*Journal of Homotopy and Related Structures, vol. 2(2), 2007, pp.119–170*

# PSEUDO ALGEBRAS AND PSEUDO DOUBLE CATEGORIES

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(*communicated by Ronald Brown*)

*Abstract*

As an example of the categorical apparatus of pseudo algebras over 2-theories, we show that pseudo algebras over the 2-theory of categories can be viewed as pseudo double categories with folding or as appropriate 2-functors into bicategories. Foldings are equivalent to connection pairs, and also to thin structures if the vertical and horizontal morphisms coincide. In a sense, the squares of a double category with folding are determined in a functorial way by the 2-cells of the horizontal 2-category. As a special case, strict 2-algebras with one object and everything invertible are crossed modules under a group.

## 1. Introduction

Recent years have seen widespread applications of *categorification*. The term categorification refers to a process of turning algebraic notions on sets into algebraic notions on categories as explained in [**6**]. Generally speaking, one takes a set-based algebraic notion, then replaces sets by categories, functions by functors, and equations by natural isomorphisms which satisfy certain coherence diagrams.

For example, a monoid (group without inverses) is a set-based algebraic concept. Its categorified notion is a monoidal category, that is a category $M$ equipped with a functor $\otimes : M \times M \longrightarrow M$ and a unit which satisfy the monoid axioms up to coherence isos. These coherence isos must satisfy certain coherence conditions, such as the familiar pentagon diagram. The commutativity of these diagrams in turn implies that all diagrams in a certain class commute, as Mac Lane proved in [**58**]. A familiar example of a monoidal category is the category of complex vector spaces under the operation of tensor product with unit $\mathbb{C}$.

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The author extends his gratitude to Peter May and Igor Kriz for fruitful conversations, and also to John Baez, Ronnie Brown, and Michael Shulman, for some helpful comments on an earlier draft. The author is indebted to the referee whose careful suggestions greatly improved this paper.
The author was supported by the Mathematical Sciences Postdoctoral Research Fellowship of the National Science Foundation. This article was completed during a visit at the Fields Institute during the Thematic Program on Geometric Applications of Homotopy Theory, and the author thanks the Fields Institute for its support.
Received June 17, 2006, revised May 6, 2007; published on December 27, 2007.
2000 Mathematics Subject Classification: 18D05, 18C10, 81T40.
Key words and phrases: 2-algebra, 2-category, 2-group, 2-theory, algebra, bicategory, coherence, connection pair, crossed module, double category, double group, folding, holonomy, pseudo algebra, pseudo double category, thin structure.

Another example of categorification is the notion of a *bicategory*, which is a categorification of the algebraic concept of category. In a bicategory the hom-sets are categories and composition is a functor. Composition is unital and associative up to coherence isomorphisms which satisfy coherence diagrams like those of a monoidal category. This similarity is not a coincidence: one-object bicategories are monoidal categories in the same way that one-object categories are monoids. A familiar example of a bicategory consists of rings, bimodules, and bimodule morphisms. Bicategories were introduced in the 1960's in [**9**], [**10**], [**35**], and [**37**]. Since then, they (and their variants) have appeared in diverse areas, such as homotopy theory and high energy physics.

However, the question arises: what exactly does one mean by "coherence isos satisfying certain coherence diagrams"? Which coherence isos and which coherence diagrams does one require? This question already suggests that there may be more than one way to categorify a given concept, such as category. Indeed, there already are a dozen or so different definitions of weak $n$-category, many of which are described in [**23**] and [**55**].

Lawvere theories and 2-theories provide one answer to this question. Lawvere theories, first introduced in [**53**], abstractly encode algebraic structure. For most familiar algebraic structures there is a Lawvere theory. For example, there is a Lawvere theory of monoids, and algebras over this theory are precisely the monoids. A Lawvere theory $T$ is simply a category whose objects are $0, 1, 2, \ldots$ such that $n$ is the product of $n$ copies of 1 with specified projection maps. If $T$ is the theory which encapsulates a certain algebraic structure, then a set $X$ with that algebraic structure is an *algebra over the theory* $T$. This means that $X$ is equipped with a morphism $\Phi : T \longrightarrow End(X)$ of theories from $T$ to the endomorphism theory on $X$. To each abstract word $w : n \longrightarrow 1$ a morphism assigns a function $\Phi(w) : X^n \longrightarrow X$ in a uniform way.

Similarly a category $X$ is a *pseudo algebra over a theory* $T$ if it is equipped with a pseudo morphism of theories $\Phi : T \longrightarrow End(X)$. To each abstract word $w : n \longrightarrow 1$ a pseudo morphism assigns a functor $\Phi(w) : X^n \longrightarrow X$. Additionally, for each operation of theories, there is a coherence isomorphism and for each relation of theories, there is a coherence diagram which these coherence isomorphisms must satisfy. This is a well-defined procedure which specifies exactly which coherence isomorphisms and coherence diagrams are appropriate, no matter if one is interested in monoids, semi-rings, rings, etc. A *pseudo monoid, pseudo semi-ring,* or *pseudo ring* is simply a pseudo algebra over the appropriate theory. There is a systematic way to leave out some coherence diagrams to encompass more examples [**40**].

However, Lawvere theories only axiomatize algebraic structures on a single set. There is no Lawvere theory of categories, since a category consists of two sets with composition defined in terms of pullback. For algebraic structures on several sets, one can use limit theories, sketches, and multi-sorted theories as in [**1**], [**11**], or [**12**], or schemes of operators as in [**47**]. In this paper we consider categories as algebras over a *2-theory.* This adds a new ingredient to categorification that we do not see in the one-object case of Lawvere theories. For example, pseudo algebras over the 2-theory of categories have an object category $I$ instead of an object set, as we shall see.

This version of categorification in terms of pseudo algebras over 2-theories was introduced in [**49**], and further developed in [**38**] and [**50**], to give a completely rigorous approach to conformal field theory with $n$-dimensional modular functor. Pseudo algebras over the *2-theory of commutative monoids with cancellation* make the symmetric approach to conformal field theory outlined in [**71**] rigorous. The notion of 2-theory was the main ingredient for a well-defined procedure of passing from a strict algebraic structure on a *family* of sets to a pseudo algebraic structure on a *family* of categories, such as the pseudo algebraic structure of disjoint union and gluing on the class of worldsheets (rigged surfaces). This procedure gave a well-defined machine for generating the coherence isos and coherence diagrams that were missing from conformal field theory until that point. Already in 1991, Mac Lane suggested a study of coherence in the context of conformal field theory in [**59**]. The foundations of pseudo algebras over theories and 2-theories were written in [**38**], as well as theorems relevant for application to conformal field theory. Among other things, it was shown that 2-categories of pseudo algebras admit pseudo limits and bicolimits, and forgetful 2-functors of pseudo algebras admit left biadjoints.

In the present article we apply this version of categorification to the fundamental algebraic structure of category and compare the resulting concept of pseudo category to weak double categories and also pseudo functors $I \longrightarrow \mathcal{C}$. One might expect that a pseudo category would neatly fit into one of the two prevailing approaches to categorification: enrichment and internalization. This however is not true, a pseudo category is neither a bicategory, nor a weak double category. Instead we arrive at an intermediate notion: a pseudo category can be 2-equivalently described as a weak double category with weak folding or as a bicategory equipped with a pseudo functor from a 1-category. Our pseudo categories are slightly different from the pseudo categories in [**63**], so we will call them pseudo $I$-categories instead.

We first treat the categorified strict case by reviewing strict categories and double categories in Section 2 and Section 3, and prove the strict versions of our desired result in Theorems 4.6, 4.8, and 4.9. Foldings, used in [**20**], are introduced to facilitate the 2-equivalence of strict 2-algebras over the 2-theory of categories with underlying category $I$ ($I$-categories for short) and certain double categories. It turns out that foldings, which have Ehresmann's quintets as their motivating example, are equivalent to Brown and Spencer's connection pairs, and also thin structures in the edge-symmetric case, as recounted in Theorem 3.28 (Lemmas 3.24-3.27) and Corollary 3.33. In light of this, Theorem 4.6 is an $I$-category analogue of the equivalence in [**20**] and [**73**] between small 2-categories and edge-symmetric double categories with thin structure.

In the case of one object with everything invertible, strict 2-algebras (not necessarily edge symmetric) are equivalent to crossed modules under a group as in Theorem 5.2 and Theorem 5.13. We generalize Brown and Spencer's equivalence in [**21**] between edge-symmetric double groups with connection pair and crossed modules. In Theorem 5.15 we prove that double groups (not necessarily edge symmetric) with folding are 2-equivalent to crossed modules under groups. The paper [**19**] contains a substantial generalization of [**21**] by giving an equivalence of 'core diagrams' to double groupoids with certain filling conditions. Double groupoids have recently found application in the theory of weak Hopf algebras in [**4**] and [**5**].

The pseudo double categories of [**45**] are reviewed in Section 6. We finally prove in Theorem 7.10 and Theorem 7.11, under the assumption of strict units, the 2-equivalence of pseudo algebras over the 2-theory of categories (pseudo $I$-categories for short), pseudo double categories with folding, and strict 2-functors[1] from a groupoid into a bicategory. The latter two 2-categories remain 2-equivalent even if $I$ is merely a category.

Theorem 4.9 and Theorem 7.11 may also be considered a special case of Theorem 6.5 of the comparison article [**39**]. That article relates the commutative-monoid-with-cancellation approach to conformal field theory in [**49**] (outlined in [**71**] in terms of trace) to the cobordism approach.

## 2. Strict $I$-Categories

A category consists of a family of sets with an algebraic structure. Namely, if $\mathcal{C}$ is a category with object set $I$, then the associated family of sets $X_{A,B} := Hom_{\mathcal{C}}(A,B)$ is parametrized by $I^2$. On this family of sets, we have the algebraic structure of composition and identity. Thus we can view a category as a functor $X : I^2 \longrightarrow Sets$ with certain algebraic operations, where $I^2$ is considered as a discrete category.

From this point of view, a category is an algebra $X : I^2 \longrightarrow Sets$ over the *2-theory of categories*. This is the notion that we categorify. In this article we do not write down the 2-theory of categories, since it suffices to directly define 2-algebras and pseudo algebras over this 2-theory. The operations are given in terms of the generating words $\circ$ and $\eta$ rather than abstract operations of 2-theories. The underlying theory of the 2-theory of categories is the theory of sets. We take the following description as a definition, and do not need the notion of 2-theory anywhere in this paper. For a development of 2-theories and their algebras, see the original paper [**49**], or the papers [**38**] and [**50**].

**Definition 2.1.** A *strict 2-algebra*[2] *over the 2-theory of categories with underlying category* $I$, called $I$*-category* for short, consists of a category $I$ and a strict 2-functor $X : I^2 \longrightarrow Cat$ with strictly 2-natural functors

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$

$$* \xrightarrow{\eta_B} X_{B,B}$$

for all $A, B, C \in I$. These functors satisfy the following relations.

[1] The term "2-functor" means strict 2-functor in this paper. Sometimes we include the word "strict" for emphasis. When we mean pseudo functor (homomorphism of bicategories), we say so.

[2] The distinction in this paper between strict 2-algebra and pseudo algebra agrees with usual 2-terminology and pseudo terminology. For example a strict 2-functor is a morphism of strict 2-categories. A pseudo functor is on the same level as a 2-functor, except a pseudo functor preserves composition and unit only up to coherent 2-cell isomorphisms. The notion of pseudo 2-algebra over a theory in [**39**] is distinct from a pseudo algebra over a theory. It should also be noted that a 2-theory is not a theory enriched in categories, nor any sort of weakened theory.

i. The composition $\circ$ is *associative.*

$$\begin{tikzcd}
(X_{C,D} \times X_{B,C}) \times X_{A,B} \arrow[r, "\circ \times 1_{X_{A,B}}"] \arrow[d, "\cong"] & X_{B,D} \times X_{A,B} \arrow[dr, "\circ"] & \\
& & X_{A,D} \\
X_{C,D} \times (X_{B,C} \times X_{A,B}) \arrow[r, "1_{X_{C,D}} \times \circ"'] & X_{C,D} \times X_{A,C} \arrow[ur, "\circ"'] &
\end{tikzcd}$$

ii. For each $B \in I$, the operation $\eta_B$ is an *identity* for $\circ$.

$$\begin{tikzcd}
* \times X_{A,B} \arrow[r, "\eta_B \times 1_{X_{A,B}}"] \arrow[dr, "pr_2"'] & X_{B,B} \times X_{A,B} \arrow[d, "\circ"] \\
& X_{A,B}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{B,C} \times * \arrow[r, "1_{X_{B,C}} \times \eta_B"] \arrow[dr, "pr_1"'] & X_{B,C} \times X_{B,B} \arrow[d, "\circ"] \\
& X_{B,C}
\end{tikzcd}$$

We denote the value of $\eta_B$ on the unique object and morphism of the terminal category by $1_B$ and $i_{1_B}$ respectively. We denote the identity morphism on an object $f$ in the category $X_{A,B}$ by $i_f$.

The term *$I$-category* is an abbreviation of *strict 2-algebra over the 2-theory of categories with underlying category $I$.* The strict morphisms of $I$-categories and their 2-cells below are the strict morphisms and 2-cells in the 2-category of strict 2-algebras over the 2-theory of categories with the same underlying groupoid $I$ as in [**38**], [**49**], and [**50**].

The term *$I$-category* agrees with existing usage of the term *$O$-category* to mean a category with the object set $O$. Indeed, if $I$ is a discrete category (*i.e.* a set) and $X$ takes values in *Sets*, then an $I$-category is precisely an ordinary category with object set $I$. More generally for groupoids $I$, we will see that $I$-categories are "categories with object groupoid $I$" in a precise sense.

**Definition 2.2.** A *strict morphism $F : X \longrightarrow Y$ of $I$-categories* is a strict 2-natural transformation $F : X \Longrightarrow Y$ which preserves composition and identity strictly.

**Definition 2.3.** A *2-cell* $\sigma : F \Longrightarrow G$ in the 2-category of $I$-categories is a modification $\sigma : F \rightsquigarrow G$ compatible with composition and identity. More specifically, a 2-cell $\sigma$ consists of natural transformations $\sigma_{A,B} : F_{A,B} \Longrightarrow G_{A,B}$ for all $A, B \in I$ such that

$$Y_{j,k}(\sigma^f_{A,B}) = \sigma^{X_{j,k}(f)}_{C,D}$$

$$\sigma^g_{B,C} \circ \sigma^f_{A,B} = \sigma^{g \circ f}_{A,C}$$

$$\sigma^{1_A}_{A,A} = i_{1_A}$$

for all $(j, k) : (A, B) \longrightarrow (C, D)$ in $I^2$, $f \in X_{A,B}$, and $g \in X_{B,C}$. Here $\circ$ denotes the composition functor of the strict 2-algebra, *not* the composition in the categories $X_{A,B}$.

**Lemma 2.4.** *If $X : I^2 \longrightarrow Cat$ is an $I$-category and $I$ is a discrete category, then $X$ is a strict 2-category with object set $I$. A morphism between two such $I$-categories is simply a strict 2-functor which is the identity on objects. A 2-cell $\sigma : F \Longrightarrow G$ is an oplax natural transformation with identity components. If additionally $X$ takes values in $Sets$, then $X$ is simply a category with object set $I$ in the usual sense. A morphism between such $I$-categories is simply a functor which is the identity on objects. There is at most a trivial 2-cell between any two such morphisms.*

*Proof.* This is just a matter of definitions. The category of morphisms from $A$ to $B$ is $X_{A,B}$. $\square$

An $I$-category is *not* an internal category in $Cat$, *nor* a $Cat$-enriched category, since we have taken as our starting point a different description of category. More specifically, if one takes as a starting point the definition of category as an object set $C_0$ and an arrow set $C_1$ along with four maps defining source, target, identity, and composition satisfying the relevant axioms, then one indeed arrives at the notion of internal category in $Cat$ as described on pages 267-270 of [**60**]. An internal category in $Cat$ is the same as a double category, which is described in an elementary way in the next section. The choice of starting point is crucial for higher-dimensional category theory. As seen in [**23**], equivalent definitions of category lead to quite different notions of higher category. We will see that the 2-cells of $I$-categories correspond to certain vertical natural transformations between double functors.

The notion of $I$-category lies between the notions of internal category in $Cat$ and $Cat$-enriched category, so how far away is an $I$-category from a 2-category? The following Lemma shows how to associate to an $I$-category a strict 2-functor $P : I \longrightarrow \mathcal{C}$. More importantly, in the presence of the other 2-cell axioms, we obtain a simplification of the requirement that 2-cells $\sigma$ be modifications in terms of compatibility with $P$. The 2-equivalence of $I$-categories to such strict 2-functors is Theorem 4.9. We will also apply the following Lemma in the comparison with double categories with folding in Theorem 4.8.

**Lemma 2.5.** *Suppose $I$ is a groupoid, $X$ and $X'$ are strict $I$-categories, and $F, G : X \longrightarrow X'$ are strict morphisms. We associate to $X$ a 2-category $\mathcal{C}$ with $Obj\ \mathcal{C} := Obj\ I$ and $Mor_{\mathcal{C}}(A, B) := X_{A,B}$. We denote the identity on $A$ in the category $I$ by $1^v_A$ while we denote the identity on $A$ in $\mathcal{C}$ by $1^h_A$. The identity 2-cell in $\mathcal{C}$ on a morphism $f$ is $i_f$. Let $P : I \longrightarrow \mathcal{C}$ be the strict 2-functor which is the identity on objects and*

$$P(j) := X_{j^{-1},1^v_C}(1^h_C) = X_{1^v_A,j}(1^h_A)$$

*for morphisms $j \in I(A, C)$. Let $\mathcal{C}'$ and $P' : I \longrightarrow \mathcal{C}'$ be the 2-category and strict 2-functor associated analogously to $X'$. Suppose further we have for each $A, B \in I$ a natural transformation $\sigma_{A,B} : F_{A,B} \Longrightarrow G_{A,B}$ such that*

$$\sigma^g_{B,C} \circ \sigma^f_{A,B} = \sigma^{g \circ f}_{A,C}$$

$$\sigma_{A,A}^{1_A^h} = i_{1_A^h}$$

*for all $f \in X_{A,B}$ and $g \in X_{B,C}$. Then the following are equivalent.*

*i. For all $(j,k) : (A,B) \longrightarrow (C,D)$ in $I^2$ and all $f \in X_{A,B}$ we have $Y_{j,k}(\sigma_{A,B}^f) = \sigma_{C,D}^{X_{j,k}(f)}$.*

*ii. For all $j : A \longrightarrow C$ in $I$ we have $\sigma_{A,C}^{P(j)} = i_{P'(j)}$.*

*Proof.* The naturality of the identity implies

$$X_{j^{-1},1_C^v}(1_C^h) = X_{1_A^v,j}(1_A^h).$$

The map $P$ preserves compositions $A \xrightarrow{j} C \xrightarrow{k} E$ in $I$ because

$$\begin{aligned} P(k \circ j) &= X_{1_A^v,k\circ j}(1_A^h) \\ &= X_{1_A^v,k}(X_{1_A^v,j}(1_A^h)) \\ &= X_{1_A^v,k}(P(j)) \\ &= X_{1_A^v,k}(1_C^h \circ P(j)) \\ &= (X_{1_C^v,k}(1_C^h)) \circ P(j) \\ &= P(k) \circ P(j) \end{aligned}$$

by the naturality diagram below.

$$\begin{tikzcd} X_{C,C} \times X_{A,C} \arrow[r, "\circ"] \arrow[d, "X_{1_C^v,k} \times X_{1_A^v,1_C^v}"'] & X_{A,C} \arrow[d, "X_{1_A^v,k}"] \\ X_{C,E} \times X_{A,C} \arrow[r, "\circ"'] & X_{A,E} \end{tikzcd}$$

It is clear that $P(1_A^v) = X_{1_A^v,1_A^v}(1_A^h) = 1_A^h$, so we indeed have a 2-functor $P$.

For $f \in X_{A,B}$ and $j \in I(A,C)$, note that $f \circ P(j^{-1}) = X_{j,1_B^v}(f)$ by the naturality diagram

$$\begin{tikzcd} X_{A,B} \times X_{C,A} \arrow[r, "\circ"] \arrow[d, "X_{j,1_B^v} \times X_{1_C^v,j}"'] & X_{C,B} \arrow[d, "X_{1_C^v,1_B^v}"] \\ X_{C,B} \times X_{C,C} \arrow[r, "\circ"'] & X_{C,B} \end{tikzcd}$$

and similarly $P(k) \circ f = X_{1_A^v,k}(f)$ for $k \in I(B,D)$. Similar statements hold for 2-cells of $\mathcal{C}$. Thus

$$X_{j,k}(f) = P(k) \circ f \circ P(j^{-1})$$

$$X_{j,k}(\alpha) = i_{P(k)} \circ \alpha \circ i_{P(j^{-1})}.$$

We use $\circ$ to denote the horizontal composition of 2-cells in a 2-category, in addition to the composition of morphisms.

Let $\sigma_{A,B}$ be a natural transformation for each $A, B \in I$ such that $\sigma$ is compatible with composition and identity. Suppose $\sigma$ satisfies (i). Then

$$\begin{aligned} \sigma^{P(j)}_{A,C} &= \sigma^{X_{1^v_A, j}(1^h_A)}_{A,C} \\ &= Y_{1^v_A, j}(\sigma^{1^h_A}_{A,A}) \\ &= Y_{1^v_A, j}(i_{1^h_A}) \\ &= i_{P'(j)} \end{aligned}$$

and (ii) holds.

Suppose $\sigma$ satisfies (ii). Then

$$\begin{aligned} Y_{j,k}(\sigma^f_{A,B}) &= i_{P'(k)} \circ \sigma^f_{A,B} \circ i_{P'(j^{-1})} \\ &= \sigma^{P(k)}_{B,D} \circ \sigma^f_{A,B} \circ \sigma^{P(j^{-1})}_{C,A} \\ &= \sigma^{P(k)\circ f \circ P(j^{-1})}_{C,D} \\ &= \sigma^{X_{j,k}(f)}_{C,D} \end{aligned}$$

and (i) holds. □

## 3. Double Categories with Folding

Ehresmann introduced double categories in [**35**] and [**37**]. After a long gestation period, a full theory of double categories is beginning to emerge. Classics in the subject include [**8**], [**21**], [**22**], [**32**]-[**37**], and [**57**]. For recent work on double categories and related topics, see [**3**], [**13**]-[**20**], [**25**]-[**31**], [**44**]-[**46**], [**56**], and [**62**]-[**65**].

We recall double categories and foldings, as well as their morphisms and transformations. Foldings allow us to compare double categories with $I$-categories in the next section. In Theorem 3.28 we show that foldings are equivalent to connection pairs, as a corollary they are also equivalent to thin structures in the edge-symmetric case.

**Definition 3.1.** A *double category* $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ is a category object in the category of small categories. This means $\mathbb{D}_0$ and $\mathbb{D}_1$ are categories equipped with functors

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r] & \mathbb{D}_1 \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{D}_0 \arrow[l, "\eta"'] \end{tikzcd}$$

that satisfy the usual axioms of a category. We call the objects and morphisms of $\mathbb{D}_0$ the *objects* and *vertical morphisms* of $\mathbb{D}$, and we call the objects and morphisms of $\mathbb{D}_1$ the *horizontal morphisms* and *squares* of $\mathbb{D}$.

We can expand this definition as in [**51**]. A *double category* $\mathbb{D}$ consists of a set of *objects*, a set of *horizontal morphisms*, a set of *vertical morphisms*, and a set of *squares* equipped with various sources, targets, and associative and unital compositions as follows. Objects are denoted with capital Latin letters $A, B, \ldots$, horizontal morphisms are denoted with lower-case Latin letters $f, g, \ldots$, vertical morphisms are denoted with lower-case Latin letters $j, k, \ldots$, and squares are denoted with lower-case Greek letters $\alpha, \beta, \ldots$ with source and target as indicated below.

$$
A \xrightarrow{f} B \qquad \begin{tikzcd} A \arrow[d, "j"'] \\ C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \tag{1}
$$

In particular, $\alpha$ has vertical source and target $f$ and $g$, and horizontal source and target $j$ and $k$ respectively. The objects and vertical morphisms form a category with composition denoted

$$
j_2 \circ j_1 = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}
$$

and identities denoted $1^v_A$. The objects and horizontal morphisms also form a category, with composition denoted

$$
f_2 \circ f_1 = [f_1 \ f_2]
$$

and identities $1^h_A$. The vertical morphisms and squares form a category under horizontal composition of squares, with horizontal identity squares denoted

$$
\begin{tikzcd} A \arrow[r, "1^h_A"] \arrow[d, "j"'] \arrow[dr, phantom, "i^h_j"] & A \arrow[d, "j"] \\ C \arrow[r, "1^h_C"'] & C. \end{tikzcd} \tag{2}
$$

If $\alpha$ and $\beta$ are horizontally composable squares, then their composition is denoted

$$
[\alpha \ \beta].
$$

The horizontal morphisms and squares form a category under vertical composition of squares, with vertical identity squares denoted

$$
\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "i^v_f"] & B \arrow[d, "1^v_B"] \\ A \arrow[r, "f"'] & B. \end{tikzcd} \tag{3}
$$

If $\alpha$ and $\beta$ are vertically composable squares, then their composition is denoted

$$
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}.
$$

The identity squares are compatible with horizontal and vertical composition.

$$
\begin{bmatrix} i^v_{f_1} & i^v_{f_2} \end{bmatrix} = i^v_{[f_1 \ f_2]} \qquad\qquad \begin{bmatrix} i^h_{j_1} \\ i^h_{j_2} \end{bmatrix} = i^h_{\left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]}.
$$

Lastly, the *interchange law* holds, *i.e.* in the situation

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\gamma"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd} \tag{4}$$

we have

$$\begin{bmatrix} [\alpha \quad \beta] \\ [\gamma \quad \delta] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} & \begin{bmatrix} \beta \\ \delta \end{bmatrix} \end{bmatrix}$$

and this composition is denoted

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \tag{5}$$

**Remark 3.2.** A few comments about composition in a double category are in order. A *compatible arrangement* is intuitively a pasting diagram of squares in a double category. It was shown in [**30**] that if a compatible arrangement has a composite, then this composite does not depend on the order of composition, although there may be compatible arrangements in a given double category that do not admit a composite at all. We show in Corollary 3.29 that all compatible arrangements in a double category with folding admit a unique composite. We implicitly use this existence and uniqueness throughout.

**Remark 3.3.** The assignments $j \mapsto i^h_j$ and $f \mapsto i^v_f$ preserve compositions. Preservation of units follows from the other axioms: an application of the interchange law to the diagram of identity morphisms

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "i^v_{1^h_A}"] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "i^h_{1^v_A}"] & A \arrow[d] \\ A \arrow[r] \arrow[d] \arrow[dr, phantom, "i^h_{1^v_A}"] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "i^v_{1^h_A}"] & A \arrow[d] \\ A \arrow[r] & A \arrow[r] & A \end{tikzcd}$$

shows that $i^h_{1^v_A} = i^v_{1^h_A}$. We abbreviate this identity square with identity boundary simply by $i_A$. This proof does not work for pseudo double categories, so $i^h_{1^v_A} = i^v_{1^h_A}$ is an axiom in Definition 6.1.

**Remark 3.4.** As a last comment about composition we remark that double categories are not required to admit mixed compositions between horizontal and vertical morphisms. Typically, horizontal and vertical morphisms are different, as Example 6.3 shows.

**Definition 3.5.** Let $\mathbb{D}$ be a double category. Then $\mathbf{H}\mathbb{D}$ denotes the *horizontal 2-category* of $\mathbb{D}$. Its objects are the objects of $\mathbb{D}$, its morphisms are the horizontal morphisms of $\mathbb{D}$, and its 2-cells are the squares of $\mathbb{D}$ which have vertical identities on the left and right sides. The underlying 1-category of $\mathbf{H}\mathbb{D}$ is denoted $(\mathbf{H}\mathbb{D})_0$. The *vertical 2-category* $\mathbf{V}\mathbb{D}$ of $\mathbb{D}$ and its underlying 1-category $(\mathbf{V}\mathbb{D})_0$ are defined analogously.

**Definition 3.6.** If $\mathbb{D}$ and $\mathbb{E}$ are double categories, a *double functor* $F : \mathbb{D} \longrightarrow \mathbb{E}$ is an *internal functor in Cat*. This consists of functors $F_0 : \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ and $F_1 : \mathbb{D}_1 \longrightarrow \mathbb{E}_1$ such that the diagrams

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r] \arrow[d, "F_1 \times_{F_0} F_1"'] & \mathbb{D}_1 \arrow[d, "F_1"] & \mathbb{D}_0 \arrow[l, "\eta"'] \arrow[d, "F_0"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r] & \mathbb{E}_1 & \mathbb{E}_0 \arrow[l, "\eta"]
\end{tikzcd}$$

commute. In other words, a double functor consists of functions

$$Obj\ \mathbb{D} \longrightarrow Obj\ \mathbb{E}$$

$$Hor\ \mathbb{D} \longrightarrow Hor\ \mathbb{E}$$

$$Ver\ \mathbb{D} \longrightarrow Ver\ \mathbb{E}$$

$$Squares\ \mathbb{D} \longrightarrow Squares\ \mathbb{E}$$

which preserve all sources, targets, compositions, and units.

**Example 3.7.** We can obtain double categories from a 2-category $\mathbf{C}$ in several ways. The double category $\mathbb{H}\mathbf{C}$ has the same objects as $\mathbf{C}$, horizontal morphisms are the morphisms of $\mathbf{C}$, the vertical morphisms are all trivial, and the squares are the 2-cells of $\mathbf{C}$. The double category $\mathbb{V}\mathbf{C}$ is defined similarly, only this time all horizontal morphisms are trivial. Any 2-functor $\mathbf{B} \longrightarrow \mathbf{C}$ induces double functors $\mathbb{H}\mathbf{B} \longrightarrow \mathbb{H}\mathbf{C}$ and $\mathbb{V}\mathbf{B} \longrightarrow \mathbb{V}\mathbf{C}$.

**Example 3.8.** Another double category associated to a 2-category $\mathbf{C}$ is Ehresmann's double category $\mathbb{Q}\mathbf{C}$ of *quintets of* $\mathbf{C}$. Its objects are the objects of $\mathbf{C}$, horizontal and vertical morphisms are the morphisms of $\mathbf{C}$, and the squares $\alpha$ as in (1) are the 2-cells $\alpha : k \circ f \Longrightarrow g \circ j$. Any 2-functor $\mathbf{B} \longrightarrow \mathbf{C}$ induces a double functor $\mathbb{Q}\mathbf{B} \longrightarrow \mathbb{Q}\mathbf{C}$. Note that the horizontal 2-category $\mathbf{H}\mathbb{Q}\mathbf{C}$ is $\mathbf{C}$. The vertical 2-category $\mathbf{V}\mathbb{Q}\mathbf{C}$ is $\mathbf{C}$ with the 2-cells reversed. We could just as well have chosen our quintets to consist of 2-cells $g \circ j \Longrightarrow k \circ f$ instead, but then the roles of $\mathbf{H}\mathbb{Q}\mathbf{C}$ and $\mathbf{V}\mathbb{Q}\mathbf{C}$ would be switched. In this article we use the former convention because compatibility with $\mathbf{H}$ is important for folding. If $I$ is a 1-category viewed as a 2-category with only trivial 2-cells, then $\mathbb{Q}I$ is the *double category* $\Box I$ *of commutative squares in* $I$. A boundary admits a unique square if and only if the boundary is a commutative square. The double categories $\mathbb{Q}\mathbf{C}$ are *edge-symmetric* double categories as in [**20**] because the horizontal and vertical edge categories coincide.

**Example 3.9.** An *adjunction* in a 2-category $\mathbf{C}$ consists of two morphisms $j_1 : A \longrightarrow C$ and $j_2 : C \longrightarrow A$ and two 2-cells $\eta : 1_C \Longrightarrow j_1 \circ j_2$ and $\varepsilon : j_2 \circ j_1 \Longrightarrow 1_A$ which satisfy the familiar triangle identities. Here $j_1$ is the *right* adjoint and $j_2$ is the *left* adjoint. The adjunctions in $\mathbf{C}$ form a double category $\mathbb{A}\mathrm{d}\mathbf{C}$ with objects the objects of $\mathbf{C}$, horizontal morphisms the morphism of $\mathbf{C}$, vertical morphisms the adjunctions in $\mathbf{C}$ (with direction given by the right adjoint), and squares $\alpha$ as in (1) the 2-cells $\alpha : k_1 \circ f \Longrightarrow g \circ j_1$. This double category (with squares reversed) was reviewed in [**51**] to describe the sense in which *mates under adjunctions* are compatible with composition and identity. There is a forgetful double functor $\mathbb{A}\mathrm{d}\mathbf{C} \longrightarrow \mathbb{Q}\mathbf{C}$. A related double category of certain *adjoint squares* was introduced and studied in [**67**] and [**68**].

We will also have occasion to use double natural transformations. There are two types: horizontal and vertical.

**Definition 3.10.** If $F, G : \mathbb{D} \longrightarrow \mathbb{E}$ are double functors, then a *horizontal natural transformation* $\theta : F \Longrightarrow G$ as in [**45**] assigns to each object $A$ a horizontal arrow $\theta A : FA \longrightarrow GA$ and assigns to each vertical morphism $j$ a square

$$\begin{tikzcd}
FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[d, "Gj"] \\
FC \arrow[r, "\theta C"'] & GC
\end{tikzcd}$$

such that:

i. For all $A \in \mathbb{D}$, we have $\theta 1^v_A = i^v_{\theta A}$,

ii. For composable vertical morphisms $j$ and $k$,

$$\begin{tikzcd}
FA \arrow[r, "\theta A"] \arrow[d, "F\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]"'] \arrow[dr, phantom, "\theta\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]"] & GA \arrow[d, "F\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]"] \\
FE \arrow[r, "\theta E"'] & GE
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[d, "Gj"] \\
FC \arrow[r, "\theta C"] \arrow[d, "Fk"'] \arrow[dr, phantom, "\theta k"] & GC \arrow[d, "Gk"] \\
FE \arrow[r, "\theta E"'] & GE,
\end{tikzcd}$$

iii. For all $\alpha$ as in (1),

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "Fj"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[r, "\theta B"] \arrow[d, "Fk"] \arrow[dr, phantom, "\theta k"] & GB \arrow[d, "Gk"] \\
FC \arrow[r, "Fg"'] & FC \arrow[r, "\theta C"'] & GD
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[r, "Gf"] \arrow[d, "Gj"] \arrow[dr, phantom, "G\alpha"] & GB \arrow[d, "Gk"] \\
FC \arrow[r, "\theta C"'] & GC \arrow[r, "Gg"'] & GD.
\end{tikzcd}$$

A horizontal natural transformation is the same as an *internal natural transformation in Cat*. We also have vertical natural transformations:

**Definition 3.11.** If $F, G : \mathbb{D} \longrightarrow \mathbb{E}$ are double functors, then a *vertical natural transformation* $\sigma : F \Longrightarrow G$ as in [**45**] assigns to each object $A$ a vertical arrow $\sigma A : FA \longrightarrow GA$ and assigns to each horizontal morphism $f$ a square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\sigma A"'] \arrow[dr, phantom, "\sigma f"] & FB \arrow[d, "\sigma B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

such that:

i. For all objects $A \in \mathbb{D}$, we have $\sigma 1^h_A = i^h_{\sigma A}$,

ii. For all composable horizontal morphisms $f$ and $g$,

$$\sigma[f\ \ g] = [\sigma f \quad \sigma g],$$

iii. For all $\alpha$ as in (1),

$$\begin{bmatrix} F\alpha \\ \sigma g \end{bmatrix} = \begin{bmatrix} \sigma f \\ G\alpha \end{bmatrix}.$$

**Example 3.12.** An oplax natural transformation between 2-functors $\mathbf{B} \longrightarrow \mathbf{C}$ is the same as a vertical natural transformation between the induced double functors $\mathbb{H}\mathbf{B} \longrightarrow \mathbb{H}\mathbf{C}$. The components are necessarily trivial.

To compare $I$-categories (and more generally pseudo $I$-categories) with certain double categories, we extend Brown and Mosa's notion of *folding* to non-edge-symmetric double categories. We prove that a folding is equivalent to a *connection pair* in Lemmas 3.24-3.27 and Theorem 3.28. In the case of edge-symmetric double categories, a connection pair (with trivial holonomy) is the same as a *thin structure* as shown in [**20**], and in higher dimensions in [**48**]. Edge-symmetric foldings were used already in [**15**] to prove that the category of crossed complexes is equivalent to the category of cubical $\omega$-groupoids, and were generalized to all dimensions in [**2**]. More recently, foldings found important applications in [**3**] and [**48**]. To define foldings, we recall Brown and Spencer's notion of *holonomy* in [**21**]:

**Definition 3.13.** A *holonomy* for a double category $\mathbb{D}$ is a 2-functor $(\mathbf{V}\mathbb{D})_0 \longrightarrow \mathbf{H}\mathbb{D}$ which is the identity on objects. In other words, a *holonomy* associates to a vertical morphism a horizontal morphism with the same domain and range in a functorial way.

**Remark 3.14.** If a double category is equipped with a holonomy, then we can define a composition of vertical morphisms $j$ with morphisms and 2-cells of $\mathbf{H}\mathbb{D}$ by

$$f \circ j := f \circ \overline{j}$$

$$j \circ g := \overline{j} \circ g$$

$$\alpha \circ j := \alpha \circ i^v_{\overline{j}}$$

$$j \circ \beta := i^v_{\overline{j}} \circ \beta$$

to obtain morphisms and 2-cells of $\mathbf{H}\mathbb{D}$. Here $\circ$ on the right-hand side designates horizontal composition of morphisms and 2-cells of the 2-category $\mathbf{H}\mathbb{D}$. These mixed compositions satisfy the obvious axioms: associativity, unitality, and the usual axioms of left and right whiskering (see [**74**]). Conversely, a double category with a mixed composition satisfying these axioms admits a holonomy defined by $j \mapsto 1_C \circ j = j \circ 1_A$. These two procedures are inverse, thus holonomies are the same as such mixed compositions.

**Remark 3.15.** Given a double category $\mathbb{D}$ equipped with a holonomy, or equivalently with mixed composition, one can construct a new double category $\mathbb{D}'$ with an inclusion holonomy. The objects and vertical 1-categories of $\mathbb{D}$ and $\mathbb{D}'$ are the same, while the set of horizontal morphisms of $\mathbb{D}'$ is the disjoint union of the sets of horizontal and vertical morphisms of $\mathbb{D}$. Composition of horizontal morphisms in $\mathbb{D}'$ is the mixed composition, with identities the included vertical identities. The squares of $\mathbb{D}'$ are the squares of $\mathbb{D}$ along with vertical identity squares for the horizontal morphisms of $\mathbb{D}'$ which come from vertical morphisms of $\mathbb{D}$. We equip $\mathbb{D}'$ with a holonomy by including the vertical morphisms, so that the double functor $\mathbb{D}' \longrightarrow \mathbb{D}$ preserves the holonomies. We will apply this construction in Example 6.10 and Example 7.12.

**Definition 3.16.** A *folding on a double category* $\mathbb{D}$ is a double functor $\Lambda : \mathbb{D} \longrightarrow \mathbb{Q}\mathbf{H}\mathbb{D}$ which is the identity on the horizontal 2-category $\mathbf{H}\mathbb{D}$ of $\mathbb{D}$ and is faithfully full on squares. More specifically, a folding consists of a holonomy $j \longmapsto \overline{j}$ and bijections $\Lambda^{f,k}_{j,g}$ from squares in $\mathbb{D}$ with boundary

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \tag{6}$$

to squares in $\mathbb{D}$ with boundary

$$\begin{tikzcd} A \arrow[r, "{[f\ \overline{k}]}"] \arrow[d, "1^v_A"'] & D \arrow[d, "1^v_D"] \\ A \arrow[r, "{[\overline{j}\ g]}"'] & D, \end{tikzcd} \tag{7}$$

such that:

i. $\Lambda$ is the identity if $j$ and $k$ are vertical identity morphisms.

ii. $\Lambda$ preserves horizontal composition of squares, *i.e.*

$$\Lambda\left(\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "f_2"] \arrow[d, "k"] \arrow[dr, phantom, "\beta"] & C \arrow[d, "\ell"] \\ D \arrow[r, "g_1"'] & E \arrow[r, "g_2"'] & F \end{tikzcd}\right) = \begin{tikzcd} A \arrow[r, "{[f_1\ f_2\ \overline{\ell}]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "{[i^v_{f_1}\ \Lambda(\beta)]}"] & F \arrow[d, "1^v_F"] \\ A \arrow[r, "{[f_1\ \overline{k}\ g_2]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "{[\Lambda(\alpha)\ i^v_{g_2}]}"] & F \arrow[d, "1^v_F"] \\ A \arrow[r, "{[\overline{j}\ g_1\ g_2]}"'] & F. \end{tikzcd}$$

iii. $\Lambda$ preserves vertical composition of squares, *i.e.*

$$\Lambda\left(\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j_1"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "k_1"] \\ C \arrow[r, "g"] \arrow[d, "j_2"'] \arrow[dr, phantom, "\beta"] & D \arrow[d, "k_2"] \\ E \arrow[r, "h"'] & F, \end{tikzcd}\right) = \begin{tikzcd} A \arrow[r, "{[f\ \overline{k}_1\ \overline{k}_2]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "{[\Lambda(\alpha)\ i^v_{\overline{k}_2}]}"] & F \arrow[d, "1^v_E"] \\ A \arrow[r, "{[\overline{j}_1\ g\ \overline{k}_2]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "{[i^v_{\overline{j}_1}\ \Lambda(\beta)]}"] & F \arrow[d, "1^v_E"] \\ A \arrow[r, "{[\overline{j}_1\ \overline{j}_2\ h]}"'] & F. \end{tikzcd}$$

iv. $\Lambda$ preserves identity squares, *i.e.*

$$\Lambda\left(\begin{tikzcd} A \arrow[r, "1^h_A"] \arrow[d, "j"'] \arrow[dr, phantom, "i^h_j"] & A \arrow[d, "j"] \\ B \arrow[r, "1^h_B"'] & B \end{tikzcd}\right) = \begin{tikzcd} A \arrow[r, "{[1^h_A\ \overline{j}]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "i^v_{\overline{j}}"] & B \arrow[d, "1^v_B"] \\ A \arrow[r, "{[\overline{j}\ 1^h_B]}"'] & B. \end{tikzcd}$$

**Definition 3.17.** Let $\mathbb{D}$ and $\mathbb{E}$ be double categories with folding. A *morphism of double categories with folding* $F : \mathbb{D} \longrightarrow \mathbb{E}$ is a double functor such that

$$F(\overline{j}) = \overline{F(j)}$$

$$F(\Lambda^{\mathbb{D}}(\alpha)) = \Lambda^{\mathbb{E}}(F(\alpha))$$

for all vertical morphisms $j$ and squares $\alpha$ in $\mathbb{D}$. This is a double functor $F$ such that

$$
\begin{tikzcd}
\mathbb{D} \arrow[r, "F"] \arrow[d] & \mathbb{E} \arrow[d] \\
\mathbb{Q}\mathbf{H}\mathbb{D} \arrow[r, "\mathbb{Q}\mathbf{H}F"'] & \mathbb{Q}\mathbf{H}\mathbb{E}
\end{tikzcd}
$$

commutes.

**Definition 3.18.** Let $F, G : \mathbb{D} \longrightarrow \mathbb{E}$ be morphisms of double categories with folding. A horizontal natural transformation $\theta : F \Longrightarrow G$ is *compatible with folding* if for all vertical morphisms $j$ the following equation holds.

$$
\Lambda\left(
\begin{tikzcd}
FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[d, "Gj"] \\
FC \arrow[r, "\theta C"'] & GC
\end{tikzcd}
\right)
=
\begin{tikzcd}
FA \arrow[r, "{[\theta A\ G\overline{j}]}"] \arrow[d, "1^v_{FA}"'] \arrow[dr, phantom, "i^v_{[\theta A\ G\overline{j}]}"] & GC \arrow[d, "1^v_{GC}"] \\
FA \arrow[r, "{[G\overline{j}\ \theta C]}"'] & GC
\end{tikzcd}
$$

A vertical natural transformation $\sigma : F \Longrightarrow G$ is *compatible with folding* if for all vertical morphisms $j$ the following equation holds.

$$
\Lambda\left(
\begin{tikzcd}
FA \arrow[r, "F\overline{j}"] \arrow[d, "\sigma A"'] \arrow[dr, phantom, "\sigma\overline{j}"] & FC \arrow[d, "\sigma C"] \\
GA \arrow[r, "G\overline{j}"'] & GC
\end{tikzcd}
\right)
=
\begin{tikzcd}
FA \arrow[r, "{[F\overline{j}\ \overline{\sigma C}]}"] \arrow[d, "1^v_{FA}"'] \arrow[dr, phantom, "i^v_{[F\overline{j}\ \sigma C]}"] & GC \arrow[d, "1^v_{GC}"] \\
FA \arrow[r, "{[\overline{\sigma A}\ G\overline{j}]}"'] & GC
\end{tikzcd}
$$

**Remark 3.19.** The compatibility of a horizontal natural transformation with folding implies that it is entirely determined by its restriction to the horizontal 2-category. Even more is true, any 2-natural transformation between the underlying horizontal 2-functors of two morphisms gives rise to a horizontal natural transformation compatible with folding, since the compatibility defines $\theta j$ and the folding axioms guarantee (ii) and (iii) of Definition 3.10. The analogous remark for vertical natural transformations does not hold, since compatibility only concerns $\sigma\overline{j}$ and not the more general $\sigma f$.

**Definition 3.20.** Let $\mathbb{D}$ be a double category equipped with two foldings $\Lambda_1, \Lambda_2 : \mathbb{D} \longrightarrow \mathbb{Q}\mathbf{H}\mathbb{D}$. Then a *morphism of foldings* $\theta : \Lambda_1 \longrightarrow \Lambda_2$ is a *horizontal* natural transformation

$$\theta : \Lambda_1 \Longrightarrow \Lambda_2$$

with identity components. Equivalently, writing $j \mapsto \overline{j}$ for the holonomy of $\Lambda_1$ and $j \mapsto \overline{j}$ for the holonomy of $\Lambda_2$, a morphism of foldings $\theta$ assigns to each vertical morphism $j$ a square

$$\begin{tikzcd} A \arrow[r, "\overline{j}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "\theta j"] & C \arrow[d, "1^v_C"] \\ A \arrow[r, "\overline{\overline{j}}"'] & C \end{tikzcd} \tag{8}$$

such that:

i. $\theta$ preserves identities, *i.e.* $\theta 1^v_A = i_A$.

ii. $\theta$ preserves compositions, *i.e.* $\theta\begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = [\theta j_1 \;\; \theta j_2]$.

iii. $\theta$ is natural, *i.e.*

$$\begin{tikzcd} A \arrow[r, "{[f\ \overline{k}]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "\Lambda_1(\alpha)"] & D \arrow[d, "1^v_D"] \\ A \arrow[r, "{[\overline{j}\ g]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "{[\theta j\ i^v_g]}"] & D \arrow[d, "1^v_D"] \\ A \arrow[r, "{[\overline{\overline{j}}\ g]}"'] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "{[f\ \overline{k}]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "{[i^v_f\ \theta k]}"] & D \arrow[d, "1^v_D"] \\ A \arrow[r, "{[f\ \overline{\overline{k}}]}"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "\Lambda_2(\alpha)"] & D \arrow[d, "1^v_D"] \\ A \arrow[r, "{[\overline{\overline{j}}\ g]}"'] & D. \end{tikzcd}$$

It may appear that a morphism of foldings is a vertical natural transformation because of Diagram (8). But this is not so, since $\theta j$ is a square in the $\mathbb{Q}\mathbf{H}\mathbb{D}$ with trivial horizontal components, and such a square is precisely of the form (8). One could alternatively interpret (8) to be an oplax natural transformation with identity components between the 2-functors that constitute the holonomies, though the naturality of this comparison 2-cell is not equivalent to the full naturality of (iii).

A double category with folding is determined by its vertical 1-catego-ry, its horizontal 2-category, and the holonomy. Vice-a-versa, one can construct a double category with folding from a 2-category equipped with a 2-functor resembling a holonomy. This will be made precise in Theorem 4.6, which states the key feature of foldings: the 2-category of double categories with folding is 2-equivalent to the 2-category of certain 2-functors. The pseudo counterpart of Theorem 4.6 is Theorem 7.9).

The squares of a double category with folding are determined by the 2-cells of the underlying horizontal 2-category via the folding. A folding *horizontalizes* a double category in the sense that it maps a double category to its underlying horizontal 2-category in a functorial way in terms of quintets. Thus, the quintessential example of a double category with folding is the double category of quintets of a 2-category as follows.

**Example 3.21.** Let $\mathbf{C}$ be a 2-category and $\mathbb{Q}\mathbf{C}$ the double category of quintets of $\mathbf{C}$ as in Example 3.8. The holonomy is the inclusion of the vertical 1-category $\mathbf{C}_0$

into the horizontal 2-category **C**, and the folding maps are the identity: the squares of $\mathbb{Q}\mathbf{C}$ with boundary (6) are by definition the 2-cells in **C** with boundary (7). In fact $\mathbb{Q}$ is a 2-functor from the 2-category of small 2-categories to the 2-category of double categories with folding, morphisms, and horizontal natural transformations compatible with folding. As a special case of $\mathbb{Q}\mathbf{C}$, the double category $\square I$ of commutative squares in a 1-category $I$ admits a folding. The folding on $\square I$ is unique. In fact, we'll see in Theorem 3.30 that foldings are unique up to isomorphism.

**Example 3.22.** The double category $\mathbb{A}\mathrm{d}\mathbf{C}$ of Example 3.9 admits a canonical folding: the holonomy sends an adjunction to its right adjoint part. The forgetful double functor $\mathbb{A}\mathrm{d}\mathbf{C} \longrightarrow \mathbb{Q}\mathbf{C}$ is an example of a morphism of double categories with folding.

In Section 6 we will extend the notion of folding to pseudo double categories. The extended notion has more examples, such as the pseudo double categories $\mathbb{R}\mathrm{ng}$ and $\mathbb{W}$ of bimodules and worldsheets.

Connection pairs on double categories can be found in [**20**] and [**73**]. In the terminology of [**46**], a connection pair is a functorial choice of a so-called orthogonal companion for each vertical morphism.

**Definition 3.23.** A *connection pair* on a double category consists of a holonomy $j \longmapsto \overline{j}$ and an assignment of a pair of squares

$$\begin{tikzcd} A \arrow[r, "\overline{j}"] \arrow[d, "j"'] \arrow[dr, phantom, "\Gamma(j)"] & C \arrow[d, "1^v_C"] \\ C \arrow[r, "1^h_C"'] & C \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, "1^h_A"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "\Gamma'(j)"] & A \arrow[d, "j"] \\ A \arrow[r, "\overline{j}"'] & C \end{tikzcd}$$

to each vertical morphism $j$ such that:

i. $\Gamma$ and $\Gamma'$ preserve identities.

$$\Gamma(1^v_A) = i_A \qquad\qquad \Gamma'(1^v_A) = i_A$$

ii. $\Gamma$ and $\Gamma'$ preserve compositions, *i.e.* the *transport laws* hold.

$$\Gamma\left(\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}\right) \quad = \quad \begin{tikzcd} \cdot \arrow[r, "\overline{j}_1"] \arrow[d, "j_1"'] \arrow[dr, phantom, "\Gamma(j_1)"] & \cdot \arrow[r, "\overline{j}_2"] \arrow[d] \arrow[dr, phantom, "i^v_{\overline{j}_2}"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d, "j_2"'] \arrow[dr, phantom, "i^h_{j_2}"] & \cdot \arrow[r, "\overline{j}_2"] \arrow[d, "j_2"] \arrow[dr, phantom, "\Gamma(j_2)"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

$$\Gamma'\left(\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}\right) = \begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Gamma'(j_1)"] & \cdot \arrow[r] \arrow[d, "j_1"] \arrow[dr, phantom, "i^h_{j_1}"] & \cdot \arrow[d, "j_1"] \\
\cdot \arrow[r, "\overline{j}_1"] \arrow[d] \arrow[dr, phantom, "i^v_{\overline{j}_1}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Gamma'(j_2)"] & \cdot \arrow[d, "j_2"] \\
\cdot \arrow[r, "\overline{j}_1"'] & \cdot \arrow[r, "\overline{j}_2"'] & \cdot
\end{tikzcd}$$

(unlabelled arrows are the identities).

iii.

$$i^v_{\overline{j}} = \begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Gamma'(j)"] & \cdot \arrow[r, "\overline{j}"] \arrow[d, "j"] \arrow[dr, phantom, "\Gamma(j)"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\overline{j}"'] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\qquad
i^h_j = \begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Gamma'(j)"] & \cdot \arrow[d, "j"] \\
\cdot \arrow[r, "\overline{j}"] \arrow[d, "j"'] \arrow[dr, phantom, "\Gamma(j)"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

(unlabelled arrows are the identities).

We now work towards a proof of Theorem 3.28, which states that the data for a connection pair is equivalent to the data for a folding. This proof is essentially a slight generalization of an argument in Section 5 of [**20**]. The idea goes back to Spencer in [**73**] and to the quintets of Ehresmann in [**36**]. If a double category admits a folding, then that folding is unique up to isomorphism.

**Lemma 3.24.** *If* $(\Gamma, \Gamma')$ *is a connection pair on a double category, then*

$$\Lambda^{f,k}_{j,g}(\alpha) := \begin{bmatrix} \Gamma'(j) & \alpha & \Gamma(k) \end{bmatrix}$$

*defines a folding.*

*Proof.* First we show that the holonomy is part of a double functor $D \longrightarrow \mathbb{Q}\mathbf{H}\mathbb{D}$.

i. Follows from Definition 3.23 (i).

ii. In the notation of Definition 3.23, we have

$$\begin{aligned}
\Lambda(\begin{bmatrix} \alpha & \beta \end{bmatrix}) &= \begin{bmatrix} \Gamma'(j) & \alpha & \beta & \Gamma(\ell) \end{bmatrix} \\
&= \begin{bmatrix} i^v_1 & i^v_{f_1} & \Gamma'(k) & \beta & \Gamma(\ell) \\ \Gamma'(j) & \alpha & \Gamma(k) & i^v_{g_2} & i^v_1 \end{bmatrix} \\
&= \begin{bmatrix} i^v_{f_1} & \Lambda(\beta) \\ \Lambda(\alpha) & i^v_{g_2} \end{bmatrix}
\end{aligned}$$

by way of Definition 3.23 (iii).

iii. This proof is similar to the proof just given in (ii).

iv.

$$\begin{aligned}\Lambda(i^h_j) &= \begin{bmatrix}\Gamma'(j) & i^h_j & \Gamma(j)\end{bmatrix}\\ &= \begin{bmatrix}\Gamma'(j) & \Gamma(j)\end{bmatrix}\\ &= i^v_{\overline{j}}\end{aligned}$$

by Definition 3.23 (iii).

The double functor $\Lambda$ is surjective on squares, since

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "i^v_f"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Gamma'(k)"] & {} \arrow[d, "k"] \\
{} \arrow[r, "f"] \arrow[d] \arrow[drr, phantom, "\delta"] & {} \arrow[r, "\overline{k}"] & {} \arrow[d] \\
{} \arrow[r, "\overline{j}"] \arrow[d, "j"'] \arrow[dr, phantom, "\Gamma(j)"] & {} \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "i^v_g"] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r, "g"'] & {}
\end{tikzcd}$$

maps to $\delta$ by the left half of Definition 3.23 (iii). The right half of Definition 3.23 (iii) shows that

$$\begin{tikzcd}
{} \arrow[rr, "f"] \arrow[d] \arrow[drr, phantom, "i^v_f"] & & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Gamma'(k)"] & {} \arrow[d, "k"] \\
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Gamma'(j)"] & {} \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & {} \arrow[r, "\overline{k}"] \arrow[d, "k"'] \arrow[dr, phantom, "\Gamma(k)"] & {} \arrow[d] \\
{} \arrow[r, "\overline{j}"] \arrow[d, "j"'] \arrow[dr, phantom, "\Gamma(j)"] & {} \arrow[r, "g"] \arrow[d] \arrow[drr, phantom, "i^v_g"] & {} \arrow[r] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[rr, "g"'] & & {}
\end{tikzcd}$$

equals $\alpha$, so that $\Lambda$ is injective on squares. □

We now prove the converse to Lemma 3.24:

**Lemma 3.25.** *If $\Lambda$ is a folding on a double category, then*

$$\Gamma(j) := (\Lambda^{\overline{j},1}_{j,1})^{-1}(i^v_{\overline{j}}) \qquad \Gamma'(j) := (\Lambda^{1,j}_{1,\overline{j}})^{-1}(i^v_{\overline{j}})$$

*defines a connection pair.*

*Proof.* i. Follows because $\Lambda$ is the identity on squares with identity boundary.

ii. An application of $\Lambda^{\overline{[j_1 j_2]},1}_{\left[\begin{smallmatrix} j_1\\ j_2\end{smallmatrix}\right],1}$ to the right side of the equation for $\Gamma$ in (ii) of Definition 3.23 yields the following.

$$\Lambda^{\overline{[j_1 j_2]},1}_{\left[\begin{smallmatrix} j_1\\ j_2\end{smallmatrix}\right],1}\left(\begin{bmatrix}\Gamma(j_1) & i^v_{\bar{j}_2}\\ i^h_{j_2} & \Gamma(j_2)\end{bmatrix}\right) = \Lambda\left(\begin{bmatrix}\begin{bmatrix}\Gamma(j_1) & i^v_{\bar{j}_2}\end{bmatrix}\\ \begin{bmatrix}i^h_{j_2} & \Gamma(j_2)\end{bmatrix}\end{bmatrix}\right)$$

$$= \begin{tikzcd} \cdot \arrow[r, "{[\bar{j}_1\ \bar{j}_2]}"] \arrow[d] \arrow[dr, phantom, "{\Lambda([\Gamma(j_1)\ i^v_{\bar{j}_2}])}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "{[\bar{j}_1\ \bar{j}_2]}"] \arrow[d] \arrow[dr, phantom, "{i^v_{\bar{j}_1}\Lambda([i^h_{j_2}\ \Gamma(j_2)])}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "{[\bar{j}_1\ \bar{j}_2]}"] & \cdot \end{tikzcd}$$

$$= \begin{bmatrix} \begin{bmatrix} i^v_{\bar{j}_1} & \Lambda(i^v_{\bar{j}_2})\end{bmatrix}\\ \begin{bmatrix}\Lambda(\Gamma(j_1)) & i^v_{\bar{j}_2}\end{bmatrix}\\ \begin{bmatrix} i^v_{\bar{j}_1} & \Lambda(\Gamma(j_2))\end{bmatrix}\\ \begin{bmatrix} i^v_{\bar{j}_1} & \Lambda(i^h_{j_2})\end{bmatrix}\end{bmatrix}$$

$$= i^v_{[\bar{j}_1\ \bar{j}_2]}$$

From this we conclude $\Gamma\left(\begin{bmatrix} j_1\\ j_2\end{bmatrix}\right) = \begin{bmatrix}\Gamma(j_1) & i^v_{\bar{j}_2}\\ i^h_{j_2} & \Gamma(j_2)\end{bmatrix}$. A similar argument works for $\Gamma'$.

iii. An application of $\Lambda^{\bar{j},1}_{1,\bar{j}}$ to $[\Gamma'(j)\ \ \Gamma(j)]$ in (iii) of Definition 3.23 yields the following.

$$\begin{aligned} \begin{bmatrix}\Gamma'(j) & \Gamma(j)\end{bmatrix} &= \Lambda^{\bar{j},1}_{1,\bar{j}}\left(\begin{bmatrix}\Gamma'(j) & \Gamma(j)\end{bmatrix}\right)\\ &= \begin{bmatrix}\Lambda(\Gamma(j))\\ \Lambda(\Gamma'(j))\end{bmatrix}\\ &= \begin{bmatrix} i^v_{\bar{j}}\\ i^v_{\bar{j}}\end{bmatrix}\\ &= i^v_{\bar{j}} \end{aligned}$$

A similar argument shows $\begin{bmatrix}\Gamma'(j)\\ \Gamma(j)\end{bmatrix} = i^h_j$.

□

**Lemma 3.26.** *Let $(\Gamma, \Gamma')$ be a connection pair on a double category with associated folding $\Lambda$ as in Lemma 3.24. Then the connection pair associated to $\Lambda$ as in Lemma 3.25 is the connection pair we started with.*

*Proof.* By Definition 3.23 (iii) we see that

$$(\Lambda^{\bar{j},1}_{j,1})(\Gamma(j)) = \begin{bmatrix}\Gamma'(j) & \Gamma(j) & i^v_1\end{bmatrix} = i^v_{\bar{j}}$$

$$(\Lambda^{1,j}_{1,\bar{j}})(\Gamma'(j)) = \begin{bmatrix}i^v_1 & \Gamma'(j) & \Gamma(j)\end{bmatrix} = i^v_{\bar{j}}.$$

□

**Lemma 3.27.** *Let $\Lambda$ be a folding on a double category with associated connection pair $(\Gamma, \Gamma')$ as in Lemma 3.25. Then the folding associated to $(\Gamma, \Gamma')$ as in Lemma 3.24 is the folding we started with.*

*Proof.* The square $\begin{bmatrix}\Gamma'(j) & \alpha & \Gamma(k)\end{bmatrix}$ has trivial vertical edges, and is therefore preserved by $\Lambda$ as in (i) of Definition 3.16.

$$\begin{aligned}\begin{bmatrix}\Gamma'(j) & \alpha & \Gamma(k)\end{bmatrix} &= \Lambda\left(\begin{bmatrix}\Gamma'(j) & \alpha & \Gamma(k)\end{bmatrix}\right)\\ &= \Lambda\left(\begin{bmatrix}\begin{bmatrix}\Gamma'(j) & \alpha\end{bmatrix} & \Gamma(k)\end{bmatrix}\right)\\ &= \begin{bmatrix}\begin{bmatrix}i^v_f & \Lambda(\Gamma(k))\end{bmatrix}\\ \begin{bmatrix}\Lambda(\begin{bmatrix}\Gamma'(j) & \alpha\end{bmatrix})\end{bmatrix}\end{bmatrix}\\ &= \begin{bmatrix}\begin{bmatrix}i^v_f & & i^v_{\bar{k}}\end{bmatrix}\\ \begin{bmatrix}\Lambda(\alpha)\end{bmatrix}\\ \begin{bmatrix}\Lambda(\Gamma'(j)) & i_g\end{bmatrix}\end{bmatrix}\\ &= \begin{bmatrix}\begin{bmatrix}i^v_f & & i^v_{\bar{k}}\end{bmatrix}\\ \begin{bmatrix}\Lambda(\alpha)\end{bmatrix}\\ \begin{bmatrix}i^v_{\bar{j}} & & i_g\end{bmatrix}\end{bmatrix}\\ &= \Lambda(\alpha).\end{aligned}$$

□

**Theorem 3.28.** *The notions of connection pair and folding on a double category are equivalent.*

*Proof.* This follows from Lemmas 3.24-3.27. □

**Corollary 3.29.** *Any compatible arrangement in a double category with folding admits a unique composite.*

*Proof.* We imitate the proof of the edge-symmetric case in [**20**]. Theorem 4.1 and Theorem 5.1 in [**30**] provide a useful criterion for every compatible arrangement of a double category $\mathbb{D}$ to admit a unique composite. Suppose that every square $\alpha$ as in (1) satisfies the following condition. If either the horizontal source $j$ or the horizontal target $k$ admits a (vertical) factorization, then that factorization extends to a vertical factorization

$$\alpha = \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}.$$

In this situation, every compatible arrangement of $\mathbb{D}$ admits a unique composite.

We claim that any double category $\mathbb{D}$ with folding satisfies this criterion. Let $(\Gamma, \Gamma')$ be the connection pair associated to the folding. If $j = \left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]$, then

$$\alpha \quad = \quad \begin{tikzcd} {} \arrow[r, equal] \arrow[d, "j_1"'] \arrow[dr, phantom, "i^h_{j_1}"] & {} \arrow[r, "f"] \arrow[d, "j_1"] & {} \arrow[dd, "k"] \\ {} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Gamma'(j_2)"] & {} \arrow[d, "j_2"] \arrow[ur, phantom, "\alpha"] & \\ {} \arrow[r, "\overline{j_2}"] \arrow[d, "j_2"'] \arrow[dr, phantom, "\Gamma(j_2)"] & {} \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "i^v_g"] & {} \arrow[d, equal] \\ {} \arrow[r, equal] & {} \arrow[r, "g"'] & {} \end{tikzcd}$$

is a vertical factorization of $\alpha$ extending the factorization of $j$. A similar proof works for factorizations of $k$. □

If a double category admits a folding, then the folding is essentially unique:

**Theorem 3.30.** *Any two foldings on a double category $\mathbb{D}$ are isomorphic.*

*Proof.* Suppose $(\Lambda_1, j \mapsto \overline{j})$ and $(\Lambda_2, j \mapsto \overline{\overline{j}})$ are foldings on $\mathbb{D}$ with respective associated connection pairs $(\Gamma_1, \Gamma_1')$ and $(\Gamma_2, \Gamma_2')$. We define a morphism $\theta : (\Lambda_1, j \mapsto \overline{j}) \longrightarrow (\Lambda_2, j \mapsto \overline{\overline{j}})$ of foldings by $\theta j := \Lambda_2(\Gamma_1(j))$. This is natural for $\alpha$ as in (1) because

$$\begin{aligned} \begin{bmatrix} \Lambda_1(\alpha) \\ \begin{bmatrix} \Lambda_2(\Gamma_1(j)) & i^v_g \end{bmatrix} \end{bmatrix} &= \begin{bmatrix} i^v_1 & \Gamma_1'(j) & \alpha & \Gamma_1(k) \\ \Gamma_2'(j) & \Gamma_1(j) & i^v_g & i^v_1 \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_2'(j) & \alpha & \Gamma_1(k) \end{bmatrix} \\ &= \begin{bmatrix} i^v_1 & i^v_f & \Gamma_2'(k) & \Gamma_1(k) \\ \Gamma_2'(j) & \alpha & \Gamma_2(k) & i^v_1 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} i^v_f & \Lambda_2(\Gamma_1(k)) \end{bmatrix} \\ \Lambda_2(\alpha) \end{bmatrix}. \end{aligned}$$

An inverse to $\theta j$ is given by $\theta^{-1} j := \Lambda_2(\Gamma_1'(j))$.

$$\begin{tikzcd} \cdot \arrow[r, "\overline{j}"] \arrow[d] \arrow[dr, phantom, "\theta j"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\overline{\overline{j}}"] \arrow[d] \arrow[dr, phantom, "\theta^{-1} j"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\overline{j}"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "\overline{j}"] \arrow[d] \arrow[dr, phantom, "\Lambda_2(\Gamma_1(j))"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\overline{\overline{j}}"] \arrow[d] \arrow[dr, phantom, "\Lambda_2(\Gamma'_1(j))"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\overline{j}"'] & \cdot \end{tikzcd}$$

$$\begin{aligned} &= \Lambda_2(\begin{bmatrix}\Gamma'_1(j) & \Gamma_1(j)\end{bmatrix}) \\ &= \Lambda_2(i^v_{\overline{j}}) \\ &= i^v_{\overline{j}} \end{aligned}$$

The other direction $\begin{bmatrix}\theta^{-1}j \\ \theta j\end{bmatrix} = i^v_{\overline{\overline{j}}}$ follows similarly from

$$\begin{bmatrix}\Gamma'_1(j) \\ \Gamma_1(j)\end{bmatrix} = i^h_j.$$

□

A related structure on an edge-symmetric double category is a thin structure as in [**20**]:

**Definition 3.31.** Let $\mathbb{D}$ be an edge-symmetric double category. Then a *thin structure* on $\mathbb{D}$ is a double functor

$$\Theta : \square(\mathbf{H}\mathbb{D})_0 \longrightarrow \mathbb{D}$$

which is the identity on objects and morphisms. Here $\square(\mathbf{H}\mathbb{D})_0$ is the double category of commutative squares of morphisms of $\mathbb{D}$. The squares of $\mathbb{D}$ in the image of $\Theta$ are called *thin*. Clearly, any commutative boundary in $\mathbb{D}$ has a unique thin filler and any composition of thin squares is thin.

**Theorem 3.32.** *(Brown-Mosa in [**20**]) A thin structure and a connection pair with trivial holonomy on an edge-symmetric double category determine each other.*

**Corollary 3.33.** *The notions of folding with trivial holonomy and thin structure on an edge-symmetric double category are equivalent.*

After introducing double categories, foldings, connection pairs, and thin structures in this section, we put them to use in an alternate description of $I$-categories in the next section.

## 4. $I$-Categories and Double Categories with Folding

Although an $I$-category is not the same thing as an internal category in $Cat$, it is of course a related concept. In this section we show how $I$-categories are related to double categories with folding in an explicit and elementary way. Surprisingly, both notions are equivalent to the simple notion of a strict 2-functor $I \longrightarrow \mathcal{C}$ that is the identity on objects. We introduce three 2-categories $\mathcal{X}^{strict}, \mathcal{Y}^{strict}, \mathcal{Z}^{strict}$ and show that they are 2-equivalent if $I$ is a groupoid. We also prove that $\mathcal{Y}^{strict}$ and $\mathcal{Z}^{strict}$ remain 2-equivalent even if $I$ is merely a category. Unless stated otherwise, $I$ denotes a fixed category.

**Notation 4.1.** Let $\mathcal{X}^{strict}$ denote the 2-category of $I$-categories as defined in Section 2. The morphisms in $\mathcal{X}^{strict}$ are strict.

**Notation 4.2.** Let $\mathcal{Y}^{strict}$ denote the 2-category whose objects are double categories $\mathbb{D}$ with folding such that $(\mathbf{V}\mathbb{D})_0 = I$. A morphism in $\mathcal{Y}^{strict}$ is a morphism of double categories with folding which is also the identity on $(\mathbf{V}\mathbb{D})_0$.

The 2-cells of $\mathcal{Y}^{strict}$ are vertical natural transformations that are compatible with folding and also have identity components. More precisely, recall that $\mathbf{H}\mathbb{D}(A,B)$ denotes the category whose objects are the horizontal morphisms from $A$ to $B$ in $\mathbb{D}$ and whose morphisms are 2-cells of $\mathbf{H}\mathbb{D}$ with source and target such horizontal morphisms. If $F : \mathbb{D} \longrightarrow \mathbb{E}$ is a morphism, we denote its restriction to $\mathbf{H}\mathbb{D}(A,B)$ by $\mathbf{H}F_{A,B} : \mathbf{H}\mathbb{D}(A,B) \longrightarrow \mathbf{H}\mathbb{E}(FA,FB) = \mathbf{H}\mathbb{E}(A,B)$. If $F, G : \mathbb{D} \longrightarrow \mathbb{E}$ are morphisms in $\mathcal{Y}^{strict}$, then a 2-cell $\sigma : F \Longrightarrow G$ in $\mathcal{Y}^{strict}$ assigns to each pair $(A,B) \in I^2$ a natural transformation $\sigma_{A,B} : \mathbf{H}F_{A,B} \Longrightarrow \mathbf{H}G_{A,B}$ such that

$$\sigma^{\overline{j}}_{A,C} = i^v_{\overline{j}}$$

$$[\ \sigma^f_{A,B} \quad \sigma^g_{B,C}\ ] = \sigma^{[f\ g]}_{A,C}$$

$$\sigma^{1^h_A}_{A,A} = i^v_{1^h_A}$$

for all vertical morphisms $j : A \longrightarrow C$, composable horizontal morphisms $f$,$g$, and all objects $A$. With these definitions, $\mathcal{Y}^{strict}$ is a 2-category.

**Notation 4.3.** Let $\mathcal{Z}^{strict}$ denote the 2-category of 2-categories $\mathcal{C}$ with object set $Obj\ I$ and equipped with a strict 2-functor $P : I \longrightarrow \mathcal{C}$ from the fixed category $I$ to $\mathcal{C}$ which is the identity on objects.

A morphism from $P : I \longrightarrow \mathcal{C}$ to $P' : I \longrightarrow \mathcal{C}'$ in $\mathcal{Z}^{strict}$ is a strict 2-functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ such that

$$\begin{tikzcd} I \arrow[r, "P"] \arrow[dr, "P'"'] & \mathcal{C} \arrow[d, "F"] \\ & \mathcal{C}' \end{tikzcd}$$

strictly commutes. We see that any morphism $F$ is the identity on objects.

If $F, G : P \longrightarrow P'$ are morphisms in $\mathcal{Z}^{strict}$, then a 2-cell $\sigma : F \Longrightarrow G$ is *not* a 2-natural transformation from $F$ to $G$, but instead consists of natural transformations $\sigma_{A,B} : F_{A,B} \Longrightarrow G_{A,B}$ for all $A, B \in Obj\ \mathcal{C}$ such that

$$\sigma_{A,C}^{P(j)} = i_{P'(j)}$$

$$\sigma_{B,C}^{g} \circ \sigma_{A,B}^{f} = \sigma_{A,C}^{g \circ f}$$

$$\sigma_{A,A}^{1_A^h} = i_{1_A^h}$$

for all $j \in I(A,C), f \in \mathcal{C}(A,B), g \in \mathcal{C}(B,C)$, and objects $A$. Here $i_{P'(j)}$ denotes the identity 2-cell on the morphism $P'(j)$ in the 2-category $\mathcal{C}'$. The notation $\circ$ denotes the horizontal composition of 2-cells as well as the composition of morphisms.

The 2-category $\mathcal{Z}^{strict}$ is similar to the 2-category of pseudo 2-algebras over a theory in [**39**].

**Remark 4.4.** If $I$ is a discrete category, then the objects and morphisms of $\mathcal{X}^{strict}$, $\mathcal{Y}^{strict}$, and $\mathcal{Z}^{strict}$ are simply 2-categories with object set $I$ and 2-functors that are the identity on objects. The 2-cells are oplax natural transformations with identity components, which are better viewed as vertical natural transformations with identity components as in Section 3. We extend this identification to general groupoids $I$ in Theorem 4.8 and Theorem 4.9 below, and then also to the weak situation in Theorem 7.10 and Theorem 7.11.

**Definition 4.5.** A 2-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a 2-equivalence if there exists a 2-functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and 2-natural isomorphisms $1_{\mathcal{C}} \cong GF$ and $FG \cong 1_{\mathcal{D}}$. The notion of 2-equivalence is the same as equivalence of $Cat$-enriched categories, *i.e.* a 2-functor which is surjective on objects up to isomorphism and locally an isomorphism of categories.

First we compare $\mathcal{Y}^{strict}$ and $\mathcal{Z}^{strict}$. This 2-equivalence is the essential feature of foldings:

**Theorem 4.6.** *Let $I$ be a category. The 2-category $\mathcal{Y}^{strict}$ of double categories $\mathbb{D}$ with folding such that $(\mathbf{V}\mathbb{D})_0 = I$ is 2-equivalent to the 2-category $\mathcal{Z}^{strict}$ of strict 2-functors $P : I \longrightarrow \mathcal{C}$ that are the identity on objects as in Notation 4.2 and Notation 4.3.*

*Proof.*
We define two strict 2-functors $\mathcal{L} : \mathcal{Y}^{strict} \longrightarrow \mathcal{Z}^{strict}$ and $\mathcal{M} : \mathcal{Z}^{strict} \longrightarrow \mathcal{Y}^{strict}$ and show that $\mathcal{M}$ is surjective on objects up to isomorphism and locally an isomorphism.

From $\mathbb{D} \in \mathcal{Y}^{strict}$, we define $\mathcal{C} := \mathbf{H}\mathbb{D}$ and take the functor $P : I \longrightarrow \mathcal{C}$ to be the holonomy, in other words $\mathcal{L}(\mathbb{D}) := P$. For a morphism $F$ and a 2-cell $\sigma$ in $\mathcal{Y}^{strict}$, $\mathcal{L}(F) := \mathbf{H}(F)$ and $\mathcal{L}\sigma := \sigma$.

For a strict 2-functor $P : I \longrightarrow \mathcal{C}$ in $\mathcal{Z}^{strict}$, the double category $\mathcal{M}(P)$ has vertical 1-category $I$ and horizontal 2-category $\mathcal{C}$. The squares $\alpha$ of the form (1) are 2-cells $P(k) \circ f \Longrightarrow g \circ P(j)$ in $\mathcal{C}$. The holonomy is $P$ and the folding bijection is the identity. Horizontal and vertical composition of squares, and the respective units, are defined by the folding axioms in Definition 3.16. For a morphism $F$ in $\mathcal{Z}^{strict}$, the double functor $\mathcal{M}(F)$ is the identity on the vertical 1-category and $F$ on the horizontal 2-category. This in fact also defines $F$ on squares. Lastly $\mathcal{M}(\sigma) = \sigma$ for all 2-cells $\sigma$ in $\mathcal{Z}^{strict}$.

The 2-functor $\mathcal{M}$ is surjective on objects up to isomorphism, since $\mathcal{M}\mathcal{L}\mathbb{D} \cong \mathbb{D}$ for all $\mathbb{D} \in \mathcal{Y}^{strict}$. The vertical 1-categories, horizontal 2-categories, and holonomies of $\mathcal{M}\mathcal{L}\mathbb{D}$ and $\mathbb{D}$ are in fact the same, and the squares correspond under the bijections $\Lambda^{f,k}_{j,g}$.

Lastly, we verify that $\mathcal{M}$ is locally an isomorphism. Clearly,

$$\mathcal{M}_{P,P'} : \mathcal{Z}^{strict}(P, P') \longrightarrow \mathcal{Y}^{strict}(\mathcal{M}P, \mathcal{M}P')$$

is injective on objects and locally injective. If $F \in \mathcal{Y}^{strict}(\mathcal{M}P, \mathcal{M}P')$, then $\mathcal{M}\mathcal{L}F = F$, and similarly for the morphisms in $\mathcal{Y}^{strict}(\mathcal{M}P, \mathcal{M}P')$. $\square$

**Remark 4.7.** Theorem 4.6 is an $I$-category analogue of the equivalence in [**20**] and [**73**] between the category of edge-symmetric double categories with thin structure and the category of small 2-categories.

**Theorem 4.8.** *Let $I$ be a groupoid. The 2-category $\mathcal{X}^{strict}$ of $I$-catego-ries (strict 2-algebras over the 2-theory of categories with underlying groupoid $I$) is 2-equivalent to the 2-category $\mathcal{Y}^{strict}$ of double categories $\mathbb{D}$ with folding such that $(\mathbf{V}\mathbb{D})_0 = I$ as defined in Notation 4.1 and Notation 4.2.*

*Proof.* We construct a 2-equivalence $\mathcal{J} : \mathcal{X}^{strict} \longrightarrow \mathcal{Y}^{strict}$. Suppose $X$ is an object of $\mathcal{X}^{strict}$. From the strict 2-functor $X : I^2 \longrightarrow Cat$ we define $(\mathbf{V}\mathcal{J}(X))_0 := I$ and $\mathbf{H}\mathcal{J}(X)(A, B) := X_{A,B}$. For $j$ as in (1) the holonomy is $\overline{j} := X_{1^v_A, j}(1^h_A)$, which is the same as $P$ in Lemma 2.5. The squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

are the morphisms from $\overline{k} \circ f$ to $g \circ \overline{j}$ in $X_{A,D}$, so that the bijection $\Lambda^{f,k}_{j,g}$ of Definition 3.16 is the identity map. The horizontal and vertical compositions of squares are defined by the axioms for $\Lambda$ in Definition 3.16.

It follows from the definitions that $\mathbf{H}\mathcal{J}(X)$ and $\mathbf{V}\mathcal{J}(X)$ are 2-catego-ries. The associativity axioms, identity axioms, and interchange law axiom for composition of squares of $\mathcal{J}(X)$ follow from the analogous axioms for the underlying 2-category $\mathbf{H}\mathcal{J}(X)$ of $X$ by Definition 3.16. In fact, the verifications are formally similar to the analogous verifications for the double category of quintets of a 2-category. Thus $\mathcal{J}(X)$ is a double category with folding and belongs to $\mathcal{Y}^{strict}$.

The strict 2-functor $\mathcal{J}$ is defined similarly on morphisms. For any morphism $F : X \longrightarrow Y$ in $\mathcal{X}^{strict}$, then double functor $\mathcal{J}(F)$ is defined as the identity on

$I$. On $\mathbf{H}\mathcal{J}(X)(A,B)$ it is $F_{A,B}$, which extends to a definition on squares via $\Lambda$. A naturality argument shows $\mathcal{J}(F)(\overline{k}) = \overline{\mathcal{J}(F)(k)}$, so that $\mathcal{J}(F)$ is a morphism in $\mathcal{Y}^{strict}$.

If $\sigma : F \Longrightarrow G$ is a 2-cell in $\mathcal{X}^{strict}$, then $\mathcal{J}(\sigma)_{A,B}$ is simply $\sigma_{A,B}$. Since $\sigma$ is a modification, we have

$$Y_{j,k}(\sigma^f_{A,B}) = \sigma^{X_{j,k}(f)}_{C,D},$$

which implies

$$\sigma^{\overline{j}}_{A,C} = i^v_{\overline{j}}$$

by Lemma 2.5. Furthermore, $\mathcal{J}(\sigma)$ is compatible with horizontal composition and square identity because $\sigma$ is. This concludes the definition of the strict 2-functor $\mathcal{J}$.

We claim that $\mathcal{J}$ is surjective on objects up to isomorphism. Let $\mathbb{D}$ be an object of $\mathcal{Y}^{strict}$. Then $\mathbf{V}\mathbb{D}(A,B) = I(A,B)$. For $A, B \in Obj\ I$, let $X_{A,B}$ be the category whose objects are horizontal morphisms $f : A \longrightarrow B$ in $\mathbb{D}$ and whose morphisms are the squares of $\mathbb{D}$ with left and right vertical morphisms $1^v_A$ and $1^v_B$ respectively. For $f \in X_{A,B}$, a square $\alpha$ in $X_{A,B}$, and vertical morphisms $j : A \longrightarrow C$ and $k : B \longrightarrow D$, define

$$X_{j,k}(f) = [\overline{j^{-1}} \quad f \quad \overline{k}] \qquad X_{j,k}(\alpha) := [i_{\overline{j^{-1}}} \quad \alpha \quad i_{\overline{k}}].$$

Then $X : I^2 \longrightarrow Cat$ is a clearly a 2-functor by the properties of $\mathbb{D}$, and even an $I$-category. Moreover, $\mathbb{D}$ is isomorphic to $\mathcal{J}(X)$ in $\mathcal{Y}^{strict}$ (squares of $\mathbb{D}$ are mapped to squares of $\mathcal{J}(X)$ using $\Lambda^{\mathbb{D}}$).

The 2-functor $\mathcal{J}$ is locally an isomorphism by inspection. Hence $\mathcal{J}$ is a 2-equivalence. $\square$

Next we compare $\mathcal{X}^{strict}$ and $\mathcal{Z}^{strict}$. The result is a strict version of Theorem 5.2 with trivial $T$ in [**39**] improved from biequivalence to 2-equivalence in Theorem 4.9. It is a corollary of Theorem 4.6 and Theorem 4.8, but we present a direct proof:

**Theorem 4.9.** *Let $I$ be a groupoid. The 2-category $\mathcal{X}^{strict}$ of $I$-catego-ries (strict 2-algebras over the 2-theory of categories with underlying groupoid $I$) is 2-equivalent to the 2-category $\mathcal{Z}^{strict}$ of strict 2-functors $P : I \longrightarrow \mathcal{C}$ that are the identity on objects as in Notation 4.1 and Notation 4.3.*

*Proof.* We construct a 2-equivalence $\mathcal{K} : \mathcal{X}^{strict} \longrightarrow \mathcal{Z}^{strict}$. For an object $X : I^2 \longrightarrow Cat$ of $\mathcal{X}$, we obtain a strict 2-functor

$$\mathcal{K}(X) = P : I \longrightarrow \mathcal{C}$$

that is the identity on objects as in Lemma 2.5.

We define $\mathcal{K}$ compatibly on morphisms and 2-cells. Let $F : X \longrightarrow X'$ be a morphism in $\mathcal{X}^{strict}$, *i.e.* $F$ is a strict 2-natural transformation from $X$ to $X'$ which preserves composition and identity. We define the 2-functor $\mathcal{K}(F) : \mathcal{C} \longrightarrow \mathcal{C}'$ to be the identity on objects, and as $F_{A,B} : X_{A,B} \longrightarrow X'_{A,B}$ on $Mor_{\mathcal{C}}(A,B) = X_{A,B}$. Then $P' = \mathcal{K}(F) \circ P$ because $F$ is 2-natural and preserves identity morphisms. For a 2-cell $\sigma : F \Longrightarrow G$ in $\mathcal{X}^{strict}$, the natural transformation

$$\mathcal{K}(\sigma)_{A,B} : \mathcal{K}(F)_{A,B} \longrightarrow \mathcal{K}(G)_{A,B}$$

is simply $\sigma_{A,B}$. One can easily check that $\mathcal{K}$ is a strict 2-functor.

The strict 2-functor $\mathcal{K}$ is surjective on objects. If $P : I \longrightarrow \mathcal{C}$ is an object of $\mathcal{Z}^{strict}$, then $X_{A,B} := Mor_{\mathcal{C}}(A,B)$ and $X_{j,k}(f) := P(k) \circ f \circ P(j^{-1})$ defines an object of $\mathcal{X}^{strict}$ which maps to $P$.

The strict 2-functor $\mathcal{K}$ is locally an isomorphism of categories. It is clearly injective on morphisms and 2-cells. If $\overline{F}$ is a morphism in $\mathcal{Z}^{strict}$ from $\mathcal{K}(X)$ to $\mathcal{K}(X')$, then a pre-image is necessarily defined by $F_{A,B} := \overline{F}_{A,B}$, the 2-naturality of which is easily verified:

$$\begin{aligned}\overline{F}X_{j,k}(f) &= F(P(k) \circ f \circ P(j^{-1})) \\ &= FP(k) \circ F(f) \circ FP(j^{-1}) \\ &= P'(k) \circ F(f) \circ P'(j^{-1}) \\ &= X'_{j,k}\overline{F}(f).\end{aligned}$$

If $\overline{\sigma} : \overline{F} \Longrightarrow \overline{G}$ is a 2-cell in $\mathcal{Z}^{strict}$, then a pre-image is defined by $\sigma_{A,B} := \overline{\sigma}_{A,B}$. Since $\overline{\sigma}_{A,C}^{P(j)} = i_{P'(j)}$, we know that $\sigma$ is a modification by Lemma 2.5. The modification $\sigma$ is clearly compatible with composition and identity because $\overline{\sigma}$ is.

We conclude that $\mathcal{K}$ is a 2-equivalence of 2-categories. □

**Remark 4.10.** A strict 2-functor $P : I \longrightarrow \mathcal{C}$ from a 1-category $I$ to a 2-category $\mathcal{C}$ (with the same object set) that is the identity on objects is a special case of the notion *weak equipment* in [**75**]. There Verity constructs a *double bicategory of squares* from a weak equipment, which essentially defines our 2-functor $\mathcal{M} : \mathcal{Z}^{strict} \longrightarrow \mathcal{Y}^{strict}$ and a 2-functor $\mathcal{Z} \longrightarrow \mathcal{Y}$ between the 2-categories defined in Section 7, though the 2-cells are different in present paper. Here we have constructed 2-equivalences between $\mathcal{X}$ and $\mathcal{Y}$ as well as between $\mathcal{X}$ and $\mathcal{Z}$ using connection pairs and foldings (in the strict and pseudo cases).

In this section we proved the strict version of our desired result: an $I$-category can be viewed as a double category with folding or as a 2-functor from a 1-category into a 2-category. Since foldings are equivalent to connection pairs, and edge-symmetric double groups with connection pair[3] are equivalent to crossed modules, one can expect that Theorem 4.8 has implications for crossed modules. Indeed, we pursue this in the next section.

## 5. 2-Groups, Double Groups, and Crossed Modules

It is often useful to investigate one-object cases of categorical concepts and compare them with more familiar concepts. For example, a one-object category is simply

[3]Whenever an edge-symmetric double category is equipped with a connection pair, we assume the holonomy to be trivial.

a monoid, and a one-object groupoid is a group. In this section, we investigate one-object $I$-categories with everything invertible and compare them with other notions in the literature: crossed modules and double groups.

To see how our comparison of one-object-everything-invertible $I$-categories will work, consider 2-groupoids. These are 2-categories in which every morphism and every 2-cell is invertible. We call a one-object 2-groupoid a *2-group.*[4] This is the same as a group object in $Cat$, or *categorical group* as in Theorem 5.5. Though the notion of 2-group is no more familiar than the notion of 2-groupoid, we can compare it to something familiar. Brown and Spencer proved in [**22**] (and attribute the result to Verdier) that categorical groups (and hence 2-groups) are equivalent to *crossed modules*. This last concept is much more familiar to topologists than 2-groups. Whitehead first introduced crossed modules in [**76**] and proved with Mac Lane that they model path-connected homotopy 2-types in [**61**]. The survey [**69**] contains an account of the use of crossed modules and their higher-dimensional analogues to model homotopy types. Recently, 2-groups have been studied in [**7**]. Our comparison of one-object $I$-categories with everything invertible will build on this result of Brown and Spencer. In fact, Brown and Spencer obtained a 2-equivalence, and we will in Theorem 5.13 as well.

In addition to the comparison with crossed modules, we also compare with double groups. A *double groupoid* is a double category in which all morphisms and squares are iso. In particular, squares are required to be isos under both vertical composition and horizontal composition. In analogy to 2-groups, we shall call a one-object double groupoid a *double group*[5]. Brown and Spencer proved that edge-symmetric double groups with connection[6] are equivalent to crossed modules in [**21**]. We extend this to a 2-equivalence between general double groups with folding and crossed modules under groups in Theorem 5.15. Brown and Higgins showed in [**15**] that so-called crossed modules over groupoids (*not* in the sense of an over category) are equivalent to edge-symmetric double groupoids with connection. In [**19**], Brown and Mackenzie substantially generalized [**21**] to an equivalence between locally trivial (not necessarily edge-symmetric) double Lie groupoids and so-called locally trivial core diagrams. This is an equivalence between double groupoids with certain filling conditions and core diagrams. Their Theorem 4.2 treats the discrete case as well. Double groupoids have recently found application in the theory of weak Hopf algebras in [**4**] and [**5**].

In most cases, our 2-cells are the vertical natural transformations. Double categories provide a good context for the 2-equivalence of categorical groups, 2-groups, and crossed modules. The 2-natural transformations of 2-groups do *not* correspond to the *homotopies* in the 2-category of crossed modules, instead one needs the vertical transformations. Theorem 5.11 and Theorem 5.15 also hold for the horizontal natural transformations after adjusting the notion of 2-cell in the 2-category of crossed modules.

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[4]In this article all 2-groupoids are strict.

[5]A group object in the category of groups is simply an abelian group by Eckmann-Hilton. Thus double groups and group objects in the category of groups are not the same.

[6]The holonomies here are trivial.

We begin by stating Theorems 4.6, 4.8, and 4.9 in the special case that $I$ is a group and all morphisms in each structure are invertible. We call these sub-2-categories $\mathcal{X}^{inv}, \mathcal{Y}^{inv}$, and $\mathcal{Z}^{inv}$ in Notation 5.1. After recalling 2-groups, categorical groups, crossed modules, and the proof by Brown and Spencer, we show that $\mathcal{Z}^{inv}$ is 2-equivalent to the 2-category $\mathcal{W}^{inv}$ of crossed modules under $\{e\} \longrightarrow I$. Therefore $I$-categories which have only one object and everything invertible are 2-equivalent to crossed modules under $I$. The 2-equivalence of $\mathcal{Y}^{inv}$ and $\mathcal{W}^{inv}$ says that the 2-category of double groups $\mathbb{D}$ with folding such that $(\mathbf{V}\mathbb{D})_0 = I$ is 2-equivalent to the 2-category of crossed modules under $I$. Lastly we turn to double groups with folding.

**Notation 5.1.** In this section, $I$ denotes a one-object groupoid, *i.e.* a group. Let $\mathcal{X}^{inv}$ denote the 2-category of $I$-categories $X$ with $X_{*,*}$ a groupoid whose objects and morphisms are invertible with respect to the 2-algebra composition $\circ$. Let $\mathcal{Y}^{inv}$ denote the 2-category of double groups $\mathbb{D}$ with folding such that $(\mathbf{V}\mathbb{D})_0 = I$. Let $\mathcal{Z}^{inv}$ denote the 2-category of strict 2-groups $\mathcal{C}$ equipped with a strict 2-functor $P : I \longrightarrow \mathcal{C}$. Morphisms and 2-cells of the 2-categories $\mathcal{X}^{inv}, \mathcal{Y}^{inv}$, and $\mathcal{Z}^{inv}$ are as in the respective categories of Notations 4.1, 4.2, and 4.3.

**Theorem 5.2.** *The 2-categories $\mathcal{X}^{inv}, \mathcal{Y}^{inv}$, and $\mathcal{Z}^{inv}$ are 2-equivalent.*

*Proof.* The 2-equivalence of $\mathcal{X}^{inv}, \mathcal{Y}^{inv}$, and $\mathcal{Z}^{inv}$ follows from Theorems 4.6, 4.8, and 4.9. □

**Definition 5.3.** A one-object 2-category in which all 1-cells and all 2-cells are invertible is called a *2-group*. We view a 2-group $\mathcal{C}$ as a double category with one object, trivial vertical morphisms, and with horizontal morphisms and squares given by the 1-cells and 2-cells of $\mathcal{C}$. In other words, we view a 2-group $\mathcal{C}$ as $\mathbb{H}\mathcal{C}$. A morphism of 2-groups is a 2-functor. This is the same as a double functor between the associated double categories. A *2-cell* is a vertical natural transformation, *not* a 2-natural transformation. We denote this 2-category by 2-*Gp*. It is a sub-2-category of $Cat(Cat)_v$, the 2-category of double categories, double functors, and vertical natural transformations.

**Definition 5.4.** A *categorical group* is a group object in *Cat*. This is a category $(X_0, X_1)$ equipped with a functor

$$(X_0, X_1) \times (X_0, X_1) \longrightarrow (X_0, X_1)$$

which strictly satisfies the axioms of a group. A *morphism of categorical groups* is a functor compatible with group structure. A *2-cell* is a natural transformation compatible with group structure. We denote the 2-category of group objects in *Cat* by $Gp(Cat)$.

**Theorem 5.5.** *The 2-category of categorical groups, morphisms, and 2-cells is 2-equivalent to the 2-category of 2-groups, 2-functors, and vertical natural transformations.*

*Proof.* The inclusion of $Gp$ into $Cat$ induces an inclusion of 2-categories $Gp(Cat) \longrightarrow Cat(Cat)_v$. This assigns a categorical group $(X_0, X_1)$ to the double category with one object, no nontrivial vertical morphisms, horizontal morphisms $X_0$, and squares $X_1$. The horizontal composition is the group operation, and the vertical composition of squares is composition in the category $(X_0, X_1)$. Morphisms and 2-cells of $Gp(Cat)$ are mapped to double functors and *vertical* natural transformations. Thus $Gp(Cat)$ is contained in 2-$Gp$. Every 2-group is isomorphic to one in $Gp(Cat)$, so $Gp(Cat)$ is 2-equivalent to 2-$Gp$. $\square$

**Definition 5.6.** A *crossed module* $\partial : H \longrightarrow G$ consists of

- groups $H$ and $G$
- a group homomorphism $\partial : H \longrightarrow G$
- a left action of $G$ on $H$ by automorphisms written $(g, \alpha) \mapsto {}^g\alpha$ such that:
  - i. $\partial({}^g\alpha) = g\partial(\alpha)g^{-1}$ for all $\alpha \in H$ and $g \in G$,
  - ii. ${}^{\partial(\beta)}\alpha = \beta\alpha\beta^{-1}$ for all $\alpha, \beta \in H$.

**Definition 5.7.** If $(H, G, \partial)$ and $(H', G', \partial')$ are crossed modules, then a *morphism* $(p, q) : (H, G, \partial) \longrightarrow (H', G', \partial')$ consists of group homomorphisms $p$ and $q$ such that the following diagram commutes

$$\begin{tikzcd} H \arrow[r, "\partial"] \arrow[d, "p"'] & G \arrow[d, "q"] \\ H' \arrow[r, "\partial'"'] & G' \end{tikzcd}$$

and $p({}^g\alpha) = {}^{q(g)}p(\alpha)$ for all $g \in G$ and $\alpha \in H$.

**Definition 5.8.** If $(p_1, q_1), (p_2, q_2) : (H, G, \partial) \longrightarrow (H', G', \partial')$ are morphisms of crossed modules, then a *homotopy* $\nu : (p_1, q_1) \Longrightarrow (p_2, q_2)$ is a function $\nu : G \longrightarrow H'$ such that $(\partial'\nu(f))q_1(f) = q_2(f)$ and:

i. For all $f, g \in G$ and $\alpha \in H$ such that $\partial(\alpha)f = g$, we have

$$p_2(\alpha)\nu(f) = \nu(g)p_1(\alpha),$$

ii. For all $f, g \in G$, the *derivation rule* holds:

$$\nu(g) \cdot {}^{q_1(g)}\nu(f) = \nu(gf).$$

The vertical composition of homotopies

$$(p_1, q_1) \overset{\nu_1}{\Longrightarrow} (p_2, q_2) \overset{\nu_2}{\Longrightarrow} (p_3, q_3)$$

is $f \longmapsto \nu_2(f)\nu_1(f)$. The horizontal composition of homotopies

$$\begin{tikzcd}[column sep=huge] (H_1, G_1, \partial_1) \arrow[r, bend left, "{(p_1,q_1)}"] \arrow[r, bend right, "{(p_1',q_1')}"'] \arrow[r, phantom, "\Downarrow \nu_1"] & (H_2, G_2, \partial_2) \arrow[r, bend left, "{(p_2,q_2)}"] \arrow[r, bend right, "{(p_2',q_2')}"'] \arrow[r, phantom, "\Downarrow \nu_2"] & (H_3, G_3, \partial_3) \end{tikzcd}$$

is $f \longmapsto \nu_2(q_1'(f)) \cdot p_2(\nu_1(f))$.

Crossed modules, morphisms, and homotopies form a 2-category denoted $XMod$. For more on crossed modules as internal categories and their 2-cells, see [**24**]. Homotopies and derivations for more general crossed modules as needed for a 2-dimensional notion of holonomy are considered in [**17**].

**Example 5.9.** An example of a crossed module is the inclusion of a normal subgroup $H$ into a group $G$ where the action is conjugation by elements of $G$. In particular, $\{e\} \longrightarrow I$ is a crossed module for any group $I$. We abbreviate $\{e\} \longrightarrow I$ by $I$.

**Example 5.10 (Whitehead).** Let $(X, A, *)$ be a pair of based spaces. Then the boundary map $\partial : \pi_2(X, A, *) \longrightarrow \pi_1(A, *)$ is a crossed module with action given by the standard action of the fundamental group. Crossed modules are known to model pointed path-connected weak homotopy 2-types algebraically. A proof is sketched in [**13**].

In preparation for our theorem, we summarize Brown and Spencer's proof as recounted in [**41**]. Brown and Spencer originally showed that categorical groups are 2-equivalent to crossed modules, crossed module morphisms, and homotopies. The 2-category of categorical groups is 2-equivalent to the 2-category of 2-groups, functors, and vertical natural transformations. Since we are interested in double groups, we work with the latter 2-category of 2-groups instead of categorical groups. See [**16**] for the analogue of Theorem 5.11 in arbitrary dimensions.

**Theorem 5.11 (Brown-Spencer in [22]).** *The 2-category 2-Gp of 2-groups, functors, and vertical natural transformations is 2-equivalent to the 2-category $XMod$ of crossed modules, crossed module morphisms, and homotopies.*

*Proof.* Let $\mathcal{C}$ be a 2-group. We obtain a crossed module from $\mathcal{C}$ as follows. The group $G$ consists of the objects of $Mor_{\mathcal{C}}(*, *)$. In particular, $e_G$ is the identity morphism. The group $H$ consists of 2-cells $\alpha$ in $\mathcal{C}$ whose source is $e_G$ and $\partial$ is the target map.

$$\begin{tikzcd} * \arrow[r, bend left=50, "e_G"] \arrow[r, bend right=50, "\partial\alpha"'] \arrow[r, phantom, "\Downarrow\alpha"] & * \end{tikzcd}$$

The group $G$ acts on $H$ on the left by conjugation, in other words ${}^{g}\alpha$ has the form below.

$$\begin{tikzcd} * \arrow[r, bend left=50, "g^{-1}"] \arrow[r, bend right=50, "g^{-1}"'] \arrow[r, phantom, "\Downarrow i_{g^{-1}}"] & * \arrow[r, bend left=50, "e_G"] \arrow[r, bend right=50, "\partial\alpha"'] \arrow[r, phantom, "\Downarrow\alpha"] & * \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow i_g"] & * \end{tikzcd}$$

If $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a 2-functor, then we obtain a morphism of crossed modules by restricting $F$ to $G$ and $H$. A 2-cell $F_1 \Longrightarrow F_2$ in the 2-category of 2-groups is a natural transformation

$$\sigma : F_1|_{Mor_{\mathcal{C}}(*,*)} \Longrightarrow F_2|_{Mor_{\mathcal{C}'}(*,*)}$$

that is compatible with the horizontal composition of 1- and 2-cells. We obtain a homotopy $\nu : G \longrightarrow H'$ by defining $\nu(g) := \sigma^g i_{(F_1 g)^{-1}}$. Here concatenation denotes the horizontal composition of 2-cells. The naturality corresponds to (i) and the compatibility with horizontal composition corresponds to (ii) in Definition 5.8.

Next we describe how to get a 2-group $\mathcal{C}$ from a crossed module $\partial : H \longrightarrow G$. The set of morphisms of $\mathcal{C}$ is $G$ and the set of 2-cells of $\mathcal{C}$ is $H \rtimes G$. The source and target of the 2-cell $(\alpha, g)$ are $g$ and $\partial(\alpha)g$ respectively. Horizontal composition of 2-cells is given by the group operation in $H \rtimes G$ and vertical composition is

$$(\alpha_2, \partial(\alpha_1)g_1) \odot (\alpha_1, g_1) := (\alpha_2\alpha_1, g_1).$$

The vertical identity is $i_g := (e_H, g) : g \Longrightarrow g$.

If $(p, q)$ is a morphism of crossed modules, we obtain a 2-functor $\mathcal{C} \longrightarrow \mathcal{C}'$ as $q$ on 1-cells and $(p, q)$ on 2-cells. A homotopy $\nu$ in the 2-category of crossed modules gives rise to a natural transformation $\sigma : F_1|_{Mor_{\mathcal{C}}(*,*)} \Longrightarrow F_2|_{Mor_{\mathcal{C}'}(*,*)}$ by defining $\sigma^g := (\nu(g), q_1(g))$. Further, this natural transformation is a vertical natural transformation: the derivation rule for homotopies guarantees that $\sigma$ is compatible with composition of horizontal morphisms as in Definition 3.11 (ii), since

$$\begin{aligned}
\sigma^g \sigma^f &= (\nu(g), q_1(g))(\nu(f), q_1(f)) \\
&= (\nu(g) \cdot {}^{q_1(g)}\nu(f), q_1(g)q_1(f)) \\
&= (\nu(gf), q_1(gf)) \\
&= \sigma^{gf}.
\end{aligned}$$

The composite 2-functor from crossed modules to 2-groups and to crossed modules back again is 2-naturally isomorphic to the identity. On the other hand, if we start with a 2-group $\mathcal{C}$, and take the associated crossed module, note that the group of 2-cells of $\mathcal{C}$ (under horizontal composition) is isomorphic to $H \rtimes G$ by the map which sends

$$\begin{tikzcd}
* \arrow[r, bend left=50, "g_1"{name=U}] \arrow[r, bend right=50, "g_2"'{name=D}] & * \arrow[Rightarrow, from=U, to=D, "\gamma"]
\end{tikzcd}$$

to $(\gamma i_{g_1^{-1}}, g_1)$. Using this map, one can see that the composite 2-functor from 2-groups to crossed modules and to 2-groups back again is 2-naturally isomorphic to the identity. □

**Notation 5.12.** The objects of the 2-category $\mathcal{W}^{inv}$ are crossed modules under $I$. These are crossed modules $\partial : H \longrightarrow G$ equipped with a morphism of crossed modules

$$\begin{tikzcd} \{e\} \arrow[r] \arrow[d] & I \arrow[d, "P"] \\ H \arrow[r, "\partial"'] & G. \end{tikzcd}$$

Morphisms of $\mathcal{W}^{inv}$ are morphisms $(p, q)$ of crossed modules under $I$, in other words

$$\begin{tikzcd} I \arrow[r, "P"] \arrow[dr, "P'"'] & G \arrow[d, "q"] \\ & G' \end{tikzcd}$$

commutes. A 2-cell in $\mathcal{W}^{inv}$ is a homotopy $\nu$ such that $\nu(P(j)) = e_{H'}$ for all $j \in I$.

**Theorem 5.13.** *The 2-category $\mathcal{W}^{inv}$ of crossed modules under $I$ is 2-equivalent to the 2-category $\mathcal{Z}^{inv}$ of 2-groups under $I$.*

*Proof.* The 2-equivalence from crossed modules to 2-groups in Theorem 5.11 extends to a 2-equivalence $\mathcal{N} : \mathcal{W}^{inv} \longrightarrow \mathcal{Z}^{inv}$. A strict 2-functor $I \longrightarrow \mathcal{C}$ is the same as a morphism of crossed modules from $I$ into the crossed module associated to $\mathcal{C}$. Morphisms of crossed modules under $I$ are the same as morphisms of 2-groups under $I$.

We observe that the 2-cells $\nu : (p_1, q_1) \Longrightarrow (p_2, q_2)$ in $\mathcal{W}^{inv}$ are precisely the 2-cells $\mathcal{N}(p_1, q_1) \Longrightarrow \mathcal{N}(p_2, q_2)$ in $\mathcal{Z}^{inv}$. From Theorem 5.11 we know that the homotopies $(p_1, q_1) \Longrightarrow (p_2, q_2)$ in $XMod$ correspond to the 2-cells $\mathcal{N}(p_1, q_1) \Longrightarrow \mathcal{N}(p_2, q_2)$ in 2-*Gp*. It suffices to show that $\nu(P(j)) = e_{H'}$ if and only if its associated 2-group 2-cell $g \mapsto \sigma^g = (\nu(g), q_1(g))$ satisfies $\sigma^{P(j)} = i_{P'(j)}$. But this is the case, since

$$\sigma^{P(j)} = (\nu(P(j)), q_1 P(j)) = (\nu(P(j)), P'(j))$$

$$i_{P'(j)} = (e_{H'}, P'(j)).$$

Therefore $\mathcal{N}$ is a 2-functor that is essentially surjective on objects and locally an isomorphism, *i.e.* a 2-equivalence.

Alternatively, one could use the 2-equivalence 2-$Gp \longrightarrow XMod$ and similarly show that a 2-cell $\sigma$ in 2-*Gp* satisfies $\sigma^{P(j)} = i_{P'(j)}$ if and only if the associated homotopy $\nu(g) = \sigma^g i_{(F_1 g)^{-1}}$ satisfies $\nu(P(j)) = e_{H'}$. □

With these notions we can extend Brown and Spencer's equivalence between crossed modules and edge-symmetric double groups with connection to the non-edge-symmetric case. The nontrivial holonomy corresponds to a morphism of crossed modules from the vertical group into the crossed module associated to the horizontal 2-group. In the rest of this section, we no longer consider fixed $I$. Our proof builds on the proof of Theorem 5.11.

**Notation 5.14.** Let $Gp/XMod$ denote the 2-category of crossed modules under groups. An object consists of a crossed module $(H, G, \partial)$ and a group $I$ equipped with a crossed module morphism

$$\begin{tikzcd} \{e\} \arrow[r] \arrow[d] & I \arrow[d] \\ H \arrow[r, "\partial"'] & G. \end{tikzcd}$$

A morphism in $Gp/XMod$ is a morphism in the arrow category of crossed modules. A 2-cell $(r_1, p_1, q_1) \Longrightarrow (r_2, p_2, q_2)$ in $Gp/XMod$ is a homotopy $\nu : (p_1, q_1) \Longrightarrow (p_2, q_2)$. Note that all homotopies of crossed module morphisms $I \longrightarrow I'$ are trivial, so we do not include this in the data for a 2-cell in $Gp/XMod$.

Let $DblGpFold$ denote the 2-category of double groups with folding. The morphisms are morphisms of double categories with folding as in Definition 3.17. The 2-cells $F_1 \Longrightarrow F_2$ are *vertical* natural transformations between the restrictions of $F_1$ and $F_2$ to the sub-double category with trivial vertical morphisms. We do *not* require that the vertical natural transformations are compatible with folding.

The 2-category $DblGpFold$ is like $\mathcal{Y}^{inv}$, except that we allow $I$ to vary and do not require the 2-cells to be compatible with folding. To extend the equivalence in [**21**] between edge-symmetric double groups with connection and crossed modules to a 2-equivalence, one is forced to take vertical transformations with identity components as the 2-cells between morphisms of edge-symmetric double groups (2-equivalences are local isomorphisms). Likewise, in the non-edge-symmetric case of $DblGpFold$, the vertical transformations are not required to be compatible with folding: any vertical transformation in $DblGpFold$ with identity components that is compatible with folding (as in Definition 3.18) is necessarily trivial.

Our choice of 2-cell in $DblGpFold$ is compatible with the degree 1 part of the internal hom for cubical $\omega$-groupoids constructed in [**16**]: an edge-symmetric double groupoid with connection is a 2-truncated cubical $\omega$-groupoid as defined in [**15**]. We now extend the equivalence in [**21**] between edge-symmetric double groups with connection and crossed modules to the non-edge-symmetric setting and upgrade it to a 2-equivalence:

**Theorem 5.15.** *The 2-category $Gp/XMod$ of crossed modules under groups is 2-equivalent to the 2-category $DblGpFold$ of double groups with folding, morphisms, and vertical transformations.*

*Proof.* We define a 2-equivalence $\mathcal{R} : DblGpFold \longrightarrow Gp/XMod$. For a double group $\mathbb{C}$ with folding, we define $I$ to be the group of vertical morphisms, $(H, G, \partial)$ to be the crossed module associated to the horizontal 2-group, and the homomorphism $I \longrightarrow G$ to be the holonomy, so that $\mathcal{R}(\mathbb{C}) = (I, H, G, \partial)$. If $F : \mathbb{C} \longrightarrow \mathbb{C}'$ is a morphism, then $\mathcal{R}(F)$ is the restriction of $F$ to $I, H$, and $G$. If $\sigma$ is a vertical transformation $F_1 \Longrightarrow F_2$ with identity components, then $\mathcal{R}(\sigma)$ is the homotopy associated to the restriction of $\sigma$ to the horizontal 2-group as in Theorem 5.11.

The 2-functor $\mathcal{R}$ is surjective on objects up to isomorphism. If $(I, H, G, \partial) \in Gp/XMod$, then we construct the 2-group $\mathcal{C}$ associated to the crossed module $(H, G, \partial)$ as in Theorem 5.11. It has horizontal morphisms $G$ and 2-cells $H \rtimes G$. The group homomorphism $I \longrightarrow G$ determines a 2-functor $I \longrightarrow C$. This data determines a double category $\mathbb{C}$ with folding as in Theorem 4.6: the vertical morphisms are $I$, the horizontal 2-category is the 2-group $\mathcal{C}$, and the squares are determined by the 2-cells of the horizontal 2-group by the folding. We see that $\mathcal{R}(\mathbb{C}) \cong (I, H, G, \partial)$.

Lastly we verify that the functor

$$\mathcal{R}_{\mathbb{C},\mathbb{C}'} : DblGpFold(\mathbb{C}, \mathbb{C}') \longrightarrow Gp/XMod(\mathcal{R}(\mathbb{C}), \mathcal{R}(\mathbb{C}'))$$

is an isomorphism of categories. Two morphisms $F_1, F_2 : \mathbb{C} \longrightarrow \mathbb{C}'$ that coincide on the vertical 1-category and the horizontal 2-category also coincide on the squares by the compatibility with folding. Similarly, a morphism $(r, p, q) : \mathcal{R}(\mathbb{C}) \longrightarrow \mathcal{R}(\mathbb{C}')$ has a pre-image because the folding defines a morphism on general squares from the horizontal 2-functor associated to $(p, q)$ (as in Theorem 5.11) and the vertical 1-functor $r$. Thus $\mathcal{R}_{\mathbb{C},\mathbb{C}'}$ is bijective on objects. The 2-cells $\mathcal{R}(F_1) \Longrightarrow \mathcal{R}(F_2)$ are the 2-cells between the underlying crossed-module morphisms of $\mathcal{R}(F_1)$ and $\mathcal{R}(F_2)$. By Theorem 5.11, the latter are in bijective correspondence with the vertical transformations between the restrictions of the double functors $F_1$ and $F_2$ to the sub-double categories of $\mathbb{C}$ and $\mathbb{C}'$ with trivial vertical morphisms. These are precisely the 2-cells of *DblGpFold*. Hence $\mathcal{R}_{\mathbb{C},\mathbb{C}'}$ is fully faithful and an isomorphism of categories.

The 2-functor $\mathcal{R}$ is locally an isomorphism and surjective up to isomorphism on objects, so that $\mathcal{R} : DblGpFold \longrightarrow Gp/XMod$ is a 2-equivalence.

□

**Remark 5.16.** Theorem 5.11 used vertical transformations as the 2-cells in 2-*Gp* and homotopies as the 2-cells in $XMod$. One could just as well work with horizontal transformations in 2-*Gp* to obtain a 2-equivalence. However, the notion of 2-cell in $XMod$ must be changed appropriately. A 2-cell $w : (p_1, q_1) \Longrightarrow (p_2, q_2)$ in this approach is an element $w \in G'$ such that:

i. $wq_1(g)w^{-1} = q_2(g)$ for all $g \in G$,

ii. ${}^{w}p(h) = p'(h)$ for all $h \in H$.

If we use these 2-cells in $Gp/XMod$ and horizontal transformations *compatible with folding* as 2-cells in $DblGpFold$, then we obtain an analogue of Theorem 5.15. The proof is very similar, but relies on Remark 3.19 in the discussion of 2-cells. I thank Simona Paoli for pointing out to me that the two notions of 2-cells correspond to horizontal and vertical natural transformations.

This concludes our discussion of strict structures.

## 6. Pseudo Double Categories with Folding

Next we turn our attention to weak structures and work towards pseudo versions of Theorems 4.6, 4.8, and 4.9. There are various ways of weakening a double category. Recall that a double category contains two 2-categories, namely its horizontal and vertical 2-categories as in Definition 3.5. One can weaken either or both of these to a bicategory, but in many applications only one direction is typically weak. In this paper, we prefer to make the horizontal 2-category into a horizontal bicategory. This corresponds to the passage from category object in $Cat$ to pseudo category object in $Cat$. Pseudo double categories, and more generally pseudo category objects, have been studied in [**45**], [**46**], [**63**], and [**75**]. Often one can arrange the units of the horizontal bicategory to be strict. Later we work with strict units.

**Definition 6.1.** A *pseudo double category* $\mathbb{D}$ consists of a class of *objects*, a set of *horizontal morphisms*, a set of *vertical morphisms*, and a set of *squares* with source and target as in (1). The vertical morphisms are equipped with a composition, as are the horizontal morphisms. The squares are equipped with a vertical and a horizontal composition. morphisms and squares also form a category under vertical composition with identity squares $i^v_f$ as in (3) which satisfy

$$[i^v_{f_1} \;\; i^v_{f_2}] = i^v_{[f_1 \;\; f_2]}.$$

There are also distinguished squares $i^h_j$ (not necessarily identity) as in (2) which satisfy

$$\begin{bmatrix} i^h_{j_1} \\ i^h_{j_2} \end{bmatrix} = i^h_{\left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]} \quad \text{and} \quad i^h_{1^v_A} = i^v_{1^h_A}.$$

The objects, horizontal morphisms, and squares with trivial left and right sides form a bicategory with coherence iso 2-cells

$$\begin{tikzcd} A \arrow[r, "1_B \circ f"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "\lambda_f"] & B \arrow[d, "1^v_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f \circ 1_A"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "\rho_f"] & B \arrow[d, "1^v_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h\circ(g\circ f)"] \arrow[d, "1^v_A"'] \arrow[dr, phantom, "\alpha_{h,g,f}"] & C \arrow[d, "1^v_C"] \\ A \arrow[r, "(h\circ g)\circ f"'] & C \end{tikzcd}$$

that satisfy the usual coherence triangle diagram and coherence pentagon diagram for bicategories as in the original [**9**], or in the review [**54**], or in the Appendix to [**39**]. These coherence iso 2-cells are also natural for all squares, for example

$$\begin{tikzcd} A \arrow[r, "1^h_A"] \arrow[d, "j"'] \arrow[dr, phantom, "i^h_j"] & A \arrow[r, "f"] \arrow[d, "j"] \arrow[dr, phantom, "\beta"] & B \arrow[d, "k"] \\ C \arrow[r, "1^h_C"'] \arrow[d, "1^v_C"'] \arrow[drr, phantom, "\rho_g"] & C \arrow[r, "g"'] & D \arrow[d, "1^v_D"] \\ C \arrow[rr, "g"'] & & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "1^h_A"] \arrow[d, "1^v_A"'] \arrow[drr, phantom, "\rho_f"] & A \arrow[r, "f"] & B \arrow[d, "1^v_B"] \\ A \arrow[rr, "f"] \arrow[d, "j"'] \arrow[drr, phantom, "\beta"] & & B \arrow[d, "k"] \\ C \arrow[rr, "g"'] & & D \end{tikzcd}$$

and also for
$$\begin{tikzcd} \cdot \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[r, "f_2"] \arrow[d] \arrow[dr, phantom, "\gamma"] & \cdot \arrow[r, "f_3"] \arrow[d] \arrow[dr, phantom, "\delta"] & \cdot \arrow[d] \\ \cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdot \arrow[r, "g_3"'] & \cdot \end{tikzcd}$$
we have

$$\begin{bmatrix} [\ \beta\ \gamma\ ]\ \delta \\ \alpha_{g_3,g_2,g_1} \end{bmatrix} = \begin{bmatrix} \alpha_{f_3,f_2,f_1} \\ \beta\ [\ \gamma\ \delta\ ] \end{bmatrix}.$$

Lastly, the interchange law holds as in (4) and (5). For a "one-sort formulation" of pseudo double category and mention of the subtleties in the following remark, see [**45**].

**Remark 6.2.** The weakening of the horizontal 2-category to a horizontal bicategory forces other parts of the notion of double category to weaken in a pseudo double category $\mathbb{D}$. For example, the horizontal composition of squares cannot be strictly associative if the composition of horizontal morphisms is not strictly associative. Similarly, the horizontal composition of squares cannot be strictly unital if the composition of horizontal morphisms is not strictly unital.

If the composition of horizontal morphisms is not strictly unital, then $\mathbf{V}\mathbb{D}$ is neither a 2-category nor a bicategory: the vertical composition of 2-cells in $\mathbf{V}\mathbb{D}$ (which is the horizontal composition of squares in $\mathbb{D}$) is not closed. However, if we redefine the vertical composition of 2-cells $\beta$ and $\gamma$ in $\mathbf{V}\mathbb{D}$ to be

$$\begin{tikzcd} \cdot \arrow[rr, "1^h"] \arrow[d] \arrow[drr, phantom, "\rho^{-1}_{1^h}"] & & \cdot \arrow[d] \\ \cdot \arrow[r, "1^h"] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[r, "1^h"] \arrow[d] \arrow[dr, phantom, "\gamma"] & \cdot \arrow[d] \\ \cdot \arrow[r, "1^h"'] \arrow[d] \arrow[drr, phantom, "\rho_{1^h}"] & \cdot \arrow[r, "1^h"'] & \cdot \arrow[d] \\ \cdot \arrow[rr, "1^h"'] & & \cdot \end{tikzcd}$$

then we obtain a 2-category.

If $\lambda$ and $\rho$ are the vertical identity squares (*i.e.* the composition of horizontal morphisms is strictly unital), then $i^h_j$ is a horizontal identity square by the naturality of $\lambda$ and $\rho$, and hence $\mathbf{V}\mathbb{D}$ is a 2-category without any alterations. If additionally $\alpha$ is a vertical identity, then we obtain the usual notion of double category. Whenever $\lambda$ and $\rho$ are the vertical identity squares, we say that $\mathbb{D}$ *has strict units.* Note that for pseudo double categories we must require $i^h_{1^v_A} = i^v_{1^h_A}$ even though this equality follows from the other axioms in the case of strict double categories.

**Example 6.3.** In the pseudo double category $\mathbb{R}\mathrm{ng}$, objects are commutative rings, horizontal morphisms from $A$ to $B$ are $(B, A)$-bimodules, vertical morphisms are ring homomorphisms, while squares $\beta$ with boundary as in (1) are group homomorphisms $\beta : f \longrightarrow g$ such that $\beta(bxa) = k(b)\beta(x)j(a)$ for all $b \in B, x \in f$, and $a \in A$. Composition of horizontal morphisms is tensor product of bimodules, while composition of vertical morphisms is ordinary function composition.

**Definition 6.4.** A *pseudo double functor* $F : \mathbb{D} \longrightarrow \mathbb{E}$ consists of maps

$$Obj\ \mathbb{D} \longrightarrow Obj\ \mathbb{E}$$

$$Hor\ \mathbb{D} \longrightarrow Hor\ \mathbb{E}$$

$$Ver\ \mathbb{D} \longrightarrow Ver\ \mathbb{E}$$

$$Squares\ \mathbb{D} \longrightarrow Squares\ \mathbb{E}$$

which preserve all sources and targets and are compatible with compositions and units in the following sense. The restriction $\mathbf{H}F$ to the horizontal bicategory is a homomorphism[7] of bicategories whose coherence isos are natural with respect to *all* squares, and further, the restriction of $F$ to the vertical 1-category is an ordinary functor. Equivalently, $F$ consists of functors $F_0 : \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ and $F_1 : \mathbb{D}_1 \longrightarrow \mathbb{E}_1$ and natural isomorphisms

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r] \arrow[d, "F_1 \times_{\mathbb{D}_0} F_1"'] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r] \arrow[ur, Rightarrow, "\gamma"] & \mathbb{E}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "\eta"] \arrow[d, "F_0"'] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_0 \arrow[r, "\eta"'] \arrow[ur, Rightarrow, "\delta"] & \mathbb{E}_1
\end{tikzcd}$$

whose components are squares with trivial vertical edges, and which satisfy the usual three coherence diagrams for homomorphisms of bicategories. The naturality of $\delta$ means

$$\begin{tikzcd}
FA \arrow[r, "1^h_{FA}"] \arrow[d, equal] \arrow[dr, phantom, "\delta_A"] & FA \arrow[d, equal] \\
FA \arrow[r, "F1^h_A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "F(i^h_j)"] & FA \arrow[d, "Fj"] \\
FC \arrow[r, "F1^h_C"'] & FC
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "1^h_{FA}"] \arrow[d, "Fj"'] \arrow[dr, phantom, "i^h_{Fj}"] & FA \arrow[d, "Fj"] \\
FC \arrow[r, "1^h_{FC}"] \arrow[d, equal] \arrow[dr, phantom, "\delta_C"] & FC \arrow[d, equal] \\
FC \arrow[r, "F1^h_C"'] & FC
\end{tikzcd}$$

---

[7]A *homomorphism of bicategories* preserves composition and units up to coherence isos. The composition coherence iso satisfies a hexagon diagram with the composition coherence isos of the bicategories, and the unit coherence iso satisfies two square diagrams with the unit coherence isos of the bicateogries. Some authors call this a *pseudo functor*.

for all vertical morphisms $j$. Thus if $\delta$ is trivial, then $F$ preserves horizontal identity squares. If $\gamma$ is additionally trivial and $\mathbb{D}$ and $\mathbb{E}$ are strict, then $F$ is an internal functor in $Cat$, in other words a double functor.

**Definition 6.5.** Let $\mathbb{D}$ be a pseudo double category. A *pseudo holonomy* is a homomorphism of bicategories $(\mathbf{V}\mathbb{D})_0 \longrightarrow \mathbf{H}\mathbb{D}$ which is the identity on objects.

**Definition 6.6.** Let $\mathbb{D}$ be a pseudo double category. A *pseudo folding on* $\mathbb{D}$ consists of a pseudo holonomy $j \longmapsto \overline{j}$ and bijections $\Lambda^{f,k}_{j,g}$ from squares in $\mathbb{D}$ with boundary as in (6) to squares in $\mathbb{D}$ with boundary as in (7) such that (i),(ii), (iii), and (iv) of Definition 3.16 hold after composing with the coherence iso 2-cells of the horizontal bicategory and the pseudo holonomy. If the pseudo holonomy of a pseudo folding is a strict 2-functor, then we say simply *folding* instead of pseudo folding.

**Remark 6.7.** One would like to say that a pseudo folding on a pseudo double category is a pseudo double functor $\mathbb{D} \longrightarrow \mathbb{Q}\mathbf{H}\mathbb{D}$, but the quintets of a bicategory unfortunately do not form a pseudo double category. Instead we write out what the pseudo functoriality would mean directly: the holonomy 2-functor is replaced by a homomorphism of bicategories, and composites of squares are preserved after composing with coherence isos.

**Definition 6.8.** Let $\mathbb{D}$ and $\mathbb{E}$ be pseudo double categories with pseudo folding. A *morphism of pseudo double categories with pseudo folding* $F : \mathbb{D} \longrightarrow \mathbb{E}$ is a morphism of pseudo double categories equipped with a coherence iso 2-cell

$$F(\overline{j}) \longrightarrow \overline{F(j)}$$

of the horizontal bicategory for each vertical morphism $j$ of $\mathbb{D}$ such that:

i. the coherence iso 2-cells are compatible with the coherence iso 2-cells of the pseudo holonomies,

ii. after composing with the relevant coherence iso 2-cells, we have

$$F(\Lambda^{\mathbb{D}}(\alpha)) = \Lambda^{\mathbb{E}}(F(\alpha))$$

for all squares $\alpha$ of $\mathbb{D}$.

**Example 6.9.** Consider the pseudo double category $\mathbb{R}$ng of rings, bimodules, ring homomorphisms, and squares in Example 6.3. From a ring homomorphism $j : A \longrightarrow C$ we get a $(C, A)$-bimodule $\overline{j}$ by viewing $C$ as a left $C$-module in the usual way and as a right $A$-module via $j$. We also denote this bimodule by $C_j$ as in [**64**], where such base changes are organized into a so-called closed symmetric bicategory. The map $j \longmapsto \overline{j}$ defines a pseudo holonomy which strictly preserves units, but preserves compositions only up to a coherence iso 2-cell. For a square

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "j"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, "k"] \\ C \arrow[r, "N"'] & D \end{tikzcd},$$

*i.e.* for a group homomorphism $\beta : M \longrightarrow N$ such that $\beta(bma) = k(b)\beta(m)j(a)$, define a 2-cell of $\mathbf{H}\mathbb{R}\mathrm{ng}$

$$\Lambda(\beta) : D_k \otimes_B M \Longrightarrow N \otimes_C C_j$$

$$d \otimes m \longmapsto (d \cdot \beta(m)) \otimes 1_C.$$

This is well defined because $(d, m) \longmapsto d \cdot \beta(m)$ is middle $B$-linear.

$$\begin{aligned}(d \cdot b) \cdot \beta(m) &= (dk(b)) \cdot \beta(m)\\ &= d \cdot (k(b) \cdot \beta(m))\\ &= d \cdot \beta(b \cdot m)\end{aligned}$$

Then $\Lambda$ is bijective, a pre-image of a 2-cell $\beta' : D_k \otimes_B M \Longrightarrow N \otimes_C C_j$ is given by $\beta(m) := \beta'(1_D \otimes m)$ under the identification $N \otimes_C C_j \cong N_j$. The coherence diagrams associated with (i),(ii), and (iii) of Definition 3.16 can be verified, and $\Lambda$ is a pseudo folding on $\mathbb{R}\mathrm{ng}$ with pseudo holonomy $j \longmapsto \overline{j}$. For more on this example in the context of so-called anchored bicategories, trace maps, and symmetric bicategories, see [**64**], [**70**], and [**72**].

**Example 6.10.** The pseudo double category $\mathbb{W}$ of worldsheets is relevant to conformal field theory, and admits a folding. A *worldsheet* $x$ is a real, compact, not necessarily connected, two dimensional, smooth manifold with complex structure and real analytically parametrized boundary components. A boundary component $k$ is called *inbound* if the orientation of its parametrization $f_k : S^1 \longrightarrow k$ with respect to the orientation on $k$ is the same as the orientation of the identity parametrization of the boundary of the unit disk. Otherwise $k$ is called *outbound.* We say that the inbound components of $x$ are *labelled by a finite set* $A$ if $x$ is equipped with a bijection between the set of inbound components and $A$.

The objects of $\mathbb{W}$ are finite sets, the horizontal morphisms from $A$ to $B$ are worldsheets $x$ whose inbound respectively outbound components are labelled by $A$ respectively $B$, the vertical morphisms are bijections of sets. For two finite sets $A$ and $B$ of the same cardinality, we also include in our horizontal morphisms from $A$ to $B$ unions of unparametrized circles $(a, S^1, b)$ where each $a \in A$ and each $b \in B$ appear only once, so we can view such unions as bijections. The circle $(a, S^1, b)$ is viewed as an infinitely thin annulus with inbound component labelled by $a$ and outbound component labelled by $b$. For $x$ and $y$ worldsheets, a square

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "j"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, "k"] \\ C \arrow[r, "y"'] & D \end{tikzcd}$$

consists of a holomorphic diffeomorphism $\beta : x \longrightarrow y$ which:

i. takes every inbound component of $x$ labelled by $a \in A$ to an inbound component of $y$ labelled by $j(a)$,

ii. takes every outbound component of $x$ labelled by $b \in B$ to an outbound component of $y$ labelled by $k(b)$, and

iii. preserves the boundary parametrizations.

For $x$ and $y$ unions of circles, there is a unique square $\beta$ with boundary as above if and only if $x = y$, and $j$ and $k$ are vertical identities. This unique square is necessarily the identity. There are no other squares in this double category.

The composition of horizontal morphisms is given by gluing of surfaces, and the horizontal composition of squares is defined analogously. The composition of vertical morphisms is composition of functions. The strict horizontal unit from $A$ to $A$ is the union of circles $(a, S^1, a)$ for $a \in A$. A strict holonomy is defined by mapping a function $j$ to the union of circles $(a, S^1, j(a))$.

Actually, this example is an illustration of a pseudo version of Remark 3.14 and Remark 3.15. The worldsheets admit a mixed composition with the bijections via relabelling. Including the disjoint union of circles also as horizontal morphisms corresponds to extending $\mathbb{D}$ to $\mathbb{D}'$, and this explains our choice of squares. The holonomy is the inclusion.

A folding $\Lambda$ is given by simply relabelling the outbound boundary components of $x$ via $k$ and relabelling the inbound boundary components of $y$ via $j$: the holomorphic diffeomorphism stays the same.

This example, along with [**52**] and [**66**], suggests that double categories play a role in the mathematics relating to field theories and high energy physics.

## 7. Pseudo Algebras and Pseudo Double Categories with Folding

As we have seen in Theorems 4.6, 4.8, and 4.9, the 2-categories of $I$-categories, double categories with folding, and certain strict 2-functors are 2-equivalent if $I$ is a groupoid, and the latter two remain 2-equivalent even if $I$ is merely a category. Next we work towards pseudo versions of these theorems, as stated in Theorems 7.9, 7.10, and 7.11. We prove the 2-equivalence of three 2-categories $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ as introduced[8] in Notations 7.5, 7.6, and 7.7. The 2-category $\mathcal{X}$ is the 2-category of pseudo $I$-categories as defined below, while $\mathcal{Y}$ and $\mathcal{Z}$ are the 2-categories of certain pseudo double categories with pseudo folding and certain 2-functors respectively.

**Definition 7.1.** A *pseudo algebra over the 2-theory of categories with underlying category $I$*, also called a *pseudo $I$-category* for short, is a category $I$ and a strict 2-functor $X : I^2 \longrightarrow Cat$ with strictly 2-natural functors

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$

$$* \xrightarrow{\eta_B} X_{B,B}$$

[8]For the 2-equivalences with $\mathcal{X}$ we assume $I$ is a groupoid.

for all $A, B, C \in I$ and natural isomorphisms

$$\begin{tikzcd}
X_{C,D} \times (X_{B,C} \times X_{A,B}) \arrow[r, "\circ \times 1_{X_{A,B}}"] \arrow[d, "\cong"'] & X_{B,D} \times X_{A,B} \arrow[dr, "\circ"] & \\
 & \Downarrow\alpha_{A,B,C,D} & X_{A,D} \\
(X_{C,D} \times X_{B,C}) \times X_{A,B} \arrow[r, "1_{X_{C,D}} \times \circ"'] & X_{C,D} \times X_{A,C} \arrow[ur, "\circ"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
* \times X_{A,B} \arrow[r, "\eta_B \times 1_{X_{A,B}}"] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\Downarrow\lambda_{A,B}"] & X_{B,B} \times X_{A,B} \arrow[d, "\circ"] \\
X_{A,B} \arrow[r, "1_{X_{A,B}}"'] & X_{A,B}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{B,C} \times * \arrow[r, "1_{X_{B,C}} \times \eta_B"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\Downarrow\rho_{B,C}"] & X_{B,C} \times X_{B,B} \arrow[d, "\circ"] \\
X_{B,C} \arrow[r, "1_{X_{B,C}}"'] & X_{B,C}
\end{tikzcd}$$

which are the components of modifications and satisfy the usual coherence diagrams for a bicategory as in the original [**9**], or in the review [**54**], or in the Appendix to [**39**]. The requirement that $\alpha, \lambda$, and $\rho$ be modifications means $X_{j,m}(\alpha_{h,g,f}) = \alpha_{X_{\ell,m}(h),X_{k,\ell}(g),X_{j,k}(f)}$, $X_{j,k}(\lambda_f) = \lambda_{X_{j,k}(f)}$, and $X_{k,\ell}(\rho_g) = \rho_{X_{k,\ell}(g)}$ for all

$$j : A \longrightarrow A'$$

$$k : B \longrightarrow B'$$

$$\ell : C \longrightarrow C'$$

$$m : D \longrightarrow D'.$$

If $\lambda$ and $\rho$ are identities, we say $X$ *has strict units*. We denote the value of $\eta_B$ on the unique object and morphism of the terminal category by $1_B$ and $i_{1_B}$ respectively. We denote the identity morphism on an object $f$ in the category $X_{A,B}$ by $i_f$.

**Definition 7.2.** A *morphism of pseudo $I$-categories* $F : X \longrightarrow Y$ is a strict 2-natural transformation $F : X \Longrightarrow Y$ with natural isomorphisms

$$\begin{tikzcd}
X_{B,C} \times X_{A,B} \arrow[r, "\circ"] \arrow[d, "F_{B,C} \times F_{A,B}"'] \arrow[dr, phantom, "\Uparrow\gamma_{A,B,C}"] & X_{A,C} \arrow[d, "F_{A,C}"] \\
Y_{B,C} \times Y_{A,B} \arrow[r, "\circ"'] & Y_{A,C}
\end{tikzcd}$$

$$\begin{tikzcd} * \arrow[r, "\eta^X_A"] \arrow[d] \arrow[dr, phantom, "\Uparrow\delta_A"] & X_{A,A} \arrow[d, "F_{A,A}"] \\ * \arrow[r, "\eta^Y_A"'] & Y_{A,A} \end{tikzcd}$$

which are the components of modifications and satisfy the usual coherence diagrams for homomorphisms of bicategories. The requirement that $\gamma$ and $\delta$ be modifications is equivalent to $Y_{j,\ell}(\gamma_{g,f}) = \gamma_{Y_{k,\ell}(g),Y_{j,k}(f)}$ and $Y_{j,j}(\delta_A) = \delta_{A'}$.

**Definition 7.3.** A *2-cell* $\sigma : F \Longrightarrow G$ *in the 2-category of pseudo* $I$*-categories* is a modification $\sigma : F \rightsquigarrow G$ compatible with composition and identity. More specifically, a 2-cell $\sigma$ consists of natural transformations $\sigma_{A,B} : F_{A,B} \Longrightarrow G_{A,B}$ for all $A, B \in I$ such that

$$Y_{j,k}(\sigma^f_{A,B}) = \sigma^{X_{j,k}(f)}_{C,D}$$

$$\gamma^G_{g,f} \odot (\sigma^g_{B,C} \circ \sigma^f_{A,B}) = \sigma^{g\circ f}_{A,C} \odot \gamma^F_{g,f}$$

$$\sigma^{1_A}_{A,A} \odot \delta^F_A = \delta^G_A$$

for all $(j,k) : (A,B) \longrightarrow (C,D)$ in $I^2, f \in X_{A,B}$, $g \in X_{B,C}$, and all objects $A$ of $I$. Here $\odot$ denotes the composition in the categories $X_{A,B}$ and $\circ$ denotes the composition functor of pseudo $I$-categories.

Following the convention introduced in Section 2, we use the term *pseudo* $I$*-category* to abbreviate *pseudo algebra over the 2-theory of categories with underlying category* $I$. The morphisms and 2-cells above are the morphisms and 2-cells in the 2-category of pseudo algebras over the 2-theory of categories with underlying category $I$ as in [**38**], [**49**], and [**50**]. In this section $I$ will denote a fixed category. Whenever we require $I$ to be a groupoid, we will explicitly say so.

**Remark 7.4.** If $I$ is a groupoid, units are strict, and $F$ and $G$ are morphisms of pseudo $I$-categories that strictly preserve the units, then Remark 2.5 holds. In particular, the requirement $Y_{j,k}(\sigma^f_{A,B}) = \sigma^{X_{j,k}(f)}_{C,D}$ on a 2-cell $\sigma$ can replaced by $\sigma^{P(j)}_{A,C} = i_{P'(j)}$.

**Notation 7.5.** Let $\mathcal{X}$ denote the 2-category of pseudo $I$-categories with strict units. The morphisms of $\mathcal{X}$ are morphisms of pseudo $I$-categories which preserve the units strictly.

**Notation 7.6.** Let $\mathcal{Y}$ denote the 2-category of pseudo double categories $\mathbb{D}$ with strict units equipped with a folding (*i.e.* the holonomy is strict) and such that $(\mathbf{V}\mathbb{D})_0 = I$. We further require that the associativity coherence iso $\alpha_{\overline{k},f,\overline{\ell}}$ is the identity for vertical morphisms $k$ and $\ell$ and horizontal morphisms $f$ such that $\overline{k} \circ f \circ \overline{\ell}$ exists.

A morphism in $\mathcal{Y}$ is a morphism $F$ of pseudo double categories with folding which preserves the holonomy and units strictly and is the identity on $(\mathbf{V}\mathbb{D})_0$. We further require that $\gamma^F_{\overline{k},f}$ and $\gamma^F_{f,\overline{\ell}}$ are identities.

A 2-cell $\sigma : F \Longrightarrow G$ in $\mathcal{Y}$ is a vertical natural transformation that is compatible with folding and has identity components. Less succinctly, a 2-cell assigns to each pair $(A, B) \in I^2$ a natural transformation $\sigma_{A,B} : \mathbf{H}F_{A,B} \Longrightarrow \mathbf{H}G_{A,B}$ such that

$$\sigma^{\overline{j}}_{A,C} = i^v_{\overline{j}}$$

$$\begin{bmatrix} [\ \sigma^f_{A,B} \quad \sigma^g_{B,C}\ ] \\ \gamma^G_{g,f} \end{bmatrix} = \begin{bmatrix} \gamma^F_{g,f} \\ \sigma^{[f\ \ g]}_{A,C} \end{bmatrix}$$

$$\sigma^{1^h_A}_{A,A} = i_{1^h_A}$$

for all vertical morphisms $j \in I(A, C)$, composable horizontal morphisms $f$ and $g$, and all objects $A$.

**Notation 7.7.** Another 2-category of interest is the 2-category $\mathcal{Z}$. An object of $\mathcal{Z}$ is a strict 2-functor $P : I \longrightarrow \mathcal{C}$ into a bicategory $\mathcal{C}$ with strict units which is the identity on objects. We further require that the associativity coherence iso $\alpha_{P(k),f,P(\ell)}$ of the bicategory $\mathcal{C}$ is the identity for morphisms $k$ and $\ell$ of $I$ such that $P(k) \circ f \circ P(\ell)$ exists in $\mathcal{C}$. The object set of $\mathcal{C}$ is *Obj* $I$.

A morphism from $P : I \longrightarrow \mathcal{C}$ to $P' : I \longrightarrow \mathcal{C}'$ in $\mathcal{Z}$ is a homomorphism of bicategories $F : \mathcal{C} \longrightarrow \mathcal{C}'$ which strictly preserves units and such that

$$\begin{tikzcd} I \arrow[r, "P"] \arrow[dr, "P'"'] & \mathcal{C} \arrow[d, "F"] \\ & \mathcal{C}' \end{tikzcd} \tag{9}$$

strictly commutes. We further require that $\gamma^F_{P(k),f}$ and $\gamma^F_{f,P(\ell)}$ are identities.

A 2-cell $\sigma : F \Longrightarrow G$ consists of natural transformations $\sigma_{A,B}$ for all $A, B \in$ *Obj* $\mathcal{C}$ such that

$$\sigma^{P(j)}_{A,C} = i_{P'(j)}$$

$$\gamma^G_{g,f} \odot (\sigma^g_{B,C} \circ \sigma^f_{A,B}) = \sigma^{g \circ f}_{A,C} \odot \gamma^F_{g,f}$$

$$\sigma^{1^h_A}_{A,A} = i_{1^h_A}$$

for all $j \in I(A, C), f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$, and all objects $A$ of $I$. Here $\odot$ denotes the vertical composition of 2-cells in a bicategory, while $\circ$ denotes the horizontal composition of 2-cells.

**Remark 7.8.** The requirement that units be strict in a pseudo double category is not as rigid as it first seems, since this can be arranged in most examples. The authors of [**45**] and [**46**] also assume that units are strict, and arrange it in most of their examples.

**Theorem 7.9.** *Let $I$ be a category. The 2-category $\mathcal{Y}$ of pseudo double categories $\mathbb{D}$ with strict units equipped with a folding such that $(\mathbf{V}\mathbb{D})_0 = I$ is 2-equivalent to the 2-category $\mathcal{Z}$ of strict 2-functors $I \longrightarrow \mathcal{C}$ into bicategories $\mathcal{C}$ with strict units which are the identity on objects as in Notation 7.6 and Notation 7.7.*

**Theorem 7.10.** *Let $I$ be a groupoid. The 2-category $\mathcal{X}$ of pseudo $I$-categories with strict units (pseudo algebras over the 2-theory of categories with underlying groupoid $I$ and strict units) is 2-equivalent to the 2-category $\mathcal{Y}$ of pseudo double categories $\mathbb{D}$ with strict units equipped with a folding such that $(\mathbf{V}\mathbb{D})_0 = I$ as defined in Notation 7.5 and Notation 7.6.*

**Theorem 7.11.** *Let $I$ be a groupoid. The 2-category $\mathcal{X}$ of pseudo $I$-categories with strict units (pseudo algebras over the 2-theory of categories with underlying groupoid $I$ and strict units) is 2-equivalent to the 2-category $\mathcal{Z}$ of strict 2-functors $I \longrightarrow \mathcal{C}$ into bicategories $\mathcal{C}$ with strict units which are the identity on objects as in Notation 7.5 and Notation 7.7.*

We omit the proofs of Theorems 7.9, 7.10, and 7.11 since they are straightforward but tedious elaborations of the strict Theorems 4.6, 4.8, and 4.9. The strictness of units for $X$ in $\mathcal{X}$ corresponds to the strictness of the holonomy in $\mathcal{Y}$ and the strictness of $P : I \longrightarrow \mathcal{C}$ in $\mathcal{Z}$. The fact that morphisms of $\mathcal{X}$ strictly preserve units corresponds to strict preservation of holonomy by morphisms in $\mathcal{Y}$, as well as the strict preservation of units by morphisms in $\mathcal{Z}$ and the strict commutativity of Diagram (9).

**Example 7.12.** The pseudo double category $\mathbb{R}\mathrm{ng}$ in Example 6.9 can be slightly modified to make it into an object of $\mathcal{Y}$ in Theorem 7.9 and Theorem 7.10. First we require the vertical morphisms to be isomorphisms of rings, then note that bimodules admit a mixed composition with isomorphisms of rings, and apply Remark 3.14 and Remark 3.15. Thus the horizontal morphisms of $\mathbb{R}\mathrm{ng}'_{\mathrm{iso}}$ are bimodules as well as isomorphisms of rings. A $(B, A)$-bimodule $M$ is composed with a ring isomorphism $k : B \longrightarrow D$ to give a $(D, A)$-bimodule $k \circ M$ with underlying abelian group $M$ by defining $d \cdot m := k^{-1}(d) \cdot m$. The composition $N \circ j$ is defined similarly. The squares of $\mathbb{R}\mathrm{ng}'_{\mathrm{iso}}$ are the squares of $\mathbb{R}\mathrm{ng}$ with invertible vertical sides, along with vertical identities of the isomorphisms of rings. The holonomy is then an inclusion and the horizontal bicategory is strictly unital.

**Example 7.13.** The pseudo double category $\mathbb{W}$ of worldsheets in Example 6.10 is an object of $\mathcal{Y}$ in Theorem 7.9 and Theorem 7.10 with $I$ the category of finite sets and bijections. The horizontal morphisms are worldsheets as well as bijections.

**Theorem 7.14.** *Analogues of Theorems 7.9, 7.10, and 7.11 hold for weak units and pseudo foldings, though "2-equivalence" must be replaced by "biequivalence." Pseudo $I$-categories with weak units correspond to pseudo double categories with weak units and pseudo foldings, which in turn correspond to homomorphisms of* *bicategories $P$ from the groupoid $I$ to a bicategory $\mathcal{C}$ with weak units. Morphisms of pseudo $I$-categories then correspond to morphisms of pseudo double categories that preserve the pseudo holonomy up to coherence iso, which in turn correspond to homomorphisms $F$ of bicategories such that (9) commutes on objects strictly, but has a coherence iso 2-cell $FP(j) \cong P'(j)$ for each morphism $j$ of $I$.*

*Proof.* Omitted. The proof relies on a construction like $L(P)$ in the proof of Theorem 6.5 in [**39**] to remedy

$$[[\overline{\ell}\ f]\ \overline{k}] \neq [\overline{\ell}\ [f\ \overline{k}]]$$

$$F([\overline{\ell}\ f\ \overline{k}]) \neq [F(\overline{\ell})\ F(f)\ F(\overline{k})]$$

$$(P(k) \circ f) \circ P(\ell) \neq P(k) \circ (f \circ P(\ell))$$

$$F(P(k) \circ f \circ P(\ell)) \neq F(P(k)) \circ F(f) \circ F(P(\ell)).$$

□

This completes our comparison of strict 2-algebras and pseudo algebras over the 2-theory of categories with variants of double categories and 2-functors $I \longrightarrow \mathcal{C}$.

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