.7 below). It turns out that many of the results on reparametrizations from the preceding section will be used in the proof. In a first step, we show that the map i is surjective:

Proposition 3.7. *For every path $p \in P(X)$, there exists a regular path $q \in R(X)$ and a reparametrization $\varphi \in \mathrm{Rep}_+(I)$ such that $p = q \circ \varphi$.*

Proof. For every interval $J \subseteq I$ let $m(J)$ denote its midpoint. Let $m : \Delta_p \to I$ denote the map $J \mapsto m(J)$ with image $C := m(\Delta_p) \subseteq I$. In order to arrive at a reparametrization with stop map m, we check the conditions from Proposition 2.13 for the order-preserving bijection $m : \Delta_p \to C$: Let $x_n = \max(J_n) \uparrow x \in I$ denote a strictly monotone sequence. Then the midpoints converge as well: $m(J_n) \uparrow x$. Likewise for a decreasing sequence of lower boundaries and corresponding midpoints. From Proposition 2.13 we conclude that there exists a reparametrization $\varphi \in \mathrm{Rep}_+(I)$ with $\Delta_\varphi = \Delta_p$. Hence, there is a set-theoretic factorization

$$\begin{tikzcd} I \arrow[r, "p"] \arrow[d, "\varphi"'] & X \\ I \arrow[ur, dashed, "q"'] & \end{tikzcd}$$

through a *regular* map $q : I \to X$.

To check that q is continuous, choose an open set $U \subset X$ and note that $q^{-1}(U) = \varphi p^{-1}(U)$. Combining Lemma 2.4 and Lemma 2.8.3, we conclude that $p^{-1}(U)$ is a union of maximal disjoint open intervals that are all mapped onto open intervals under φ. Hence $q^{-1}(U)$ is open as well. □

Proposition 3.8. *Two regular paths $p, q \in R(X)(x, y)$ that are reparametrization equivalent are in fact strictly reparametrization equivalent, i.e., there exists a homeomorphism $\eta \in \mathrm{Homeo}_+(I)$ such that $q = p \circ \eta$.*

The proof of Proposition 3.8 (and also that of Theorem 3.6) makes use of

Lemma 3.9. *Let $p \in R(X)(x, y)$, $p' \in P(X)(x, y)$, $\varphi, \varphi' \in \mathrm{Rep}_+(I)$ with $p \circ \varphi = p' \circ \varphi'$. Then there exists $\eta \in \mathrm{Rep}_+(I)$ with $p \circ \eta = p'$. Unless p is constant, η is unique.*

Proof. We apply Proposition 2.17 to prove the *existence* of such a reparametrization η: For every interval $J' \in \Delta_{\varphi'}$, there is a unique $J \in \Delta_{p' \circ \varphi'} = \Delta_{p \circ \varphi} = \Delta_\varphi$ such that $J' \subseteq J$. Hence, there exists a unique $\eta \in \mathrm{Rep}_+(I)$ with $\varphi = \eta \circ \varphi'$, whence $p' \circ \varphi' = p \circ \varphi = (p \circ \eta) \circ \varphi'$. Since φ' is onto, we conclude that $p' = p \circ \eta$.

To prove *uniqueness* of the factor η, suppose that $\eta_1, \eta_2 \in \mathrm{Rep}_+(I)$ with $p \circ \eta_1 = p \circ \eta_2$. If $\eta_1 \neq \eta_2$, one can choose an interval $J = [a, b]$ such that $\eta_1(a) = \eta_2(a), \eta_1(b) = \eta_2(b)$ and $\eta_1(t) < \eta_2(t)$ for $a < t < b$ (or vice versa). Given $a < t_0 = t'_0 < b$, choose an increasing sequence t_i and a decreasing sequence t'_i such that $\eta_1(t_{i+1}) = \eta_2(t_i)$, resp. $\eta_2(t'_{i+1}) = \eta_1(t_i)$. Both sequences converge to, say, $a \leqslant T' < T \leqslant b$. Since $\eta_1(T) = \eta_2(T)$ and $\eta_1(T') = \eta_2(T')$, we must have $T' = a, T = b$.

On the other hand, $p(\eta_2(t_i)) = p(\eta_1(t_{i+1})) = p(\eta_2(t_{i+1}))$ and hence $p(t_0) = p(b)$, and, by a similar argument: $p(t_0) = p(a)$. Since $t_0 \in]a, b[$ was chosen arbitrarily, p has to be constant on the interval $J = [a, b]$; this is impossible for a regular path p unless it is constant. □

Proof. (of Proposition 3.8) Let $p, q \in R(X)(x, y)$ be reparametrization equivalent, i.e., there exist reparametrizations $\varphi, \psi \in \mathrm{Rep}_+(I)$ such that $p \circ \varphi = q \circ \psi$. If one of p, q is constant, the other is as well. Assume that neither of them is constant. By Lemma 3.9, there is a reparametrization $\rho \in \mathrm{Rep}_+(I)$ with $q = p \circ \rho$. Since q is regular and non-constant, ρ has to be injective and thus a homeomorphism; in particular, p and q represent the same element in $T_R(X)(x, y)$. □

Proof. (of Theorem 3.6) From Propositions 3.7 and 3.8 we conclude that the map i from Theorem 3.6 is a continuous bijection. To see that it is also open, the following diagram will be useful:

$$\begin{tikzcd} R(X)(x,y) \arrow[r, "\subseteq"] \arrow[d, "Q_R"'] & P(X)(x,y) \arrow[d, "Q"] \\ T_R(X)(x,y) \arrow[r, "i"] & T(X)(x,y) \end{tikzcd} \tag{3.1}$$

By Lemma 3.1, a basis for the topology of $R(X)(x, y)$ is given by the sets

$$R(X)(A,\mathcal{U}; x, y) = \{p \in R(X)(x, y) \mid p([a_{j-1}, a_j]) \subseteq U_j\}$$

indexed by all partitions $A = \{0 = a_0 < \cdots < a_n = 1\}$ of I and collections of n open sets $\mathcal{U} = \{U_1, \ldots, U_n\}$, $n \in \mathbf{N}$.

The sets $Q_R(R(X)(A,\mathcal{U}; x, y))$ form a basis for the quotient topology on $T_R(X)(x, y)$ since

$$\begin{aligned} Q_R^{-1}(Q_R(R(X)(A,\mathcal{U}; x, y))) &= \bigcup\nolimits_{\rho\in \mathrm{Homeo}_+(I)} \rho \cdot R(X)(A,\mathcal{U}; x, y)) \\ &= \bigcup\nolimits_{\rho\in \mathrm{Homeo}_+(I)} R(X)(\rho^{-1}(A),\mathcal{U}; x, y)), \end{aligned}$$

and the latter set is open in $R(X)(x, y)$.

We need to show that the sets $i(Q_R(R(X)(A,\mathcal{U}; x, y))) = Q(R(X)(A,\mathcal{U}; x, y))$ are open in $T(X)(x, y)$, i.e., that the sets $Q^{-1}(Q(R(X)(A,\mathcal{U}; x, y)))$ are open in $P(X)(x, y)$. To this end, we note first that $Q^{-1}(Q(R(X)(A,\mathcal{U}; x, y))) = \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$: The inclusion $\supseteq$ is obvious. If $p \in R(X)(A,\mathcal{U}; x, y)$ and $p' \in P(X)(x, y)$, $\varphi, \varphi' \in \mathrm{Rep}_+(I)$ are such that $p \circ \varphi = p' \circ \varphi'$, then Lemma 3.9 yields $\eta \in \mathrm{Rep}_+(I)$ such that $p' = p \circ \eta$, hence $p' \in \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$.

In the next two steps we show that every element $q \in \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$ has an open neighbourhood in that set. For elements in $R(X)(A,\mathcal{U}; x, y)$ it suffices to check that $P(X)(A,\mathcal{U}; x, y) \subseteq \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$: According to Proposition 3.7, a path $p \in P(X)(A,\mathcal{U}; x, y)$ can be factored in the form $p = q \circ \varphi$ with $q \in R(X)(x, y), \varphi \in \mathrm{Rep}_+(I)$; the factors then have to satisfy $\varphi([a_{i-1}, a_i]) \subseteq q^{-1}(U_i)$. By Lemma 2.9, there is an approximation $\rho \in \mathrm{Homeo}_+(I)$ to φ satisfying $\rho([a_{i-1}, a_i]) \subseteq q^{-1}(U_i)$ for all i. We can thus factor $p = (q \circ \rho) \circ (\rho^{-1} \circ \varphi)$ with $q \circ \rho \in R(X)(A,\mathcal{U}; x, y)$, and hence $p \in \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$.

Now let $q \in R(X)(A,\mathcal{U}; x, y)$ and $\varphi \in \mathrm{Rep}_+(I)$. Let $B = \{b_0 = 0 < \cdots < b_n = 1\}$ denote any partition of I with $\varphi(B) = A$. Then $q \circ \varphi \in P(X)(B,\mathcal{U}; x, y)$, which is open in $P(X)(x, y)$. Hence it will suffice to show that $P(X)(B,\mathcal{U}; x, y) \subseteq \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$. To this end, choose $\rho \in \mathrm{Homeo}_+(I)$ such that $\rho(b_i) = a_i$. Let $p \in P(X)(B,\mathcal{U}; x, y)$ and write $p = p_1 \circ \rho$. Then $p_1 \in P(X)(A,\mathcal{U}; x, y) \subseteq \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$ by the previous step, and hence also $p \in \mathrm{Rep}_+(I) \cdot R(X)(A,\mathcal{U}; x, y)$. □

Remark 3.10. All four spaces in (3.1) are (at least) weakly homotopy equivalent: If x, y are not in the same path component, they are all empty. Otherwise, $P(X)(x, y)$ is homotopy equivalent to the loop space $\Omega(X)(x)$ based at x; likewise, $R(X)(x, y)$ is homotopy equivalent to the space of regular loops $\Omega_R(X)(x)$ based at x. Both loop spaces are fibres in fibrations $P_R(X) \downarrow X$, resp. $P(X) \downarrow X$ over X with contractible total spaces $P_R(X; x)$, resp. $P(X; x)$ of (regular) paths starting at x. The Five Lemma shows that the inclusion map $\Omega_R(X)(x) \hookrightarrow \Omega(X)(x)$ is a weak homotopy equivalence.

Note that Lemma 3.9 allows us to give the following "backwards" characterization of reparametrization equivalence:

Proposition 3.11. *Paths $p, q \in P(X)$ are reparametrization equivalent if and only if there exists $r \in P(X)$ and $\varphi, \psi \in \mathrm{Rep}_+(I)$ such that $p = r \circ \varphi$ and $q = r \circ \psi$.*

Proof. For the "only if" part, suppose $p \circ \varphi_1 = q \circ \psi_1, \varphi_1, \psi_1 \in \mathrm{Rep}_+(I)$. We use Proposition 3.7 to write $p = r_1 \circ \eta_1$ and $q = r_2 \circ \eta_2$, with $r_1, r_2 \in R(X)$ and $\eta_1, \eta_2 \in \mathrm{Rep}_+(I)$. Then $r_1 \circ \eta_1 \circ \varphi_1 = r_2 \circ \eta_2 \circ \psi_1$; by Lemma 3.9, there exists $\eta \in \mathrm{Rep}_+(I)$ such that $r_2 = r_1 \circ \eta$, whence $q = r_1 \circ \eta \circ \eta_2$.

The reverse implication is clear by Proposition 2.19; if $\varphi_2, \psi_2 \in \mathrm{Rep}_+(I)$ are such that $\varphi \circ \varphi_2 = \psi \circ \psi_2$, then $p \circ \varphi_2 = r \circ \varphi \circ \varphi_2 = r \circ \psi \circ \psi_2 = q \circ \psi_2$. □

Let us finally note the following consequence of Proposition 3.7:

Definition 3.12. A path $p : I \to X$ is called *loop-free* if $p(s) = p(t)$ for any $s < t \in I$ implies that the restriction $p\,|_{[s,t]}$ is the constant path.

Note that a loop-free *regular* path is either constant or injective.

Corollary 3.13. *A loop-free path $p : I \to X$ in a Hausdorff space X has an image $p(I) \subseteq X$ that is either a point or homeomorphic to I.*

Proof. By Proposition 3.7, there is a factorization $p = q \circ \varphi$ with q a loop-free regular and thus either constant or injective path. In the second case, q is a continuous bijection from the compact space I to its image $q(I) = p(I) \subseteq X$. The claim follows since X is Hausdorff. □

4. Directed traces

Originally motivated by models arising in concurrency theory in theoretical computer science, an investigation of topological spaces with "preferred directions" has been launched under the title "Directed Algebraic Topology". Various frameworks (local po-spaces [**6**], d-spaces [**9**], flows [**7**] and others) have been suggested, all modifying in various ways concepts from elementary algebraic topology, in particular replacing relevant groups or groupoids by categories. The d-spaces introduced by M. Grandis [**9**] have turned out to give rise to a particularly successful way to combine homotopy theoretical and categorical methods for the study of spaces with preferred directions.

The question whether one can neglect reparametrizations in a homotopy theoretical study of spaces of directed paths was one of the original motivations for this article. This is – under a certain natural condition – affirmed in Corollary 4.5, which is used as a starting point for the homotopy theoretical study of "directed spaces" in [**15**].

Definition 4.1 ([9]). A *d-space* is a topological space X together with a set $\vec{P}(X) \subseteq P(X)$ of continuous paths $I \to X$ such that

1. $\vec{P}(X)$ contains all constant paths;
2. $p \circ \varphi \in \vec{P}(X)$ for any $p \in \vec{P}(X)$ and any continuous increasing (not necessarily surjective)s map $\varphi : I \to I$;

3. for all $p, q \in \vec{P}(X)$ such that $p(1) = q(0)$, their concatenation $p * q \in \vec{P}(X)$, cf. (2.1).

Elements of $\vec{P}(X)$ are called d-*paths*. $\vec{P}(X) \subseteq P(X)$ is given the subspace topology (of the compact-open topology).

Definition 4.2. A *d-map* between d-spaces X, Y is a continuous mapping $f : X \to Y$ satisfying $p \in \vec{P}(X) \Rightarrow f \circ p \in \vec{P}(Y)$. Isomorphisms in the category of d-spaces are called *d-homeomorphisms.*

The d-interval $\vec{I}$ is given the standard d-structure $\vec{P}(I) = \mathrm{Rep}_+(I)$.

In general d-spaces, it may occur that a non-d-path becomes directed after reparametrization. To exclude this possibility, we add

Definition 4.3. A d-space X is called *saturated* if it has the following additional property:

4. If $p \in P(X), \varphi \in \mathrm{Rep}_+(I)$ and $p \circ \varphi \in \vec{P}(X)$, then $p \in \vec{P}(X)$.

In words: If a path becomes a d-path after reparametrization, then it has to be a d-path itself already. Remark that for a saturated d-space, two trace equivalent paths are either both d-paths, or neither of them is. The d-space $\vec{I}$ is saturated. It is easy to turn a given d-space X into a saturated one: one just adds all paths $p \in P(X)$ for which there is a reparametrization $\varphi \in \mathrm{Rep}_+(I)$ with $p \circ \varphi \in \vec{P}(X)$ to the d-paths in a new structure $S\vec{P}(X)$, which is easily seen to satisfy the properties of a saturated d-space. So there is no harm in assuming that a d-space is saturated right away.

Among the d-paths in X, we pay particular attention to the *regular* d-paths, cf. Definition 1.1.3; the set of all those will be denoted $\vec{R}(X) = R(X) \cap \vec{P}(X)$ and equipped with the subspace topology. Again, $\mathrm{Homeo}_+(I)$ acts (essentially freely) on $\vec{R}(X)$, and $\mathrm{Rep}_+(I)$ acts on $\vec{P}(X)$. We can now speak of spaces of (regular) traces in a saturated d-space X:

Definition 4.4.

- $\vec{T}_R(X)(x,y) := \vec{R}(X)(x,y)/_{\mathrm{Homeo}_+(I)} \subseteq T_R(X)(x,y)$
- $\vec{T}(X)(x,y) := \vec{P}(X)(x,y)/_{\mathrm{Rep}_+(I)} \subseteq T(X)(x,y)$

These form the morphisms of categories $\vec{T}_R(X)$, resp. $\vec{T}(X)$ that are investigated from a homotopy theory point of view in [**15**]. The following consequence of Theorem 3.6 tells us that it makes no difference in topology which of the two (quotient) trace spaces is chosen:

Corollary 4.5. *Let X denote a saturated d-space and let $x, y \in X$. The map $\vec{i} : \vec{T}_R(X)(x,y) \to \vec{T}(X)(x,y)$ induced by inclusion $\vec{R}(X)(x,y) \hookrightarrow \vec{P}(X)(x,y)$ is a homeomorphism.*

Remark 4.6. It is no longer clear whether the inclusion map $\vec{R}(X)(x,y) \hookrightarrow \vec{P}(X)(x,y)$ is a weak homotopy equivalence. The (weak) homotopy types of both spaces depend on the choice of x and y and it is therefore not possible to argue using loop spaces as in 3.10. From the diagram

$$\begin{tikzcd} \vec{R}(X)(x,y) \arrow[r, "\subseteq"] \arrow[d, "Q_R"', "\simeq"] & \vec{P}(X)(x,y) \arrow[d, "Q"] \\ \vec{T}_R(X)(x,y) \arrow[r, "i", "\cong"'] & \vec{T}(X)(x,y), \end{tikzcd}$$

obtained from (3.1) by restricting to d-paths, we can only deduce that the inclusion maps $\vec{R}(X)(x,y) \hookrightarrow \vec{P}(X)(x,y)$ induce injections and that the quotient maps $Q : \vec{P}(X)(x,y) \to \vec{T}(X)(x,y)$ induce surjections on all homotopy groups.

Definition 4.7.

1. [**9**] A *d-homotopy* from a d-path $p \in \vec{P}(X)$ to a d-path $q \in \vec{P}(X)$ is a d-map $H : \vec{I} \times \vec{I} \to X$ for which $H(0,\cdot) = p$, $H(1,\cdot) = q$, and $H(\cdot,0)$, $H(\cdot,1)$ are constant.
2. A d-homotopy is said to be *thin* if it factors through the d-interval $\vec{I}$, i.e. if there are d-maps $\Phi : \vec{I} \times \vec{I} \to \vec{I}$, $r : \vec{I} \to X$ such that $H = r \circ \Phi$.
3. Two d-paths $p, q \in \vec{P}(X)$ are said to be *d-homotopic*, respectively *thinly d-homotopic*, if there exists a sequence $H_1, \ldots, H_{2n+1}$ of d-homotopies, respectively thin d-homotopies, such that $H_1(0,\cdot) = p$, $H_{2n+1}(1,\cdot) = q$, $H_{2i-1}(1,\cdot) = H_{2i}(1,\cdot)$, and $H_{2i}(0,\cdot) = H_{2i+1}(0,\cdot)$.

Remark 4.8.

- The directed structure on a product of d-spaces X and Y is given by
$$P(X \times Y) \supseteq \vec{P}(X \times Y) = \vec{P}(X) \times \vec{P}(Y) \subseteq P(X) \times P(Y)$$
under the natural identification $P(X \times Y) \cong P(X) \times P(Y)$. In particular, $\vec{P}(\vec{I} \times \vec{I})$ consists of all paths $p : I \to I \times I$ that are (weakly) increasing in both coordinates.
- The relations $\preccurlyeq$, $\preccurlyeq_T$ on d-paths given by existence of (thin) d-homotopies are preorders on $\vec{P}(X)$. The relations $\simeq$, $\simeq_T$ on d-paths given by being (thinly) d-homotopic are equivalence relations on $\vec{P}(X)$; they are the symmetric, transitive closures of $\preccurlyeq$ respectively $\preccurlyeq_T$.
- In the differentiable setting, thin homotopies of differentiable loops and paths were defined (using homotopies of rank at most 1) and a resulting smooth fundamental groupoid used for holonomy considerations in [**2, 14**].

Proposition 4.9. *Two d-paths p, q in a saturated d-space are reparametrization equivalent if and only if they are thinly d-homotopic.*

This result is needed in the study [**5**] of directed squares. Note that the notion of d-homotopy factors over $\vec{T}(X)$ (and $\vec{T}_R(X)$).

Proof. We use Proposition 3.11. For the forward implication, write $p = r \circ \varphi$, $q = r \circ \psi$, and let $\eta = \max(\varphi, \psi)$. Define $\Phi, \Psi : \vec{I} \times \vec{I} \to \vec{I}$ by

$$\Phi(s,t) = (1-s)\varphi(t) + s\eta(t)$$
$$\Psi(s,t) = (1-s)\psi(t) + s\eta(t)$$

then $r \circ \Phi$, $r \circ \Psi$ are thin d-homotopies connecting p and q.

To show the back implication, it is enough to consider the case where p and q are connected by *one* thin d-homotopy H with $H(0,\cdot) = p$ and $H(1,\cdot) = q$. Write $H = r \circ \Phi : \vec{I} \times \vec{I} \to \vec{I} \to X$; by Corollary 4.5, we can assume r to be regular. Also, by reparametrizing if necessary, we can assume that $\Phi(0,0) = 0$ and $\Phi(1,1) = 1$. Then

$$r(\Phi(s,0)) = H(s,0) = H(0,0) = r(0)$$
$$r(\Phi(s,1)) = H(s,1) = H(1,1) = r(1)$$

hence by regularity of r and following the logic of the final step in the proof of Lemma 3.4, $\Phi(s,0) = 0$ and $\Phi(s,1) = 1$.

Now define $\varphi, \psi : \vec{I} \to \vec{I}$ by $\varphi(t) = \Phi(0,t)$, $\psi(t) = \Phi(1,t)$, then $\varphi, \psi \in \mathrm{Rep}_+(I)$ and $p = r \circ \varphi$, $q = r \circ \psi$. □

Finally, we modify the results from Section 3 about loop-free paths (Definition 3.12) to the d-space environment:

Definition 4.10. A d-space X is said to be *locally loop-free* provided that every point has a neighbourhood in which all non-constant d-paths are loop-free.

Note that d-spaces arising from a space with a locally partial order [**6**] are locally loop-free. The following result applies such to such spaces, in particular. For po-spaces, a result similar to the following had previously been obtained in [**12**, Thm. 5].

Corollary 4.11. *If $p \in \vec{P}(X)$ is a loop-free d-path in a locally loop-free saturated Hausdorff d-space X, then its image $p(\vec{I})$ is either a point or d-homeomorphic to $\vec{I}$.*

Proof. The statement is trivial for a constant d-path. Otherwise, Corollary 4.5 provides us with a *regular* loop-free d-path $q : \vec{I} \to X$ with $p = q \circ \varphi$, $\varphi \in \mathrm{Rep}_+(I)$ which by Corollary 3.13 yields the homeomorphism $q : I \to p(\vec{I})$. All we need to show is that its inverse $q^{-1} : p(\vec{I}) \to \vec{I}$ is a d-map.

Let $r : \vec{I} \to p(\vec{I})$ be a d-path; we need to show that $q^{-1} \circ r \in \vec{P}(I) = \mathrm{Rep}_+(I)$. Let $t_1 < t_2 \in I$ and suppose that $q^{-1}(r(t_1)) > q^{-1}(r(t_2))$.

Restricting to a smaller interval, if necessary, will ensure that $r([t_1,t_2]) \subseteq X$ is contained in a loop-free neighbourhood $U \subseteq X$. The concatenation

$$r\,|_{[t_1,t_2]} * q\,|_{[q^{-1}(r(t_2)),q^{-1}(r(t_1))]}$$

is a d-path and a loop in U and hence constant. Then $q\,|_{[q^{-1}(r(t_2)),q^{-1}(r(t_1))]}$ is constant, in contradiction to being a regular and non-constant path. Hence $q^{-1}(r(t_1)) \leqslant q^{-1}(r(t_2))$ whence q^{-1} is a d-map. □

Remark 4.12. It seems plausible that many of the methods and of the results from this article allow generalizations to maps $p : I^n \to X$, resp. $p : \vec{I}^n \to X$, from (directed) cubes to (d)-spaces. The relevant reparametrizations to investigate are the d-maps $\varphi : \vec{I}^n \to \vec{I}^n$ (monotone in every coordinate) that preserve boundaries in the following sense:

$$\varphi(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \text{ and } x_i = 0, \text{ resp. } 1 \Rightarrow y_i = 0, \text{ resp. } 1.$$

In the differentiable setting, such reparametrizations for so-called n-loops and resulting smooth homotopy groups have been defined and studied in [**3, 14**].

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